Distinguished representations of reductive p-adic groups

Fiona Murnaghan

Abstract We present a survey of results on representations of reductive p-adic groups distinguished by groups of fixed points of involutions. Topics discussed include criteria that characterize relatively supercuspidal and relative discrete series representations, formulas for spaces of invariant forms on distingished tame supercuspidal representations, and properties of spherical characters.

1 Introduction

If π is a representation of a group G, V is the space of π, H is a subgroup of G, and χ is a one-dimensional representation of H, let

$$\mathrm{Hom}_H(\pi,\chi) = \{\, \lambda \in V^* \mid \langle \lambda\,,\,\pi(h)v \rangle = \chi(h)\langle \lambda\,,\,v\rangle \ \forall\, h \in H,\ v \in V \,\}.$$

If χ is trivial, we write $\mathrm{Hom}_H(\pi,1)$. The representation π is said to be (H,χ)-*distinguished* (or simply H-*distinguished* if χ is trivial) if $\mathrm{Hom}_H(\pi,\chi)$ is nonzero.

The purpose of these notes is to describe several results pertaining to representations distinguished by symmetric subgroups of reductive groups over nonarchimedean local fields.

Let F be a nonarchimedean local field, $\boldsymbol{G}$ a connected reductive F-group and $G = \boldsymbol{G}(F)$. Throughout, we assume that the residual characteristic of F is odd. By an *involution* of G, we mean an F-automorphism of $\boldsymbol{G}$ of order two. We work in the setting of H-distinguished smooth complex representations of G, where $H = \boldsymbol{H}(F)$ and $\boldsymbol{H} = \boldsymbol{G}^\theta$ is the group of fixed points of an involution θ of G. We refer to $(\boldsymbol{G},\boldsymbol{H})$ (or (G,H)) as a *symmetric pair* and G/H as a *(reductive p-adic) symmetric space*.

We now give a brief overview of the contents of these notes. Section 2 is devoted mainly to setting up notation and terminology. In Section 3, we list a few exam-

Fiona Murnaghan
University of Toronto, e-mail: fiona@math.toronto.edu

ples of symmetric pairs. Information about tori and parabolic subgroups adapted to involutions is reviewed in Sections 4 and 5.

Various fundamental results in the representation theory of G have analogues in the context of H-distinguished representations. These include Jacquet's subrepresentation theorem, the characterization of irreducible supercuspidal representations in terms of vanishing of Jacquet modules, Casselman's criterion for square-integrability and Harish-Chandra's local integrability of characters of admissible representations. We recall these results and describe their relative (or symmetric space) analogues in Sections 6, 7, 9 and 12.

Two examples of relatively supercuspidal representations are discussed in Section 8.

In Section 10, we discuss cases where H-invariant forms are realized by way of integrating matrix coefficients.

A summary of results on distinguished tame supercuspidal representations is given in Section 11.

These notes are based on material covered in lectures given at the doctoral school. Some additional material has been included.

We emphasize that there are various important topics and results not discussed here. These include:

- The major work of Sakellaridis and Venkatesh ([38]) on the Plancherel decomposition of $L^2(\boldsymbol{X}(F))$, where $\boldsymbol{X}$ is a spherical $\boldsymbol{G}$-variety, $\boldsymbol{G}$ splits over F and F has characteristic zero. As symmetric varieties are spherical, the results are applicable in the setting of p-adic symmetric varieties. Also of note are conjectures in [38] concerning direct integral decompositions of $L^2(\boldsymbol{X}(F))$ in terms of Arthur parameters that are $\boldsymbol{X}$-distinguished in some sense.
- The results of Blanc and Delorme ([2]) giving decompositions of the spaces of invariant forms $\mathrm{Hom}_H(\pi,1)$ for representations π arising via parabolic induction from distinguished representations of Levi factors of proper θ-split parabolic subgroups.
- Local results obtained via global methods: in some situations, results on distinction of representations occurring as local components of automorphic representations may be deduced from global results on nonvanishing of period integrals.
- Connections between distinguished representations, functorial lifting and properties of local factors such as L-functions.

2 Preliminaries

The normalized absolute value on the nonarchimedean local field F will be denoted by $|\cdot|_F$. We will assume throughout that the residual characteristic of F is odd. If E is a finite extension of F, $\mathrm{Res}_{E/F}$ denotes restriction of scalars.

If $\boldsymbol{G}'$ is an algebraic F-group, let $\boldsymbol{G}'^{\circ}$ be the identity component of $\boldsymbol{G}'$ and $G' = \boldsymbol{G}'(F)$, that is, the F-rational points of $\boldsymbol{G}'$.

Recall that $\boldsymbol{G}$ denotes a connected reductive F-group. Let $\boldsymbol{A_G}$ be the maximal F-split torus in the centre of $\boldsymbol{G}$. The centre of G is denoted by Z_G. We refer to $A_G := \boldsymbol{A_G}(F)$ as the F-split component of G. If $\boldsymbol{G}'$ (respectively G') is a subgroup of $\boldsymbol{G}$ (respectively, G), let $Z_{\boldsymbol{G}}(\boldsymbol{G}')$ (respectively, $Z_G(G')$) be the centralizer of $\boldsymbol{G}'$ in $\boldsymbol{G}$ (respectively, of G' in G).

As in the introduction, we fix an involution θ of G, that is, an involutive F-automorphism of $\boldsymbol{G}$, and we set $\boldsymbol{H} = \boldsymbol{G}^\theta$. (In many contexts, it is possible to work in the slightly more general situation, where $\boldsymbol{H}$ is an F-subgroup of $\boldsymbol{G}$ such that $(\boldsymbol{G}^\theta)^\circ \subseteq \boldsymbol{H} \subseteq \boldsymbol{G}^\theta$. See [22] and [35], for example.)

A quasicharacter of G (respectively, a closed subgroup G' of G) is a smooth one-dimensional representation of G (respectively, G'). If χ is a quasicharacter of H, the notion of (H,χ)-distinguished representation of G, along with related notations $\mathrm{Hom}_H(\pi,\chi)$ and $\mathrm{Hom}_H(\pi,1)$, are as defined in the introduction.

Throughout the notes, (π,V) denotes a smooth (complex) representation of G, which, at various points, exhibits additional properties. The notation $(\widetilde{\pi},\widetilde{V})$ will be used for the contragredient (smooth dual) of π. If V^* is the dual of V, then $\langle\cdot,\cdot\rangle : V^* \times V \to \mathbb{C}$ denotes the usual G-invariant pairing (given by evaluation), which restricts to a pairing on $\widetilde{V} \times V$. Recall that the (smooth) representation (π,V) is *admissible* if for every compact open subgroup K of G, the subspace V^K of K-fixed vectors in V is finite-dimensional.

Recall that a *matrix coefficient of* π is a function of the form

$$g \mapsto \varphi_{\tilde{v},v}(g) := \langle \tilde{v}, \pi(g^{-1})v \rangle, \qquad g \in G,$$

where $\tilde{v} \in \widetilde{V}$ and $v \in V$ are fixed. Here, we are following the convention that defines matrix coefficient in such a way that, for fixed $\tilde{v}$, the map $v \mapsto \varphi_{\tilde{v},v}$ interwines π and the left regular representation of G on the space $C^\infty(G)$ of locally constant complex-valued functions on G.

For $\lambda \in \mathrm{Hom}_H(\pi,1)$ and $v \in V$, define $\varphi_{\lambda,v}(g) = \langle \lambda \, , \pi(g^{-1})v \rangle$. The function $\varphi_{\lambda,v}$, which is smooth and right H-invariant, is called a *generalized (or relative) matrix coefficient of* π, or an (H,λ)-*matrix coefficient of* π. Note that, if λ is nonzero, then λ is typically not smooth, in which case $\varphi_{\lambda,v}$ is not a matrix coefficient of π in the usual sense.

Let $C^\infty(G/H)$ be the space of locally constant complex-valued functions on G/H. Whenever convenient, we identify $C^\infty(G/H)$ with the subspace of right H-invariant functions in $C^\infty(G)$. If $\lambda \in \mathrm{Hom}_H(\pi,1)$ then the map $v \mapsto \varphi_{\lambda,v}$ interwines π with the left regular representation of G on $C^\infty(G/H)$. Viewing $C^\infty(G/H)$ as induced from the trivial representation of H, by a version of Frobenius reciprocity ([4], Theorem 2.28), if π is irreducible, then π is H-distinguished if and only if π is equivalent to a subrepresentation of $C^\infty(G/H)$.

The representation π is *supercuspidal* (the terminology *cuspidal* is also commonly used) if every matrix coefficient of π is compactly supported modulo the centre Z_G of G. If π is H-distinguished and $\lambda \in \mathrm{Hom}_H(\pi,1)$ is nonzero, then we say that π is (H,λ)-*relatively supercuspidal* if for all $v \in V$, the function $\varphi_{\lambda,v}$ is compactly supported modulo HZ_G. The representation π is H-*relatively supercusp-*

idal if π is (H,λ)-*relatively supercuspidal* for all nonzero $\lambda \in \mathrm{Hom}_H(\pi,1)$. (When H is understood, we may simply refer to relatively supercuspidal representations.) As discussed in Section 7, the H-relatively supercuspidal representations are the symmetric space analogues of the supercuspidal representations.

Suppose that π has a unitary central quasicharacter, that is, Z_G acts on V through a unitary quasicharacter. Then the absolute value of each matrix coefficient of π is a function on G/Z_G. The representation π is *square integrable (mod centre)*, or is a *discrete series* representation, if the absolute value of every matrix coefficient of π is square integrable on G/Z_G. If π is H-distinguished, then the absolute value of each relative matrix coefficient of π is a function on G/HZ_G. Given $\lambda \in \mathrm{Hom}_H(\pi,1)$, we say that π is λ-*square integrable* if for every $v \in V$, the absolute value of $\varphi_{\lambda,v}$ is square integrable on G/HZ_G. The representation π is H-*square integrable* or is a *relative discrete series* representation if π is λ-square integrable for every $\lambda \in \mathrm{Hom}_H(\pi,1)$. The relative discrete series representations are the symmetric space analogues of the discrete series representations (see Section 9).

Remark 1. In the group case (see §3), a distinguished representation $\pi \otimes \widetilde{\pi}$ of $G = G' \times G'$ is a relatively supercuspidal (respectively, relative discrete series) representation if and only if π is a supercuspidal (respectively, discrete series) representation of G'.

We close this section by noting a key property of distinguished representations:

Theorem 1. ([9], Theorem 4.5) *If π has finite length, then* $\mathrm{Hom}_H(\pi,1)$ *is finite-dimensional.*

Remark 2. Particular symmetric pairs are known to have the property that the dimension of $\mathrm{Hom}_H(\pi,1)$ is at most one for all irreducible admissible representations π of G. Examples of irreducible supercuspidal representations π for which $\mathrm{Hom}_H(\pi,1)$ has dimension two are described in §5.9 of [17].

3 Examples of symmetric pairs

Suppose that $\boldsymbol{G}'$ is a connected reductive F-group, $\boldsymbol{G} = \boldsymbol{G}' \times \boldsymbol{G}'$ and $\theta(g_1,g_2) = (g_2,g_1)$, $g_1, g_2 \in \boldsymbol{G}'$. We refer to this as the *group case*. Here, $H \cong G'$. If (π,V) and $(\dot{\pi},\dot{V})$ are irreducible admissible representations of G, then $\pi \otimes \dot{\pi}$ is H-distinguished if and only if $\dot{\pi}$ and $\widetilde{\pi}$ are equivalent. (Note that if $\dot{\pi} = \widetilde{\pi}$ then $v \otimes \tilde{v} \mapsto \langle \tilde{v}, v \rangle$, $v \in V$, $\tilde{v} \in \widetilde{V}$, extends to an element of $\mathrm{Hom}_H(\pi \otimes \widetilde{\pi},1)$.) Harmonic analysis on G/H corresponds to harmonic analysis on G' in the sense that particular properties of an H-distinguished representation $\pi \otimes \widetilde{\pi}$ of G are analogous to properties of the representation π of G' (see Remarks 1, 4 and 6).

Symmetric pairs $(\boldsymbol{G},\boldsymbol{H})$ with or $\boldsymbol{G} = \mathrm{Res}_{E/F}\, \boldsymbol{GL}_n$ for some finite extension E of F have been studied by various authors. Examples include $(\boldsymbol{GL}_{2n},\boldsymbol{Sp}_{2n})$, $(\boldsymbol{GL}_n,\boldsymbol{O}_n)$ (here, $\boldsymbol{O}_n$ denotes an $n \times n$ orthogonal group), $(\boldsymbol{GL}_n,\boldsymbol{GL}_k \times \boldsymbol{GL}_{n-k})$, $1 \le k \le n-1$, and $(\mathrm{Res}_{E/F}\,\boldsymbol{GL}_n, \boldsymbol{U}_n)$, where E is a quadratic extension of F and $\boldsymbol{U}_n$ is an $n \times n$ unitary group relative to the extension E/F.

Let E be a quadratic extension of F and let $\boldsymbol{G}'$ be a connected reductive F-group. Let θ be the involution of $\boldsymbol{G} = \mathrm{Res}_{E/F}\,\boldsymbol{G}'$ determined by the action of the nontrivial element of $\mathrm{Gal}(E/F)$. The pair $(\boldsymbol{G}'(E), \boldsymbol{G}'(F))$ is often referred to as a *Galois* symmetric pair.

If $g_0 \in G \setminus Z_G$ is such that $g_0^2 \in Z_G$ then conjugation by g_0 defines an inner involution of G. For example, the symmetric pairs $(\boldsymbol{GL}_n(F), \boldsymbol{GL}_k(F) \times \boldsymbol{GL}_{n-k}(F))$, $1 \leq k < n$, and $(\boldsymbol{Sp}_{4n}(F), \boldsymbol{Sp}_{2n}(F) \times \boldsymbol{Sp}_{2n}(F))$ arise from inner involutions. Here, $\boldsymbol{Sp}_{2n}$ denotes the $2n \times 2n$ symplectic group.

4 Tori

If $\boldsymbol{T}$ is a (maximal) F-torus of $\boldsymbol{G}$, we refer to $T = \boldsymbol{T}(F)$ as a (maximal) torus of G. We say that T splits over F whenever $\boldsymbol{T}$ splits over F.

There exist θ-stable maximal tori of G ([20]). Indeed, there exists a θ-stable maximal torus T of G such that A_T is a maximal F-split torus of G ([22], Prop. 2.3). An element $g \in G$ is said to be θ-*split* if $\theta(g) = g^{-1}$. If $\boldsymbol{T}$ is a θ-stable torus of $\boldsymbol{G}$, we say that $\boldsymbol{T}$ (or T) is θ-*split* if every element of $\boldsymbol{T}$ (or T) is θ-split. A (θ,F)-*split* F-torus is a torus that is both θ-split and F-split.

If $T = \boldsymbol{T}(F)$ is a θ-stable torus of G, let

$$\boldsymbol{T}^- = \{\, t \in \boldsymbol{T} \mid \theta(t) = t^{-1} \,\}^\circ \ \text{ and } \ T^- = \boldsymbol{T}^-(F).$$

Note that, since the centre Z_G of G is θ-stable, the F-split component A_G of G is θ-stable. We refer to A_G^- as the (θ,F)-*split component of* G.

Recall that any two maximal F-tori of $\boldsymbol{G}$ are conjugate under $\boldsymbol{G}$, although they need not be conjugate under G. Moreover, any two maximal F-split tori of G are conjugate in G. Two maximal (θ,F)-split tori of $\boldsymbol{G}$ are conjugate by an element of $\boldsymbol{H}$ ([40]). However, there may be more than one H-conjugacy class of maximal (θ,F)-split tori of G.

Proposition 1. ([22], Prop. 10.3) *Let A and A' be maximal (θ,F)-split tori of G. Let $\tilde{A} = \tilde{\boldsymbol{A}}(F)$ be a maximal F-split torus containing A. Then A and A' are conjugate under $(\boldsymbol{H}^\circ Z_{\boldsymbol{G}}(\tilde{\boldsymbol{A}}))(F)$.*

Remark 3. As shown in Lemma 4.5 of [22], any maximal F-split torus of G that contains a maximal (θ,F)-split torus of G is θ-stable.

As can be seen from the next example, it is not difficult to find particular pairs (G,H) having the property that two maximal (θ,F)-split tori need not be H-conjugate.

Example 1. Let $G = \boldsymbol{SL}_2(F)$ and let θ be the involution given by conjugation by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Fix $\gamma \in F^\times$ and define

$$T_\gamma = \left\{ \begin{pmatrix} a & b\gamma \\ b & a \end{pmatrix} \;\middle|\; a^2 - b^2\gamma = 1 \right\}.$$

Then T_γ is a maximal θ-split torus of G and T_γ is F-split if and only if $\gamma \in (F^\times)^2$. If $\gamma' \in F^\times$, then T_γ and $T_{\gamma'}$ are conjugate under an element of H (that is, under the diagonal subgroup of G) if and only if $\gamma' \in \gamma(F^\times)^4$. Since F is p-adic, $(F^\times)^2 \neq (F^\times)^4$. Hence there exist maximal (θ, F)-split tori in G that are not H-conjugate.

5 Parabolic subgroups

If $\boldsymbol{P}$ is a parabolic F-subgroup of $\boldsymbol{G}$, we refer to $P = \boldsymbol{P}(F)$ as a parabolic subgroup of G. We follow a similar convention with respect to unipotent radicals of parabolic subgroups and Levi subgroups. We sometimes denote the unipotent radical of a parabolic subgroup P of G by N_P.

Before discussing the behaviour of particular parabolic subgroups of G under θ, we recall some basic facts about parabolic subgroups. A minimal parabolic subgroup $P_0 = \boldsymbol{P}_0(F)$ of G contains a maximal F-split torus $A = \boldsymbol{A}(F)$ of G. Let $\Phi = \Phi(\boldsymbol{G}, \boldsymbol{A})$ be the root system of $\boldsymbol{G}$ with respect to $\boldsymbol{A}$. There is a unique basis of Δ of Φ corresponding to the positive system of roots $\Phi^+ = \Phi(\boldsymbol{P}_0, \boldsymbol{A})$. There exists an inclusion-preserving bijection $I \mapsto P_I$ between set of subsets $I \subseteq \Delta$ and the set of parabolic subgroups of G that contain P_0, arising as follows. For $I \subset \Delta$, let $\boldsymbol{A}_I$ be the identity component of the intersection of the kernels of the elements of I let N_I be the unipotent radical of P_I, and let $M_I = Z_G(A_I)$. Then we have a Levi factorization $P_I = M_I \ltimes N_I$ of P_I. The set of roots of A_I in the Lie algebra of N_I is equal to $\Phi^+ \setminus (\Phi^+ \cap \Phi_I)$. Here, Φ_I is the subsystem of Φ generated by I. A parabolic subgroup of G is conjugate to a unique parabolic subgroup $P_I \supseteq P_0$.

Two parabolic subgroups P and Q of G are *opposite* if $P \cap Q$ is a Levi factor of P (and also of Q). In this case, for $M = P \cap Q$, we have $P = M \ltimes N_P$, $Q = M \ltimes N_Q$ and $N_P \cap N_Q = \{1\}$. A parabolic subgroup P of G is said to be θ-*split* if P and $\theta(P)$ are opposite. Observe that in this case, θ is an involution of the Levi factor $P \cap \theta(P)$ of P and $\theta(N_P) = N_{\theta(P)}$. The θ-split parabolic subgroups of G play an important role in many results in harmonic analysis on G/H. Results about θ-split subgroups in the algebraically closed setting are found in [40], whereas [21] and [22] contain detailed information in the setting of groups over fields of characteristic not equal to 2.

Suppose that P is a θ-split parabolic subgroup of G, $M = P \cap \theta(P)$, $N = N_P$, $\mathfrak{m} = \mathrm{Lie}(M)$, $\mathfrak{n} = \mathrm{Lie}(N)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. Then (denoting the differential of θ by θ), we have $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{m} \oplus \theta(\mathfrak{n})$. Given $X \in \theta(\mathfrak{n})$, we write $X = (X + \theta(X)) - \theta(X)$ with $X + \theta(X) \in \mathfrak{m}$ and $\theta(X) \in \mathfrak{n}$. Thus $\mathfrak{g} = \mathfrak{n} + \mathfrak{m} + \mathfrak{h}$. This implies that PH is open in G.

Fix a maximal (θ, F)-split torus $S = \boldsymbol{S}(F)$ of G and a maximal F-split torus $A = \boldsymbol{A}(F)$ of G that contains S. Because $\boldsymbol{A}$ is θ-stable (see Remark 3), the root system $\Phi = \Phi(\boldsymbol{G}, \boldsymbol{A})$ is θ-stable. There exists a θ-*basis* Δ of Φ, characterized by the property that if α belongs to the set of positive roots Φ^+ determined by Δ and $\alpha \circ \theta \neq \alpha$, then $\alpha \circ \theta \notin \Phi^+$. Let $\Delta_\theta = \{\alpha \in \Delta \mid \alpha \circ \theta = \alpha\}$. As shown in Prop. 2.6 of [21], for $I \subset \Delta$, the parabolic subgroup P_I is θ-split if and only if $I \supset \Delta_\theta$ and the subsystem Φ_I of Φ generated by I is θ-stable. For this reason, such an I is called *θ-split*. The parabolic subgroup P_{Δ_θ} is a minimal θ-split parabolic subgroup of G and $M_{\Delta_\theta} = Z_G(S)$ ([22], Prop. 4.7).

Lemma 1. ([26], Lemma 2.5) *Fix $\boldsymbol{S}$, $\boldsymbol{A}$ and Δ as above. A θ-split parabolic subgroup P of G is of the form gP_Ig^{-1} for some θ-split $I \subset \Delta$ and some $g \in (\boldsymbol{H}Z_{\boldsymbol{G}}(\boldsymbol{S}))(F)$.*

Example 2. Let $G = \boldsymbol{GL}_{2n}(F)$ ($n \geq 1$) and let

$$J_n = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}$$

Then $g \mapsto \theta(g) := J_n{}^t g^{-1} J_n^{-1}$ defines an involution of G with $H \cong \boldsymbol{Sp}_{2n}(F)$. The subgroup

$$S := \left\{ \begin{pmatrix} a_1 & b_1 & & & & & \\ b_1 & a_1 & & & & & \\ & & a_2 & b_2 & & & \\ & & b_2 & a_2 & & & \\ & & & & \ddots & & \\ & & & & & a_n & b_n \\ & & & & & b_n & a_n \end{pmatrix} \;\middle|\; a_j, b_j \in F,\, a_j^2 - b_j^2 = 1,\, 1 \leq j \leq n \right\}$$

is a maximal (θ, F)-split torus of G. The unique parabolic subgroup of G that contains $Z_G(S)$ and all upper triangular matrices in G is a minimal θ-split parabolic subgroup $P_0 = M_0 \ltimes N_0$ of G. Note that $M_0 \cap H = M_0^\theta$ is isomorphic to the direct product of n copies of $\boldsymbol{SL}_2(F)$.

Remark 4. In the group case, the θ-split parabolic subgroups of $G = G' \times G'$ have the form $Q \times \overline{Q}$, where Q and $\overline{Q}$ are opposite parabolic subgroups of G'.

6 Descent of H-invariant forms to Jacquet modules

We begin this section by recalling the definition of Jacquet modules and describing Casselman's pairing on Jacquet modules. After that, we state the symmetric space analogue of Casselman's pairing (due to Kato and Takano, and, independently, Lagier), which involves invariant forms on Jacquet modules of H-distinguished admissible representations along θ-split parabolic parabolic subgroups. When working with a θ-split parabolic subgroup P of G, we always choose the unique Levi factorization $M \ltimes N$ of P such that $M = P \cap \theta(P)$.

Fix a parabolic subgroup P of G. We regard P as a standard parabolic $P_I = M_I \ltimes N_I$ with respect to a suitable choice of P_0, A, Δ and $I \subseteq \Delta$, as in §5. Set $M = M_I$ and $N = N_I$. Let $\delta_P : P \to \mathbb{R}$ be defined by $\delta_P(mn) = |\det(\mathrm{Ad}\, m)_{\mathfrak{n}}|_F$, $m \in M$ and $n \in N$. (Here, $\mathfrak{n}$ is the Lie algebra of N.) If (π, V) is a smooth representation of G, let

$$V(N) = \mathrm{Span}\{\, \pi(n)v - v \mid n \in N,\, v \in V\,\} \quad \text{and} \quad V_N = V/V(N).$$

Let $j_N : V \to V_N$ be the projection map. The *normalized Jacquet module of π along P* is the smooth representation (π_P, V_N) of P defined by

$$\pi_P(mn)\, j_N(v) = \delta_P^{-1/2}(m)\, j_N(\pi(m)v), \qquad m \in M,\, n \in N,\, v \in V.$$

We often view π_P as a representation of M.

Note that the F-split component A_M of M is equal to the torus A_I defined in §5. If $\varepsilon \in \mathbb{R}$, let

$$A_M(\varepsilon) = \{\, a \in A_M \mid |\alpha(a)|_F \le \varepsilon \ \ \forall \alpha \in \Delta \setminus I\,\}. \tag{1}$$

Let $\overline{P}$ be the parabolic opposite to P with respect to M (that is, $P \cap \overline{P} = M$) and let $\overline{N}$ be the unipotent radical of $\overline{P}$.

Proposition 2. ([8], Proposition 4.2.3) *If (π, V) is an admissible representation of G, then there exists a unique pairing $\langle \cdot, \cdot \rangle_N : \widetilde{V}_{\overline{N}} \times V_N \to \mathbb{C}$ with the following propery: given $\tilde{v} \in \widetilde{V}$ and $v \in V$, there is an $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$ such that*

$$\langle\, \tilde{v},\, \pi(a)v\,\rangle = \delta_P(a)^{1/2} \langle\, j_{\overline{N}}(\tilde{v}),\, \pi_P(a)\, j_N(v)\,\rangle_N \quad \forall a \in A_M(\varepsilon).$$

The pairing is non-degenerate and M-invariant.

Remark 5. The references [7] and [37](VI.9.6) generalize Casselman's construction to the setting of smooth representations using ideas of Bernstein.

For the remainder of this section, we assume that the maximal F-split torus A is θ-stable, A^- is a maximal (θ, F)-split torus of G, Δ is a θ-basis of Φ and I is a proper θ-split subset of Δ. Thus $P = P_I$ is a proper θ-split parabolic subgroup of G, $M = P \cap \theta(P)$ and θ restricts to an involution of M. Recall that A_M^- denotes the (θ, F)-split component of M. (We caution the reader that our notation differs somewhat from that of [26] and [28].)

If π is an H-distinguished admissible representation of G then each linear functional $\lambda \in \mathrm{Hom}_H(\pi, 1)$ descends to a uniquely defined $\lambda_P \in \mathrm{Hom}_{M^\theta}(\pi_P, 1)$ in such a way that the asymptotic behaviours of (H, λ)-matrix coefficients of π and $(M \cap H, \lambda_P)$-matrix coefficients of π_P along A_M^- are comparable.

Proposition 3. ([26], Proposition 5.5, [28], Theorem 1) *Let (π, V) be an admissible H-distinguished representation of G. and let $\lambda \in \mathrm{Hom}_H(\pi, 1)$. There exists a unique*

$\lambda_P \in \mathrm{Hom}_{M^\theta}(\pi_P, 1)$ *with the following property: for* $v \in V$*, there exists* $\varepsilon \in \mathbb{R}$ *with* $0 < \varepsilon \leq 1$ *such that*

$$\langle \lambda, \pi(a)v \rangle = \delta_P(a)^{1/2} \langle \lambda_P, \pi_P(a) j_N(v) \rangle \quad \forall\, a \in A_M^- \cap A_M(\varepsilon).$$

Moreover, the function $\lambda \mapsto \lambda_P$ *is linear.*

Remark 6. In the group case $G = G' \times G'$, let Q and $\overline{Q}$ be opposite proper parabolic subgroups of G'. Then $P := Q \times \overline{Q}$ is a θ-split parabolic subgroup of G with unipotent radical $N_Q \times N_{\overline{Q}}$. Let (π, V) be an admissible representation of G'. and define $\lambda \in \mathrm{Hom}_H(\pi \otimes \widetilde{\pi}, 1)$ using the pairing between $\widetilde{V}$ and V (as in §3). Then the descent λ_P is given by Casselman's pairing between $\widetilde{V}_{N_{\overline{Q}}}$ and V_{N_Q}.

Suppose that $P = M \ltimes N$ is a θ-split parabolic subgroup of G (where, as usual, $M = P \cap \theta(P)$). Then a θ-split parabolic subgroup of M is of the form $M \cap Q$ where Q is a θ-split parabolic subgroup of G with $Q \subseteq P$. Let π be an admissible representation of G. By transitivity of Jacquet modules, $(\pi_P)_{M \cap Q}$ and π_Q are equivalent representations of L. When we fix an equivalence and identify them, the descents of H-invariant forms along θ-split parabolics exhibit the analogous transitivity property.

Lemma 2. ([26], Prop. 5.9, [28], Theorem 3) *With notation as above,* $(\lambda_P)_{M \cap Q} = \lambda_Q$ *for* $\lambda \in \mathrm{Hom}_H(\pi, 1)$.

7 Relatively supercuspidal representations and the relative subrepresentation theorem

Suppose that $P = M \ltimes N$ is a proper parabolic subgroup of G and (τ, V) is a smooth representation of M. Let $\mathrm{Ind}_P^G \tau$ denote the representation of G obtained via normalized parabolic induction from τ. The group G acts via right translation on the space of $\mathrm{Ind}_P^G \tau$, which consists of V-valued functions f on G satisfying:

$f(mng) = \delta_P(m)^{1/2} \tau(m) f(g)$ for all $m \in M$, $n \in N$ and $g \in G$,

f is right K_f-invariant under some compact open subgroup K_f of G.

The following well-known theorem, a proof of which may be found in §5 of [8], is due to Jacquet and Harish-Chandra. The second part is known as Jacquet's subrepresentation theorem.

Theorem 2. *Let* π *be an admissible representation of* G*. Then*

1. π *is supercuspidal if and only if* $\pi_P = 0$ *for every proper parabolic subgroup* P *of* G*.*
2. *If* π *is irreducible, then either* π *is supercuspidal or there exist a proper parabolic subgroup* $P = M \ltimes N$ *of* G *and an irreducible supercuspidal representation* τ *of* M *such that* π *is equivalent to a subrepresentation of* $\mathrm{Ind}_P^G \tau$*.*

The relative version of the above theorem is expressed in terms of descents of H-invariant linear forms to θ-split parabolic subgroups and parabolic induction from θ-split parabolic subgroups. The second part of the theorem is referred to as the relative subrepresentation theorem.

Theorem 3. ([26], Theorems 6.9, 7.1) *Let π be an H-distinguished admissible representation of G. Then*

1. *π is H-relatively supercuspidal if and only if $\lambda_P = 0$ for every $\lambda \in \mathrm{Hom}_H(\pi, 1)$ and every proper θ-split parabolic subgroup P of G.*
2. *If π is irreducible, then either π is H-relatively supercuspidal or there exist a proper θ-split parabolic subgroup $P = M \ltimes N$ of G and an irreducible M^θ-relatively supercuspidal representation τ of M such that π is equivalent to a subrepresentation of $\mathrm{Ind}_P^G \tau$.*

If K is a good, special subgroup of G, the Cartan decomposition of G gives an explicit system of representatives for the double cosets $K\backslash G/K$ ([6], Proposition 4.4.3). The Cartan decomposition plays an essential role in the proofs of Theorem 2(1) and Casselman's square-integrability criterion (see §9 for the statement). The relative Cartan decomposition plays a comparable role in the proofs of Theorem 3(1) and the relative Casselman criterion (see §9). Here, we state the relative Cartan decomposition from [3]. Another version is given for split groups in Theorem 0.1 of [10].

Theorem 4. ([3], Theorem 1.1) *Let $\widetilde{S}$ be the union of a set of representatives for the H-conjugacy classes of maximal (θ, F)-split tori of G. There exists a compact subset C of G such that $G = C\widetilde{S}H$.*

Note that if π is an H-distinguished supercuspidal representation of G, the fact that $\pi_P = 0$ for every proper θ-split parabolic subgroup $P = M \ltimes N$ of G implies that $\mathrm{Hom}_{M^\theta}(\pi_P, 1) = 0$, which, since $\lambda_P \in \mathrm{Hom}_{M^\theta}(\pi_P, 1)$, implies that π is H-relatively supercuspidal. (Further comments on H-distinguished supercuspidal representations appear in §11.)

There exist symmetric pairs (G, H) having the property that no irreducible supercuspidal representation of G is H-distinguished. The pair $(\boldsymbol{GL}_{2n}(F), \boldsymbol{Sp}_{2n}(F))$ is one such example. The case $n = 1$ is easily verified directly:

Example 3. Let $G = \boldsymbol{GL}_2(F)$ and $H = \boldsymbol{SL}_2(F)$. Let (π, V) be an irreducible supercuspidal representation of G. Let $P = M \ltimes N$ be a proper parabolic subgroup of G. Fix $v \in V$. Since π is supercuspidal, by Theorem 2(1), $V_N = 0$, that is, $V = V(N)$. Thus there exist $v_i \in V$ and $n_i \in N$, $1 \le i \le m$ such that $v = \sum_{i=1}^m \pi(n_i)v_i - v_i$. Suppose that $\lambda \in \mathrm{Hom}_H(\pi, 1)$. Then, because $N \subset H$,

$$\langle \lambda, v \rangle = \sum_{i=1}^m (\langle \lambda, \pi(n_i)v_i \rangle - \langle \lambda, v_i \rangle) = 0.$$

For $n \ge 2$, the fact that no $\boldsymbol{Sp}_{2n}(F)$-distinguished irreducible admissible representation of $\boldsymbol{GL}_{2n}(F)$ is supercuspidal can be seen as follows. It is well known that an irreducible supercuspidal representation of $\boldsymbol{GL}_{2n}(F)$ has a Whittaker model. By Theorem 3.2.2 of [23], an $\boldsymbol{Sp}_{2n}(F)$-distinguished irreducible admissible representation of $\boldsymbol{GL}_{2n}(F)$ cannot have a Whittaker model.

In general, the H-relatively supercuspidal representations of G may include both supercuspidal and nonsupercuspidal representations. As discussed in Example 4 (see §8), this is the case for the symmetric pair $(\boldsymbol{GL}_4(F), \boldsymbol{GL}_2(F) \times \boldsymbol{GL}_2(F))$.

It is evident from the relative subrepresentation theorem that parametrization of H-relatively supercuspidal representations is a fundamental component of the problem of parametrization of H-distinguished representations. A construction of some H-relatively supercuspidal representations of groups G that split over tamely ramified extensions of F is carried out in [34]. Certain elementary cases of this construction are given in [33]. In these cases, the relatively supercuspidal representations are attached to G-conjugacy classes of pairs (T, ϕ) where T is a θ-stable tamely ramified maximal torus of G having the property that T^-/A_G^- is compact and T^- contains G-regular elements, and ϕ is a quasicharacter of T. The quasicharacter ϕ satisfies a specific regularity condition and is trivial on the topologically unipotent set in T^θ, that is, the maximal pro-p-subgroup of T^θ. (This is a relative version of a construction of supercuspidal representations associated to pairs (T, ϕ) where T is an elliptic maximal torus of G and ϕ is a quasicharacter of T satisfying a regularity condition.) An example is discussed in the next section.

Remark 7. In some cases, H-distinguished representations are realized as (subquotients of) representations of the form $\mathrm{Ind}_P^G \tau$, where $P = M \ltimes N$ is a proper θ-*stable* parabolic subgroup of G and τ is an irreducible supercuspidal representation of M that is (M^θ, χ)-distinguished with respect to a nontrivial quasicharacter χ of M^θ. The $\boldsymbol{GL}_{n-1}(F) \times F^\times$-relatively supercuspidal representations of $\boldsymbol{GL}_n(F)$ $(n \geq 3)$ described in Example 8.2 of [26] occur in this manner. This illustrates one way in which the study of distinguished representations may require working in the more general context of representations distinguished by nontrivial quasicharacters.

8 Two relatively supercuspidal examples

Example 4. Let $G = \boldsymbol{GL}_4(F)$ and $H = \boldsymbol{GL}_2(F) \times \boldsymbol{GL}_2(F)$, where θ is given by conjugation by the diagonal matrix $\mathrm{diag}(1,-1,1,-1)$. Fix nonsquares τ_1 and τ_2 in $F^\times$. For $j = 1, 2$, let $E_j = F(\sqrt{\tau_j})$, $\mathfrak{p}_{E_j}$ the maximal ideal in the ring of integers of E_j, σ_j the nontrivial element of $\mathrm{Gal}(E_j/F)$ and $N_{E_j/F} : E_j \to F$ the norm map. Let

$$T = \left\{ \begin{pmatrix} a & b\tau_1 & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & d\tau_2 \\ 0 & 0 & d & c \end{pmatrix} \;\middle|\; a,b,c,d \in F,\, a^2 - b^2\tau_1 \neq 0,\, c^2 - d^2\tau_2 \neq 0. \right\}$$

Note that $T \cong E_1^\times \times E_2^\times$ and, when we identify T with $E_1^\times \times E_2'$, $\theta(u_1, u_2) = (\sigma_1(u_1), \sigma_2(u_2))$ for $u_1 \in E_1^\times$ and $u_2 \in E_2^\times$. Hence $T^\theta \cong F^\times \times F^\times$, $T^- \cong \mathrm{Ker} N_{E_1/F} \times$

$\mathrm{Ker}N_{E_2/F}$ is compact (note that A_G^- is trivial), and T^- contains G-regular elements (which, in this case, are elements of T^- having distinct eigenvalues in the extension E_1E_2). For $j = 1, 2$, fix a quasicharacter ϕ_j of $E_j^\times$ such that $\phi_j\,|\,F^\times$ is trivial and $\phi_j^2\,|\,1+\mathfrak{p}_{E_j}$ is nontrivial. If $E_1 = E_2$, we impose the additional condition $\phi_2 \notin \{\,\phi_1, \phi_1\circ\sigma_1 = \phi_1^{-1}\,\}$. Let ϕ be the quasicharacter of T corresponding to the quasicharacter $\phi_1\otimes\phi_2$ of $E_1^\times\times E_2^\times$. Note that $\phi\,|\,T^\theta$ is trivial. It is straightforward to check that the regularity condition of Definition 5.3 (see also Definition 3.5) of [33] holds for ϕ. Thus the G-conjugacy class of the pair (T,ϕ) corresponds to an equivalence class of (nonsupercuspidal) positive regular representations of G (in the sense of [33]). Assuming that such a representation is H-distinguished (beyond mentioning that the property $\phi\,|\,T^\theta = 1$ is key, we do not include an explanation here), it follows from Proposition 8.3 of [33] that the representation is H-relatively supercuspidal. We remark that it is well-known that there exist H-distinguished irreducible supercuspidal representations of G (this follows from Corollary 1 in §11, for example). Thus the set of H-relatively supercuspidal representations of G contains both supercuspidal and nonsupercuspidal representations.

Example 5. Let $G = \boldsymbol{GL}_{2n}(F)$, $H = \boldsymbol{Sp}_{2n}(F)$, J_n, S and $P_0 = M_0 \ltimes N_0$ be as in Example 2. Let P be the parabolic subgroup $P = M \ltimes N$ such that

$$M = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A, B \in \boldsymbol{GL}_n(F) \right\} \quad \text{and} \quad N = \left\{ \begin{pmatrix} I_n & C \\ 0 & I_n \end{pmatrix} \mid C \in M_{n\times n}(F) \right\}.$$

(Here, I_n denotes the $n\times n$ identity matrix and $M_{n\times n}(F)$ is the set of $n\times n$ matrices with entries in F.) Let ρ be an irreducible discrete series representation of $\boldsymbol{GL}_n(F)$. We view $\tau_\rho := \rho|\det(\cdot)|_F^{1/2} \otimes \rho|\det(\cdot)|_F^{-1/2}$ as a representation of M. The induced representation $\mathrm{Ind}_P^G\,\tau_\rho$ is reducible of length 2, and, as shown in Theorem 11.1 of [23], its unique irreducible quotient π_ρ is H-distinguished.

Because the Galois cohomology of $\boldsymbol{SL}_2$ is trivial, it follows from Lemma 2.5 of [26] such that all maximal (θ,F)-split tori are H-conjugate (to S). Thus each θ-split parabolic subgroup of G is H-conjugate to a θ-split parabolic that contains P_0.

Our aim is to follow the approach of Proposition 8.3.4 of [26] to indicate why π_ρ is H-relatively supercuspidal whenever ρ is unitary and supercuspidal. By transitivity of descent of linear forms (Lemma 2), it suffices to show that if $Q = L\ltimes U$ is a maximal proper θ-split parabolic subgroup of G that contains P_0 and $\lambda \in \mathrm{Hom}_H(\pi_\rho, 1)$, then $\lambda_Q = 0$. This is done by showing that $(\pi_\rho)_Q$ (when nonzero) is not L^θ-distinguished.

By exactness of Jacquet modules, because π_ρ is a quotient of $\mathrm{Ind}_P^G\,\tau_\rho$, $(\pi_\rho)_U$ is a quotient of $(\mathrm{Ind}_P^G\,\pi_\rho)_U$. thus any $\lambda \in \mathrm{Hom}_{L^\theta}((\pi_\rho)_Q, 1)$ lifts to an element of $\mathrm{Hom}_{L^\theta}((\mathrm{Ind}_P^G\,\tau_\rho)_Q, 1)$.

The geometric lemma of Bernstein and Zelevinsky ([5]) is useful here. It is possible to choose a set $\{g\}$ of representatives for the double cosets $Q\backslash G/P$ such that for each g, $L\cap{}^gP$ is a parabolic subgroup of L with Levi decomposition $(L\cap{}^gM)\ltimes(L\cap{}^gN)$ and $Q\cap{}^gM$ is a parabolic subgroup of ${}^gM$ with Levi factorization $(L\cap{}^gM)\ltimes(U\cap{}^gM)$. Let ${}^g\tau_\rho$ be the representation of gMg^{-1} defined by

${}^g\tau_\rho(gmg^{-1}) = \tau_\rho(m)$, $m \in M$. According to the geometric lemma, the Jacquet module $(\mathrm{Ind}_P^G \tau_\rho)_Q$ (as a representation of L) has a filtration involving representations of the form

$$\mathrm{Ind}_{L\cap {}^gP}^L(({}^g\tau_\rho)_{Q\cap {}^gM}).$$

Because τ_ρ (hence ${}^g\tau_\rho$) is supercuspidal, if $Q\cap {}^gM$ is a proper parabolic subgroup of ${}^gM$, that is, if $U\cap {}^gM$ is nontrivial, we have $({}^g\tau_\rho)_{Q\cap {}^gM} = 0$ (by Theorem 2(1)). Thus only those coset representatives g for which ${}^gM \subseteq L$, that is, $Q\cap {}^gM = L$, contribute nontrivially to the filtration of $(\mathrm{Ind}_P^G \tau_\rho)_Q$. Hence $(\mathrm{Ind}_P^G \tau_\rho)_Q$ has a filtration involving representations of the form $\mathrm{Ind}_{{}^gM\ltimes(L\cap {}^gN)}^L {}^g\tau_\rho$, for $g\in G$ with ${}^gM\subseteq L$.

Recall that M_0 (which is isomorphic to the direct product of n copies of $\boldsymbol{GL}_2(F)$) is a subgroup of L. Thus L is isomorphic to a maximal Levi subgroup of the form $\boldsymbol{GL}_{2m}(F)\times \boldsymbol{GL}_{2n-2m}(F)$ for some natural number m. The Levi subgroup M is isomorphic to the maximal Levi subgroup $\boldsymbol{GL}_n(F)\times\boldsymbol{GL}_n(F)$. By maximality of the Levi subgroups, ${}^gM\subseteq L$ implies ${}^gM = L$ and $L\cap {}^gN$ is trivial. Since this cannot happen if n is odd, we have $(\mathrm{Ind}_P^G \tau_\rho)_Q = 0$ for n odd.

For n even, ${}^gM = L$ forces $n = 2m$, $U = N$, and $\mathrm{Ind}_{{}^gM\ltimes(L\cap {}^gN)}^L {}^g\tau_\rho = {}^g\tau_\rho$. Thus the filtration of $(\mathrm{Ind}_P^G \tau_\rho)_Q$ involves supercuspidal representations of the form ${}^g\tau_\rho$ for g in the normalizer of M. Observe that the restriction of θ to $M = L = \boldsymbol{GL}_{2m}(F)\times \boldsymbol{GL}_{2m}(F)$ is given by

$$\theta(A,B) = (J_m{}^tA^{-1}J_m^{-1}, J_m{}^tB^{-1}J_m^{-1}), \qquad A,B\in \boldsymbol{GL}_{2m}(F),$$

where J_m is as in Example 2. Hence $M^\theta = L^\theta \cong \boldsymbol{Sp}_{2m}(F)\times\boldsymbol{Sp}_{2m}(F)$. Recall that an irreducible supercuspidal representation of $\boldsymbol{GL}_{2m}(F)$ is not $\boldsymbol{Sp}_{2m}(F)$ distinguished. Thus $(\mathrm{Ind}_P^G \tau)_Q$ is not L^θ-distinguished. We can now conclude that π_ρ is H-relatively supercuspidal.

9 Relative discrete series and the relative Casselman criterion

Let (π,V) be a smooth representation of G and Z' a closed subgroup of Z_G. If χ is a quasicharacter of Z', the subspace

$$V_\chi := \{\, v\in V \;\mid\; \exists \ell\in\mathbb{N}\,\forall\, z\in Z',\ (\pi(z)-\chi(z))^\ell\cdot v = 0\,\}.$$

is G-invariant. If V_χ is nonzero, then χ is called an *exponent of π with respect to Z'*. Let $\mathrm{Exp}_{Z'}(\pi)$ denote the set of such exponents. If π is finitely generated and admissible, then $\mathrm{Exp}_{Z'}(\pi)$ is a finite set and $V = \bigoplus_{\chi\in\mathrm{Exp}_{Z'}(\pi)} V_\chi$ ([8], Proposition 2.1.9). If $P = M\ltimes N$ is a parabolic subgroup of G, A_M is the F-split component of M and (π_P, V_N) is the normalized Jacquet module of π along P, then elements of $\mathrm{Exp}_{A_M}(\pi_P)$ are referred to as *(normalized) exponents of π along P*.

When π has a unitary central quasicharacter, Casselman's Criterion gives necessary and sufficient conditions for square integrability of π in terms of properties of exponents of π along parabolic subgroups. If $P = M\ltimes N$ is a parabolic subgroup of

G, let A_M^1 be the maximal compact subgroup of A_M and let $A_M(1)$ be as in (1) (see §6), for $\varepsilon = 1$.

Theorem 5. ([8], Theorem 4.4.6) *With notation as above, assuming that π has a unitary central quasicharacter, the representation π is square integrable if and only if for every parabolic subgroup $P = M \ltimes N$ of G and every $\chi \in \mathrm{Exp}_{A_M}(\pi_P)$, $|\chi(a)| < 1$ for all $a \in A_M(1) \setminus Z_G A_M^1$.*

Now suppose that $P = M \ltimes N$ is a θ-split parabolic subgroup of G, A_M^- is the (θ, F)-split component of M, $\lambda \in \mathrm{Hom}_H(\pi, 1)$ and λ_P is as in Proposition 3. The set of *exponents of π_P with respect to λ_P* is defined by:

$$\mathrm{Exp}_{A_M^-}(\pi_P, \lambda_P) := \{\, \chi \in \mathrm{Exp}_{A_M^-}(\pi_P) \mid \lambda_P \,|\, (V_N)_\chi \neq 0 \,\}. \tag{2}$$

Theorem 6. ([27], Theorem 4.7) *Let (π, V) be a finitely generated admissible H-distinguished representation of G such that A_G^- acts on V by a unitary quasicharacter. Then, for $\lambda \in \mathrm{Hom}_H(\pi, 1)$, π is H-square integrable with respect to λ if and only if for every θ-split parabolic subgroup $P = M \ltimes N$ of G,*

$$\forall \chi \in \mathrm{Exp}_{A_M^-}(\pi_P, \lambda_P),\ |\chi(a)| < 1\ \forall\, a \in (A_M(1) \cap A_M^-) \setminus A_G^-(A_M^1 \cap A_M^-).$$

Remark 8. As shown in Lemma 4.6 of [27], if the above condition is satisfied for all maximal (proper) θ-split parabolics, then it holds for all θ-split parabolics.

Proposition 4. ([27], Proposition 4.10) *If π is an H-distinguished finitely generated admissible discrete series representation of G, then π is a relative discrete series representation.*

There are examples of symmetric pairs for which no discrete series representations are distinguished. As a consequence of Theorem 3.2.2 of [23] and the fact that irreducible discrete series representations of general linear groups have Whittaker models, there are no $\boldsymbol{Sp}_{2n}(F)$-distinguished irreducible discrete series representations of $\boldsymbol{GL}_{2n}(F)$ (the supercuspidal case is discussed in §7).

To date, there are very few constructions of relative discrete series representations that are neither discrete series representations nor relatively supercuspidal. Let ρ be the Steinberg representation of $\boldsymbol{GL}_2(F)$ and π_ρ the $\boldsymbol{Sp}_4(F)$-distinguished representation of $\boldsymbol{GL}_4(F)$ defined at the beginning of §7. Kato and Takano ([27], Example 5.1) verified that that π_ρ is $\boldsymbol{Sp}_4(F)$-square integrable. Although it has not been checked, it is likely that π_ρ is not relatively supercuspidal.

In some instances, relative discrete series representations that do not belong to the discrete series may be constructed as representations induced from distinguished discrete series representations of Levi factors of θ-stable parabolic subgroups. Recent work of Smith ([39]) gives this sort of construction of some families of relative discrete series representations of general linear groups.

10 Realizing H-invariant forms via integration of matrix coefficients

Let (π, V) be an irreducible admissible representation of G. If ω_π is the central quasicharacter of π and $\lambda \in \mathrm{Hom}_H(\pi, 1)$, then

$$\langle \lambda, \pi(z)v \rangle = \omega_\pi(z) \langle \lambda, v \rangle \qquad v \in V,\ z \in Z_G.$$

Hence we may (and do) assume that ω_π is trivial on $H \cap Z_G$, as this will be the case whenever π is H-distinguished.

Fix $\tilde{v} \in \widetilde{V}$. Observe that if $v \in V$, the function $h \mapsto \langle \tilde{v}, \pi(h)v \rangle$ is $H \cap Z_G$-invariant. Fix an H invariant measure on $H/H \cap Z_G$. Under certain conditions, it is possible to define $\lambda_{\tilde{v}} \in \mathrm{Hom}_H(\pi, 1)$ by

$$\langle \lambda_{\tilde{v}}, v \rangle = \int_{H/H \cap Z_G} \langle \tilde{v}, \pi(h)v \rangle \, dh^\times, \qquad v \in V. \tag{3}$$

When H is not compact modulo $H \cap Z_G$, the above integral does not necessarily converge. When convergence is not an issue, it may be the case that $\lambda_{\tilde{v}} = 0$. If the integral converges for all choices of $\tilde{v} \in \widetilde{V}$ and $v \in \widetilde{V}$, we say that the matrix coefficients of π are *H-integrable*. If π is supercuspidal, then, because all matrix coefficients of π are compactly supported modulo Z_G, all matrix coefficients of π are H-integrable.

It is a long-standing conjecture that every irreducible supercuspidal representation π is compactly induced in the sense that there exist an open subgroup K of G that is compact modulo Z_G and an irreducible smooth representation (κ, W) of K such that $\pi = \text{c-Ind}_K^G \kappa$. (Here, c-Ind denotes compact (smooth) induction.) Assume that π is of this form and denote the space V of π by $\text{c-Ind}_K^G W$. By Lemma 7 of [16],

$$\mathrm{Hom}_H(\pi, 1) \cong \bigoplus_{g \in K \backslash G/H} \mathrm{Hom}_{K \cap gHg^{-1}}(\widetilde{\kappa}, 1), \tag{4}$$

where the sum is over a set of representatives for the double cosets $K \backslash G/H$. Since K is compact modulo Z_G and κ is irreducible and smooth, κ is finite-dimensional. In particular, the dual representation κ^* is equal to the smooth dual $\widetilde{\kappa}$, and $\mathrm{Hom}_{K \cap gHg^{-1}}(\kappa, 1) \cong \mathrm{Hom}_{K \cap gHg^{-1}}(\kappa^*, 1)$ for all $g \in G$.

Remark 9. As a consequence of (4) and the above comments, if π is supercuspidal and compactly induced, then $\mathrm{Hom}_H(\pi, 1) \cong \mathrm{Hom}_H(\widetilde{\pi}, 1)$. It is worth noting that more generally this is known to hold for all irreducible unitary admissible representations, but it is not known for arbitrary irreducible admissible representations.

Fix $g \in G$ such that κ^* is $K \cap gHg^{-1}$-distinguished and fix a nonzero element w^* of $\mathrm{Hom}_{K \cap gHg^{-1}}(\kappa^*, 1)$. Denote the usual pairing between $\widetilde{W} = W^*$ and W by $\langle \cdot, \cdot \rangle_W$. Then, up to a constant depending on normalization of measures, $f \mapsto \int_{H/H \cap Z_G} \langle w^*, f(gh) \rangle_W \, dh^\times$ is the element of $\mathrm{Hom}_H(\pi, 1)$ corresponding to w^* under the isomorphism in (4).

We identify $(\widetilde{\pi}, \widetilde{V})$ with $(\text{c-Ind}_K^G \kappa^*, \text{c-Ind}_K^G W^*)$ and normalize the G-invariant measure on G/K in such a way that the pairing between $\widetilde{V} = \text{c-Ind}_K^G W^*$ and $V = \text{c-Ind}_K^G W$ is given by

$$\langle \tilde{f}, f \rangle = \int_{G/K} \langle \tilde{f}(g), f(g) \rangle_W \, dg^\times, \qquad \tilde{f} \in \text{c-Ind}_K^G W^*,\ f \in \text{c-Ind}_K^G W.$$

Here, up to a constant, the integral is a sum over a set of coset representatives for G/K. Because we are using compact induction, each function in $\text{c-Ind}_K^G W$ (or $\text{c-Ind}_K^G W^*$) is supported on finitely many left cosets of K in G, so the sum is actually finite. Let f_{w^*} be the unique element of $\widetilde{V}$ that is supported on K and satisfies $f_{w^*}(k) = \kappa(k) w^*$ for all $k \in K$. It is a straightforward exercise to verify that if $\tilde{v} = \widetilde{\pi}(g^{-1}) f_{w^*}$, then

$$\langle \lambda_{\tilde{v}}, f \rangle = \int_{H/H \cap Z_G} \langle w^*, f(gh) \rangle_W \, dh^\times, \qquad f \in V.$$

In particular $\lambda_{\tilde{v}}$ corresponds to w^* under the isomorphism of (4). It follows that every element of $\text{Hom}_H(\pi, 1)$ is of the form $\lambda_{\tilde{v}}$ for some $\tilde{v} \in \widetilde{V}$. By Theorem 1.5 of [42], this holds for all irreducible supercuspidal representations (that is, without the assumption that π is compactly induced):

Lemma 3. *If π is an irreducible supercuspidal representation of G,* $\text{Hom}_H(\pi, 1) = \{ \lambda_{\tilde{v}} \mid \tilde{v} \in \widetilde{V} \}$.

Recent results of M. Gurevich and Offen ([11], Theorem 1.1) provide a criterion for H-integrability of the matrix coefficients of an admissible representation π of G, expressed in terms of properties of exponents of π along θ-stable parabolic parabolic subgroups of G. In the group case, the criterion reduces to Casselman's criterion for square integrability of matrix coefficients. As defined in [38] (in the more general context of spherical varieties), G/H is *strongly tempered* (resp. *strongly discrete*) if the matrix coefficients of all irreducible, tempered (resp. discrete series) representations of G are H-integrable. As an application of their H-integrability criterion, Gurevich and Offen verify that certain symmetric spaces G/H are strongly tempered or strongly discrete. For example, if (G,H) is a Galois symmetric pair, then the symmetric space G/H is strongly discrete. C. Zhang ([42], Theorem 1.4) has shown that for any symmetric space G/H exhibiting a particular property that implies G/H is strongly discrete, the conclusion of Lemma 3 holds for discrete series representations of G. Zhang proves that Galois symmetric pairs have this property.

11 Distinguished tame supercuspidal representations

In this section, we assume that $\boldsymbol{G}$ splits over a tamely ramified extension of F. Generalizing an earlier construction of Adler ([1]), J.-K. Yu ([41]) gave a construction of irreducible supercuspidal representations of G. We refer to these representations as *tame* supercuspidal representations.

Each tame supercuspidal representation π is compactly induced in the sense described in §10, and we write $\pi = \text{c-Ind}_K^G \kappa$ (using the notation of that section). In view of equation 4, in order to describe $\mathrm{Hom}_H(\pi, 1)$, it suffices to determine $\mathrm{Hom}_{K \cap gHg^{-1}}(\kappa^*, 1)$ for each $g \in G$. Under a technical assumption regarding behaviour of quasicharacters, an analysis of the spaces $\mathrm{Hom}_{K \cap gHg^{-1}}(\kappa^*, 1)$ is carried out in [17], resulting in necessary conditions for distinction of π. These conditions are expressed in terms of symmetry properties of a cuspidal generic G-datum employed in Yu's construction to produce π. For a tame supercuspidal representation π associated to a datum with the requisite symmetry properties, a formula for $\mathrm{Hom}_H(\pi, 1)$ is obtained. These results, when applied to the group case, give rise to a parametrization of the equivalence classes of tame supercuspidal representations in terms of an equivalence relation on cuspidal generic G-data. Recent work of Kaletha ([25]) shows that this parametrization is valid without the technical assumption of [17]. Below we give an overview of the aforementioned results of [17], phrased in the terminology of [31].

If $\boldsymbol{G}'$ is an F-subgroup of $\boldsymbol{G}$, we say that $G' = \boldsymbol{G}'(F)$ is a (*tame*) *twisted Levi subgroup of* G if $\boldsymbol{G}'$ becomes a Levi subgroup of $\boldsymbol{G}$ over some finite (tamely ramified) extension of F. For example if E is a (tamely ramified) quadratic extension of F, then $\boldsymbol{GL}_2(E) = (\mathrm{Res}_{E/F}\, \boldsymbol{GL}_2)(F)$ is a (tame) twisted Levi subgroup of $\boldsymbol{GL}_4(F)$.

In Yu's construction, reduced, generic cuspidal G-data (see Definition 3.11 of [17]) give rise to tame supercuspidal representations of G. As in Definition 8.4 of [31], to each such G-datum, we may associate a triple (G', π', ϕ) comprising a tame twisted Levi subgroup G' of G having the property that $Z_{G'}/Z_G$ is compact, a depth-zero irreducible supercuspidal representation π' of G', and a quasicharacter ϕ of G' that is G-regular on the topologically unipotent set in G' (see [31], Definition 6.1) and is factorizable in the sense of Definitions 5.1 and 5.3 of [31]. We remark that Kaletha ([25]) has shown that under some mild conditions on p, factorizability is automatic. We say that two triples (G', π', ϕ) and $(\dot{G}, \dot{\pi}, \dot{\phi})$ are *G-equivalent* if there exists $g \in G$ such that $G' = g\dot{G}g^{-1}$ and $\pi'\phi$ is equivalent to the representation ${}^g(\dot{\pi}\dot{\phi})$ defined by ${}^g(\dot{\pi}\phi)(x) = (\dot{\pi}\dot{\phi})(g^{-1}xg)$, $x \in G'$. As shown in Theorem 8.7 of [31], G-equivalence of triples corresponds to the equivalence relation on the set of cuspidal generic G-data defined in [17]. Combining this with Corollary 3.5.5 of [25] (which allows removal of the conditions on quasicharacters imposed in [17]), we obtain the following restatement of Theorem 6.6 of [17]:

Theorem 7. *The abovementioned map (from G-data to triples) induces a bijective correspondence between the set of equivalence classes of tame supercuspidal representations and the set of G-equivalence classes of triples associated to cuspidal generic G-data.*

For the rest of this section, let $\pi = \text{c-Ind}_K^G \kappa$ be a tame supercuspidal representation of G and let (G', π', ϕ) be a triple whose G-equivalence class corresponds to (the equivalence class of) π. According to Proposition 6.8 of [29], because π' is a depth-zero irreducible supercusidal representation, $\pi' = \text{c-Ind}_{K'}^{G'} \kappa'$, where K' is the normalizer of a maximal parahoric subgroup J of G' and κ' is an irreducible smooth representation of K' whose restriction to J contains the inflation of an irreducible cuspidal representation of the finite group of Lie type arising as the quotient of J modulo its pro-unipotent radical. The next theorem rephrases Lemma 5.4, Prop. 5.7(2), Prop. 5.20 and Theorem 5.26(5) of [17]. A *quadratic character* of a group is a character whose square is trivial.

Theorem 8. *Suppose that π is H-distinguished. Then K, G', K', κ, κ' and ϕ may be chosen so that:*

1. *$K' = K \cap G'$, and the groups G' and K (hence also K') are θ-stable.*
2. *$\phi \circ \theta = \phi^{-1}$*
3. *There exists a quadratic character χ of K'^{θ} such that*

$$\mathrm{Hom}_{K^\theta}(\kappa, 1) \cong \mathrm{Hom}_{K'^\theta}(\kappa'\phi, \chi) \neq 0.$$

4. *Assuming that choices are as above, there exist involutions $\theta_1 = \theta, \theta_2, \dots, \theta_\ell$ of G that stabilize K', together with with quasicharacters $\chi_1 = \chi, \chi_2, \dots, \chi_\ell$ of $K'^{\theta_1}, \dots, K'^{\theta_\ell}$, respectively, such that*

$$\mathrm{Hom}_H(\pi, 1) \cong \bigoplus_{j=1}^{\ell} \mathrm{Hom}_{K'^{\theta_j}}(\kappa'\phi, \chi_j).$$

Remark 10. With notation and choices as in the theorem, the set $\{\, g\theta(g)^{-1} \mid g \in G\} \cap K'$ is a finite union of K'-orbits for the action $(k, y) \mapsto k \cdot y := ky\theta(k)^{-1}$. The involutions θ_j arise as follows. Choose a set $\{g_j \mid 1 \le j \le \ell\}$ of representatives for these orbits. For $1 \le j \le \ell$, let θ_j be the involution of G defined by $x \mapsto \theta_j(x) := g_j\theta(g_j^{-1}xg_j)g_j^{-1}$. We note that it can be shown that $\{g_1, \dots, g_\ell\}$ is also a set of representatives for the double cosets in $K\backslash G/H$ that contain an element g such that $g\theta(g)^{-1} \in K'$.

Remark 11. The formula for $\mathrm{Hom}_H(\pi, 1)$ given in Theorem 8 involves quadratic distinction of inflations of representations of finite groups of Lie type. This is another instance, in addition to that mentioned in Remark 7, where distinction of representations by nontrival characters arises in the study of distinguished representations. In recent work, Hakim ([15]) reworks Yu's construction. This leads to a recasting of the results of [17] which yields new information about the quadratic characters in the formula for $\mathrm{Hom}_H(\pi, 1)$.

Certain supercuspidal representations have the property that their equivalence classes are parametrized by G-conjugacy classes of pairs (T, ϕ) where T is an elliptic maximal torus of G and ϕ is a quasicharacter of T exhibiting specific regularity properties. When n is not divisible by p, this includes all irreducible supercuspidal representations of $\boldsymbol{GL_n}(F)$ ([24]). More generally, if $\boldsymbol{G}$ splits over a tamely ramified extension of F, the regular supercuspidal representations of [25], which comprise a subset of the tame supercuspidal representations, have this property. Distinction of such supercuspidal representations is reflected in properties of T and ϕ relative to the involution θ (up to conjugacy).

Recall that an element g of G is said to be G-regular if the identity component of the centralizer of g in $\boldsymbol{G}$ is a maximal torus of $\boldsymbol{G}$.

Theorem 9. ([32], Theorem 10.2) *Suppose that F' is a finite tamely ramified extension of F, $n \geq 2$ and $\boldsymbol{G} = \mathrm{Res}_{F'/F}\, \boldsymbol{GL}_n$. Let E be a tamely ramified degree n extension of F'. The following are equivalent:*

1. *There exists a θ-stable elliptic maximal torus T of G such that $T \cong E^\times$ and T contains θ-split G-regular elements.*
2. *There exists an F'-admissible quasicharacter ϕ of $T \cong E^\times$ (in the sense of [24]) such that a supercuspidal representation of G corresponding to the G-orbit of (T,ϕ) is H-distinguished.*

Corollary 1. ([32], Theorem 10.4) *Let G be as in the above theorem. The following are equivalent:*

1. *There exist H-distinguished tame supercuspidal representations of G.*
2. *There exists a θ-stable tamely ramified elliptic maximal torus of G that contains θ-split G-regular elements.*

Remark 12. In the more general setting where $\boldsymbol{G}$ splits over a tamely ramified extension of F, as shown in Proposition 6.7(2) and Theorem 6.9 of [32], similar results hold for a particular subset of the regular supercuspidal representations of G. These representations were referred to as toral supercuspidal representations in [17] and [32], and are the same as the supercuspidal positive regular representations of [33]).

There exist symmetric pairs (G,H) having the property that G has H-distinguished tame supercuspidal representations, but G contains no θ-split G-regular elements. For example, as shown in §7 of [32], if ϖ is a uniformizer of F, this is the case for the symmetric pair $(\boldsymbol{Sp}_4(F), \boldsymbol{SL}_2(F(\sqrt{\varpi}))$.

12 Spherical characters

In [35], Rader and Rallis carried over several results of Howe and Harish-Chandra to reductive p-adic symmetric spaces. Earlier work of Hakim ([12]–[14]) treated particular cases.

Recall that a distribution on G is a linear functional on the space $C_c^\infty(G)$ of locally constant, compactly supported, complex-valued functions on G. An irreducible admissible representation π of G having the property that both π and $\widetilde{\pi}$ are H-distinguished is said to be *class one*. If π is a class one representation, $\lambda_\pi \in \mathrm{Hom}_H(\pi,1)$ and $\lambda_{\widetilde{\pi}} \in \mathrm{Hom}_H(\tilde{\pi},1)$, there is an associated H-biinvariant distribution (that is, a distribution that is invariant under left and right H-translation of functions in $C_c^\infty(G)$) $\Phi_{\lambda_\pi,\lambda_{\tilde{\pi}}}$ on G, called a *spherical character* of π. The space of spherical characters of π is the span of all such distributions as λ_π and $\lambda_{\tilde{\pi}}$ range over $\mathrm{Hom}_H(\pi,1)$ and $\mathrm{Hom}_H(\widetilde{\pi},1)$, respectively. The reader may refer to [35] for the definition of $\Phi_{\lambda_\pi,\lambda_{\tilde{\pi}}}$.

The spherical characters of class one representations are the symmetric space analogues of the characters of admissible representations. We discuss a few aspects of this below.

The character Θ_π of an admissible representation π of G is a G-invariant distribution on G (with respect to conjugation). Harish-Chandra ([19]) proved that the character Θ_π of an admissible finite-length representation of G is given by integration against a locally integrable function on G. Furthermore, this function is locally constant on the regular set. We say that an element $g \in G$ is *θ-regular* if the intersection of the centralizer of $g\theta(g)^{-1}$ in $\boldsymbol{G}$ with the connected component of the identity in $\{x \in \boldsymbol{G} \mid \theta(x) = x^{-1}\}$ is a maximal θ-split torus of $\boldsymbol{G}$. The θ-regular set $G^{\theta-\mathrm{reg}}$, that is, the set of all θ-regular elements in G, is open and dense in G. The symmetric space analogue of Harish-Chandra's local integrability of characters involves restrictions of spherical characters of class one representations to the θ-regular set.

Theorem 10. ([35], Theorem 5.1, Corollary 5.2) *Let π be an irreducible class one representation of G. The restriction of a spherical character of π to $C_c^\infty(G^{\theta-\mathrm{reg}})$ is given by integration against a locally constant function on $G^{\theta-\mathrm{reg}}$.*

A spherical character is generally not realized by a function that is locally integrable on all of G.

As indicated in §4 of [35], certain results, such as Shalika germ expansions, do not have obvious symmetric space analogues. The Howe-Harish-Chandra local character expansion gives a germ expansion for the character of an irreducible admissible representation near the identity element, in terms of Fourier transforms of nilpotent orbital integrals on the Lie algebra. Theorem 7.11 of [35] is a symmetric space version of the local character expansion, but, due to the fact that nilpotent orbital integrals are often not defined in the symmetric space setting, the germ expansion for spherical characters near the identity is expressed only in terms of Fourier transforms of unspecified H-invariant distributions of nilpotent support.

Characters of supercuspidal and discrete series representations may be realized in terms of integration against matrix coefficients. This was originally proved in the supercuspidal setting by Harish-Chandra and later extended to discrete series by Rader and Silberger.

Theorem 11. ([18],[36]) *Let π be an irreducible supercuspidal or discrete series representation of G and let $d(\pi)$ be the formal degree of π. If φ is a matrix coefficient of π, then*

$$\varphi(1)\,\Theta_\pi(f) = d(\pi)\int_{G/Z}\int_G f(g)\varphi(xgx^{-1})\,dg\,dx^\times, \qquad f \in C_c^\infty(G).$$

In the supercuspidal setting, the analogous result for spherical characters takes the following form:

Theorem 12. ([30], Theorem 6.1, [42],Theorem 1.5, Corollary 1.11) *Let (π,V) be an irreducible H-distinguished supercuspidal representation of G. Let $\tilde{v}\in V$, $v\in V$ and $\varphi=\varphi_{\tilde{v},v}$. Then*

1. *The map $f\mapsto D_\varphi(f):=\int_{H/H\cap Z}\int_{H/H\cap Z}\int_G f(g)\,\varphi(h_1gh_2)\,dg\,dh_1^\times dh_2^\times$ defines an H-biinvariant distribution on G.*
2. *If $\lambda_{\tilde{v}}\in\mathrm{Hom}_H(\pi,1)$ and $\lambda_v\in\mathrm{Hom}_H(\widetilde{\pi},1)$ are defined as in* (3) *and $\Phi=\Phi_{\lambda_{\tilde{v}},\lambda_v}$ is the corresponding spherical character, then $\Phi(f)=D_\varphi(f)$. Moreover, Φ is nonzero if and only if $\lambda_{\tilde{v}}$ and λ_v are nonzero.*
3. *The set of spherical characters of π is equal to the span of the spherical characters of the form D_φ (as φ varies over the set of matrix coefficients of π).*

As shown in [42], for certain strongly discrete G/H symmetric spaces (including Galois symmetric spaces), the theorem extends to distinguished discrete series representations.

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