#1. $u_{tt} = 4u_{xx}$ $\qquad c^2 = 4 \Rightarrow c = 2$

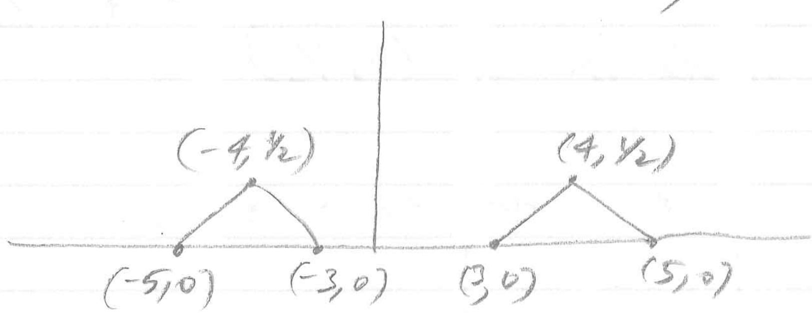
corners in the graph are located at $x=-1$, $x=0$, $x=1$. At time 2, these will be located at:

$$x=-1 \begin{cases} \nearrow x = -1 - 2\cdot 2 = -5 \\ \searrow x = -1 + 2\cdot 2 = 3 \end{cases}$$

$$x=0 \begin{cases} \nearrow x = -4 \\ \searrow x = 4 \end{cases}$$

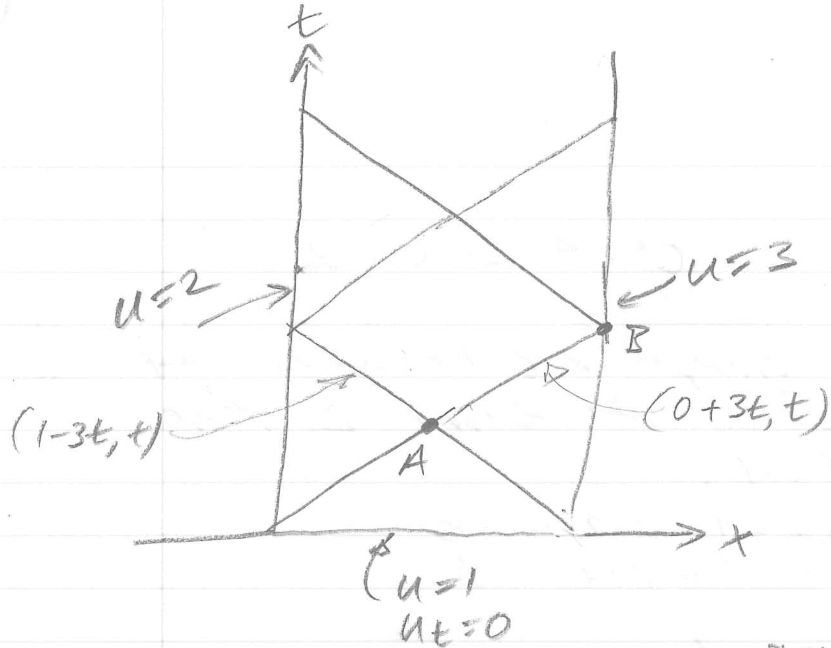
$$x=1 \longrightarrow \begin{cases} x = 1 - 4 = -3 \\ x = 1 + 4 = 5 \end{cases}$$

$$u(x,2) = \frac{1}{2}\left(u_0(x-4) + u_0(x+4)\right)$$



#2.
$$\begin{cases} u_{tt} = 9u_{xx} & \text{on } [0,1]\times(0,\infty) \\ u(0,t) = 2 & t>0 \\ u(1,t) = 3 & t>0 \\ u(x,0) = 1 & x\in[0,1] \\ u_t(x,0) = 0 & x\in[0,1] \end{cases}$$

want solution at $t = 1/12$, $t = 1/3$



$c^2 = 9$
$\Rightarrow c = 3$

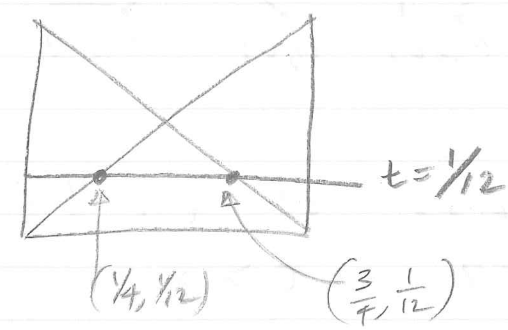
point A: $1 - 3t = 3t$
$\Rightarrow 1 = 6t$
$\Rightarrow t = 1/6$

A is the point $(1/2, 1/6)$

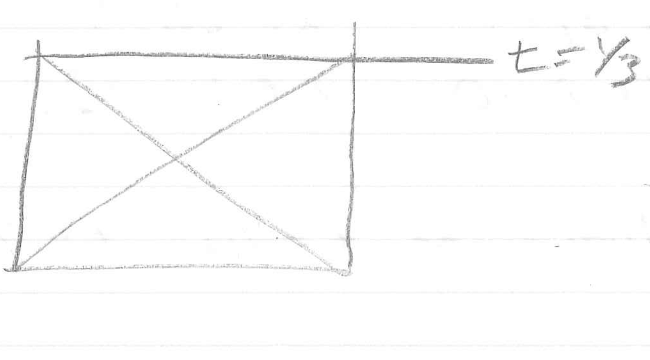
point B: $3t = 1$
$t = 1/3$

B is the point $(1, 1/3)$

seeking solution at time $t = 1/12$



Seeking solution at time $t = 1/3$



Three approaches:

1. Subtract off the problem that solves the inhomog. BCs and then
   a. Solve it by Fourier methods
   b. solve it by d'Alembert's formula.
2. Use the parallelogram rule to solve it directly.

Approach 1:

Solve
$$\begin{cases} \tilde{u}_{tt} = 9\tilde{u}_{xx} & \text{in } (0,1)\times(0,\infty) \\ \tilde{u}(0,t) = 0 & \text{for } t>0 \\ \tilde{u}(1,t) = 0 & \text{for } t>0 \\ \tilde{u}(x,0) = -1-x & \text{for } x\in[0,1] \\ \tilde{u}_t(x,0) = 0 & \text{" } \quad \text{"} \end{cases}$$
and then add the function $v(x) = x+2$ to the solution.

a) Fourier methods

$$\tilde{u}(x,t) = \sum_1^\infty \left[a_n \cos(3n\pi t) + b_n \sin(3n\pi t)\right]\sin(n\pi x)$$

$$\tilde{u}_t(x,0) = 0 \Rightarrow b_n = 0 \text{ for all } n$$

$$\tilde{u}(x,0) = -1-x = \sum_1^\infty a_n \sin(n\pi x)$$

where $a_n = 2\int_0^1 \tilde{u}(x,0)\sin(n\pi x)\,dx = 2\int_0^1 [-1-x]\sin(n\pi x)\,dx$

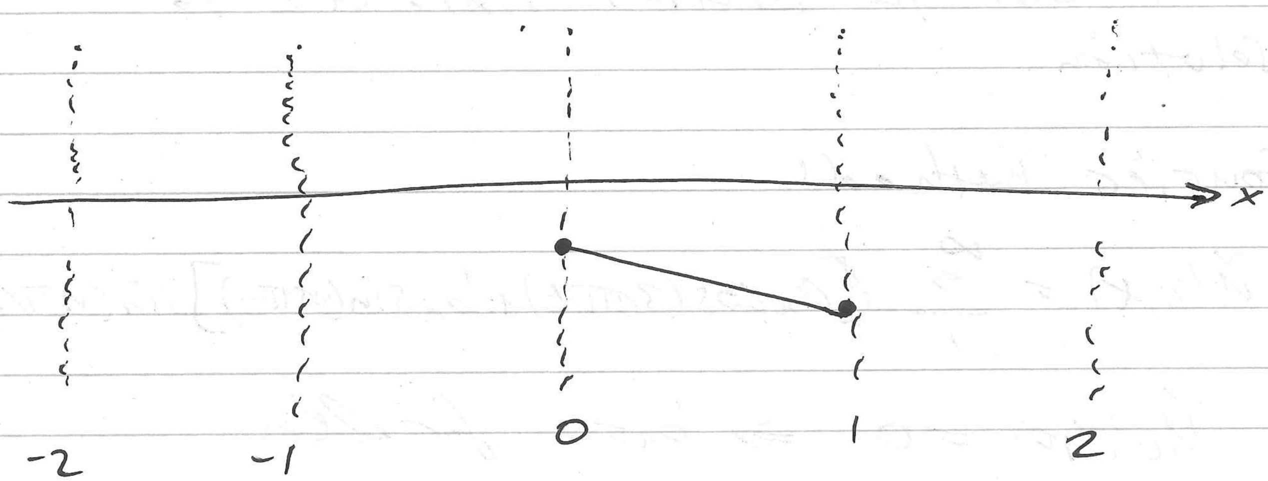
$$a_n = \frac{4(-1)^n - 2}{n\pi}$$

$\Rightarrow$ Solution to the problem is

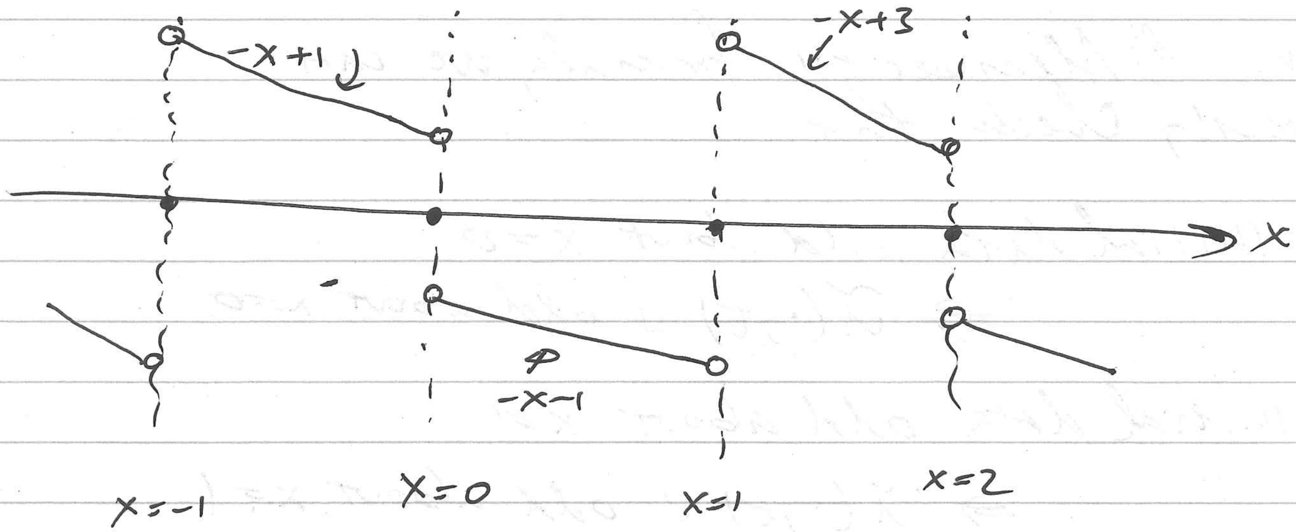
$$\boxed{u(x,t) = 2 + x + \sum_{1}^{\infty} \frac{4(-1)^n - 2}{n\pi} \sin(n\pi x)}$$

Great! But now we need to think about infinite series in order to graph this ☹ See book.

<u>Note</u>: When doing a Fourier series expansion for a fxn on $[0,1]$ we have jump discontinuities at $x=0$ and $x=1$ in this case:



What the Fourier series (well, really <u>sine</u> series) does is computes the series that converges to this function:



i.e. take u_0 on $(0,1)$
and extend it to $(-1,0)$ so that it's odd about $x=0$ $\Rightarrow$ the extension is 0 at $x=0$

then extend this from $(-1,1)$ to $(1,3)$ by creating a fxn that is odd about $x=1$
$\Rightarrow$ the extension is 0 at $x=1$.

This creates an extension to $(-1,3)$
extend to all of $\mathbb{R}$ so that we have a fxn that is periodic w/ period 2 and is odd about $x=0$ and $x=1$.

Call this function u_{extend}.

We're now ready to use d'Alembert to solve the problem

$$\begin{cases} \tilde{u}_{tt} = 9\tilde{u}_{xx} & \text{for } x \in \mathbb{R} \ t>0 \\ \tilde{u}(x,0) = u_{extend}(x) & \text{for } x \in \mathbb{R} \\ \tilde{u}_t(x,0) = 0 & \text{" "} \end{cases}$$

From d'Alembert's formula, we can directly check that

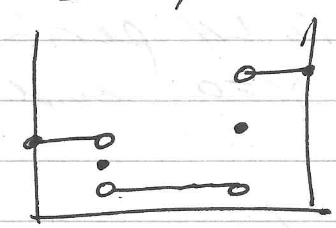
a) initial data odd about $x=0$
$$\Rightarrow \tilde{u}(\cdot,t) \text{ is odd about } x=0$$

b) initial data odd about $x=1$
$$\Rightarrow \tilde{u}(\cdot,t) \text{ is odd about } x=1$$

$$\Rightarrow u(x,t) = 2+x+\tilde{u}(x,t)$$
will satisfy the BCs because
$$\tilde{u}(0,t) = \tilde{u}(1,t) = 0 \quad \forall t.$$

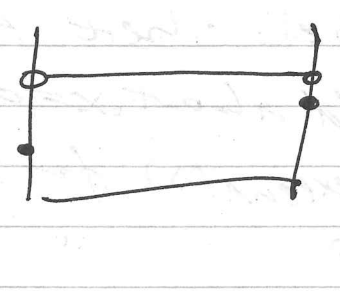
Now we plot the solution directly at times $t=1/12$ and $t=1/3$ by shifting the graph of u_{extend} by $\pm 1/4$ (for $t=1/12$) and by ± 1 (for $t=1/3$) and adding the shifts and dividing by 2: This results in

$t=1/12$



$$u(x,1/12) = \begin{cases} 2 & x\in[0,1/4) \\ 3/2 & x=1/4 \\ 1 & x\in(1/4,3/4) \\ 2 & x=3/4 \\ 3 & x\in(3/4,1] \end{cases}$$

$t=1/3$



$$u(x,1/3) = \begin{cases} 2 & x=0 \\ 4 & x\in(0,1) \\ 3 & x=1 \end{cases}$$

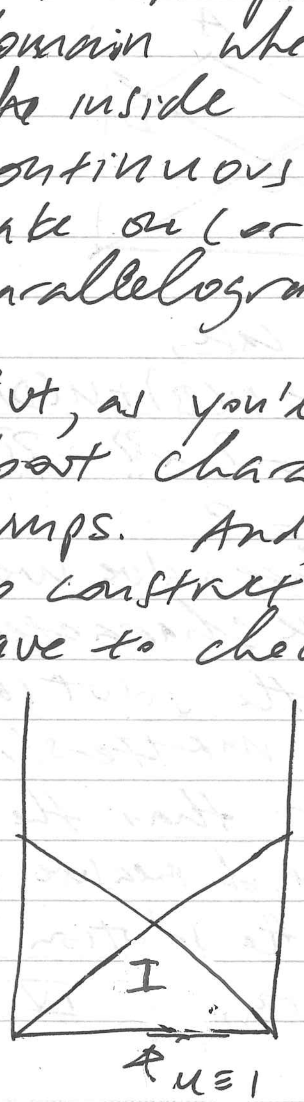
Note: the d'Alembert solution is the same as the Fourier series solution because the Fourier series converges to the average, at a jump discontinuity.

Approach 2:

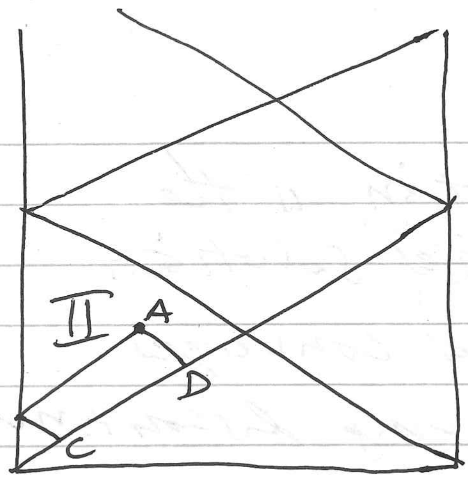
Use the parallelogram rule.

Here we need to think a little. The derivation of the parallelogram rule was implicitly assuming the solution was classical and that the parallelogram was inside the domain where the PDE holds. (In this case, who inside $(0,1)\times(0,\infty)$). If the solution is continuous on $[0,1]\times[0,\infty)$ then we can take one (or more) of the vertices of the parallelogram on the boundary of $(0,1)\times(0,\infty)$.

But, as you'll see, we have to be careful about characteristics at which the solution jumps. And, no matter what, we use it to construct a function which we then have to check is actually a weak solution.



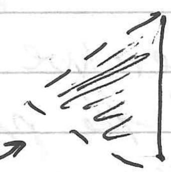
region I: in this closed triangle, we can use d'Alembert to get $u \equiv 1$.

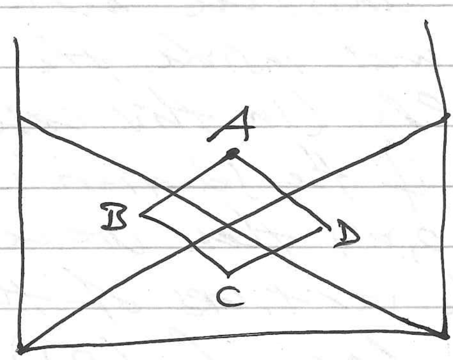


$$u(A) = u(B) + u(D) - u(C)$$
$$= 2 + ? - ?$$
$$= 2$$

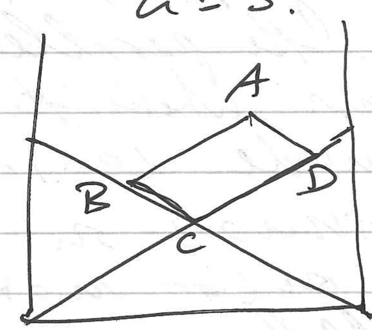
Note that the values at C & D are irrelevant because they'll cancel out.

So on this set  $u \equiv 2$.

Similarly, on this set  $u \equiv 3$.



vs



in this case

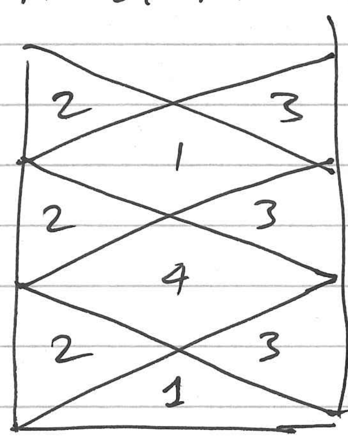
$$u(A) = u(B) + u(D) - u(C)$$
$$= 2 + 3 - 1 = 4$$

in this case,

$$u(A) = u(B) + u(D) - u(C)$$
$$= ? + ? - ?$$
$$= ?$$

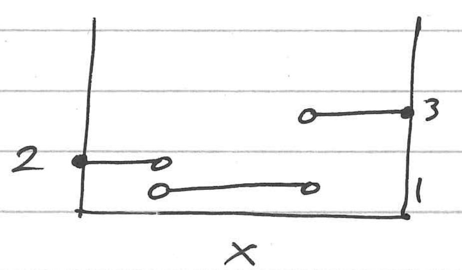
i.e. what value we assign on those characteristics where the solution jumps, matters! And we ~~get~~ that the soln on a set of measure 0 affects the solution on all of region IV.

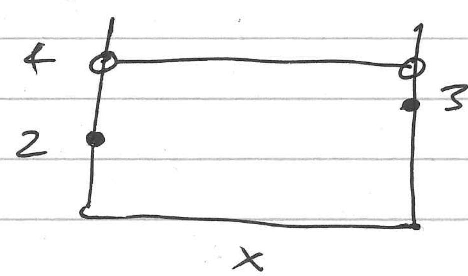
Long story short, we need that the jump across the characteristics is constant along the characteristics (see transmission condition in §2.3.b) but other than that we can assign what we want on those curves.



the values of u in various <u>open</u> triangles

plotting at times $t = 1/12$, $t = 1/3$





#3) $x^2 u_{xx} + 0 u_{xy} - y^2 u_{yy} = 0$

$a(x,y) = x^2$

$b(x,y) = 0$ $\Rightarrow b^2 - 4ac = 4x^2y^2 > 0$ if $x \neq 0$ and $y \neq 0$

$c(x,y) = -y^2$

$x \neq 0$ and $y \neq 0$ $\Rightarrow$ PDE is hyperbolic

$x \neq 0 \wedge y = 0$ or $x = 0 \wedge y \neq 0$ $\Rightarrow$ PDE is parabolic

$x = 0 \wedge y = 0$ $\Rightarrow$ PDE? what PDE?

If $x=0$ but $y\neq 0$ then
PDE becomes $-y^2 u_{yy}=0$
$\Rightarrow u_{yy}=0.$

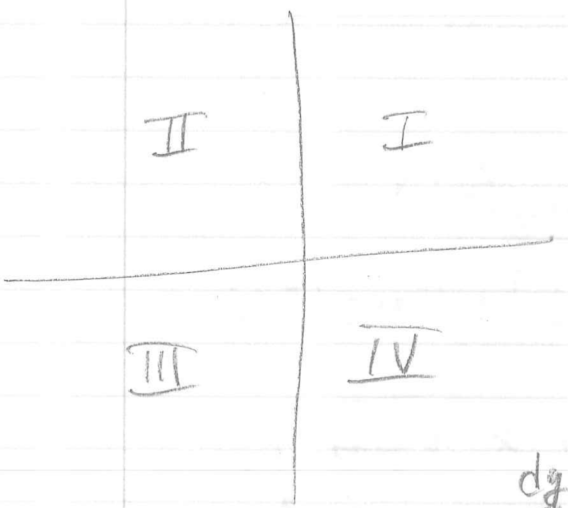
If $y=0$ but $x\neq 0$ then
PDE becomes $x^2 u_{xx}=0$
$\Rightarrow u_{xx}=0$

Assume $x\neq 0$ and $y\neq 0$.

Characteristic curves found by solving

$$\frac{dy}{dx}=\frac{b\pm\sqrt{b^2-4ac}}{2a}=\frac{0\pm\sqrt{4x^2y^2}}{2x^2}$$

$$=\pm\frac{2|x||y|}{2x^2}=\pm\frac{|y|}{|x|}$$



$I=\{x>0 \text{ and } y>0\}$
characteristics are

$$\frac{dy}{dx}=\pm y/x$$

$$\frac{dy}{dx}=\frac{y}{x}\Rightarrow y=cx\Rightarrow \mu=y/x$$

$$\frac{dy}{dx}=-y/x\Rightarrow y=\frac{c}{x}\Rightarrow \eta=yx$$

v defined via

$$u(x,y) = v(\mu(x,y), \eta(x,y))$$

$$u_x = v_\mu \mu_x + v_\eta \eta_x$$

$$= v_\mu\left(\frac{-y}{x^2}\right) + v_\eta(y)$$

$$u_{xx} = \frac{2y}{x^3} v_\mu - \frac{y}{x^2}\left[v_{\mu\mu}\left(\frac{-y}{x^2}\right) + v_{\mu\eta}(y)\right]$$

$$+ y\left[v_{\mu\eta}(-y/x^2) + v_{\eta\eta}(y)\right]$$

$$\boxed{u_{xx} = \frac{2y}{x^3} v_\mu + \frac{y^2}{x^4} v_{\mu\mu} - 2\frac{y^2}{x^2} v_{\mu\eta} + y^2 v_{\eta\eta}}$$

$$u_y = v_\mu \mu_y + v_\eta \eta_y$$

$$= v_\mu\left(\frac{1}{x}\right) + v_\eta(x)$$

$$u_{yy} = \frac{1}{x}\left[v_{\mu\mu}\left(\frac{1}{x}\right) + v_{\mu\eta}(x)\right]$$

$$+ x\left[v_{\mu\eta}\left(\frac{1}{x}\right) + v_{\eta\eta}(x)\right]$$

$$\boxed{u_{yy} = \frac{1}{x^2} v_{\mu\mu} + 2v_{\mu\eta} + x^2 v_{\eta\eta}}$$

Combining,

$$x^2 u_{xx} - y^2 u_{yy} = 0$$

$$\frac{2y}{x} v_\mu(\mu(x,y), \eta(x,y)) - 4y^2 v_{\mu\eta}(\mu(x,y), \eta(x,y)) = 0$$

(7)

recalling

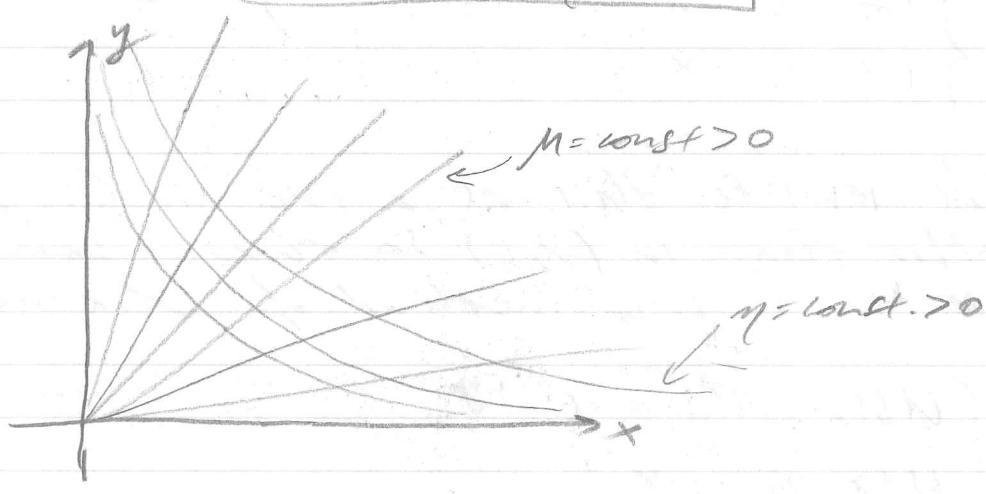
$$\mu(x,y) = \frac{y}{x} \qquad \eta(x,y) = yx$$

$$\Rightarrow \mu(x,y)\,\eta(x,y) = y^2$$

So in region I the PDE's normal form is

$$+4\mu\eta\, v_{\mu\eta} = 2\mu\, v_\mu$$

$$\Rightarrow \boxed{v_{\mu\eta} = \frac{1}{2}\frac{1}{\eta} v_\mu}$$



regions II, III, IV will be analogous

for example: II $\quad x<0,\ y>0 \qquad \frac{dy}{dx} = \pm\frac{|y|}{|x|} = \pm\frac{y}{-x} = \pm\frac{y}{x}$

$\Rightarrow \mu = y/x \quad \eta = yx$ as before

only difference is that $\mu<0,\ \eta<0$.

Still get $v_{\mu\eta} = \frac{1}{2}\frac{1}{\eta}v_\mu$.

#4. Solve
$$\begin{cases} u_t + u u_x = e^{-x} v \\ v_t + c v_x = 0 \\ u(x,0) = x \\ v(x,0) = e^x \end{cases}$$

1. Solve $v_t + c v_x = 0$
$v(x,0) = e^x$

$\Rightarrow v(x,t) = e^{x-ct}$

2. Solve
$$\begin{cases} u_t + u u_x = e^{-x}(e^{x-ct}) = e^{-ct} \\ u(x,0) = x \end{cases}$$

I'll rewrite this as a PDE in (x,y) rather than in (x,t) so that I can use t in my method of charact.

$$\begin{cases} u u_x + u_y = e^{-cy} \\ u(x,0) = x \end{cases}$$

$$\begin{cases} \frac{dx}{dt} = z(t) \\ x(0) = s \end{cases} \quad \begin{cases} \frac{dy}{dt} = 1 \\ y(0) = 0 \end{cases} \quad \begin{cases} \frac{dz}{dt} = e^{-cy} \\ z(0) = s \end{cases}$$

solving for $y(s,t) \Rightarrow y(s,t) = t$

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Solving for $z(s,t)$

$$\begin{cases} \dfrac{dz}{dt} = e^{-cy(s,t)} = e^{-ct} \\ z(0) = s \end{cases}$$

$$\Rightarrow z(s,t) = -\frac{1}{c}e^{-ct} + \left(s + \frac{1}{c}\right)$$

Solving for $x(s,t)$

$$\begin{cases} \dfrac{dx}{dt} = s + \dfrac{1}{c} - \dfrac{1}{c}e^{-ct} \\ x(0) = s \end{cases}$$

$$\Rightarrow x(s,t) = \left(s + \frac{1}{c}\right)t + \frac{1}{c^2}e^{-ct} + s - \frac{1}{c^2}$$

oof! $\begin{cases} x(s,t) = \left(s + \frac{1}{c}\right)t + \frac{1}{c^2}e^{-ct} + s - \frac{1}{c^2} \\ y(s,t) = t \end{cases}$

want to solve for $s(x,y)$, $t(x,y)$.

$$\begin{pmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} t+1 & s + \frac{1}{c} - \frac{1}{c}e^{-ct} \\ 0 & 1 \end{pmatrix}$$

$\det = (1+t) \Rightarrow$ have hope for $t > -1$

① $t = y$

② $x = (s + \frac{1}{c})y + \frac{1}{c^2}e^{-cy} + s - \frac{1}{c^2}$

$$\Rightarrow \quad s = \frac{x - y/c - 1/c^2\, e^{-cy} + 1/c^2}{y+1}$$

Now

$$u(x,y) := z(s(x,y), t(x,y))$$
$$= -\frac{1}{c}e^{-cy} + \frac{1}{c} + \left(\frac{x - y/c - \frac{1}{c^2}e^{-cy} + 1/c^2}{y+1}\right)$$

Check.

$$u(x,0) = -\frac{1}{c} + \frac{1}{c} + \frac{x - 1/c^2 + 1/c^2}{1} = x \checkmark$$

$u u_x + u_y = e^{-cy}$ (did the grungy scratchwork)

Now translate u back from (x,y) to (x,t) to get final answer

$$\boxed{\begin{aligned} u(x,t) &= -\frac{1}{c}e^{-ct} + \frac{1}{c} + \frac{x - t/c - 1/c^2\, e^{-ct} + 1/c^2}{1+t} \\ v(x,t) &= e^{x-ct} \end{aligned}}$$

(P)

#5. $F(x, y, u(x,y), u_x(x,y), u_y(x,y)) = 0$

$$\Rightarrow \frac{\partial}{\partial x} \quad " \quad " = 0$$

$$\rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} u_x + \frac{\partial F}{\partial p} u_{xx} + \frac{\partial F}{\partial q} u_{xy} = 0$$

where $z = u(x,y)$

$p = u_x(x,y)$ $\Rightarrow$ $u_{xx} = p_x$

$q = u_y(x,y)$ $\quad$ $u_{xy} = p_y$

So differentiating wrt x yields

$$\frac{\partial F}{\partial p} p_x + \frac{\partial F}{\partial q} p_y = -\frac{\partial F}{\partial x} - \frac{\partial F}{\partial z} p$$

This is a quasilinear PDE which we'd solve using the characteristics.

$$(*) \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial F}{\partial p} \\ \dfrac{dy}{dt} = \dfrac{\partial F}{\partial q} \\ \dfrac{\partial p}{\partial t} = -\dfrac{\partial F}{\partial x} - p\dfrac{\partial F}{\partial z} \end{cases}$$

Now differentiate wrt y to get a quasilinear PDE for q...

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} u_y + \frac{\partial F}{\partial p} u_{xy} + \frac{\partial F}{\partial q} u_{yy} = 0$$

recalling $u_y = q$

$u_{xy} = q_x$

$u_{yy} = q_y$

we have

$$\frac{\partial F}{\partial p} q_x + \frac{\partial F}{\partial q} q_y = -\frac{\partial F}{\partial y} - \frac{\partial F}{\partial z} q$$

So this we'd solve using characteristics

$$(**) \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial F}{\partial p} \\ \dfrac{\partial y}{\partial t} = \dfrac{\partial F}{\partial q} \\ \dfrac{\partial q}{\partial t} = -\dfrac{\partial F}{\partial y} - q\dfrac{\partial F}{\partial z} \end{cases}$$

Note! $(*)$ and $(**)$ have the same x and y characteristics!! We have 4 ODEs...

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial F}{\partial p} \\ \dfrac{dy}{dt} = F_q \\ \dfrac{dp}{dt} = -F_x - pF_z \\ \dfrac{dq}{dt} = -F_y - qF_z \end{cases}$$

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Now we just need an ODE for z to close the system. (The RHS of the 4 ODE system depends on $z(t)$.)

$$\frac{dz}{dt} = \frac{d}{dt} u(x(t), y(t))$$

$$= u_x \frac{dx}{dt} + u_y \frac{dy}{dt}$$

$$= p \frac{dx}{dt} + q \frac{dy}{dt}$$

$$= p F_p + q F_q$$

Finally we have it!

$$\begin{cases} \dfrac{dx}{dt} = F_p \\ \dfrac{dy}{dt} = F_q \\ \dfrac{dz}{dt} = pF_p + qF_q \\ \dfrac{dp}{dt} = -F_x - pF_z \\ \dfrac{dq}{dt} = -F_y - qF_z \end{cases}$$

#6) Rescale space and time.

v defined on $[0, 2\pi]$

want u defined on $[0, 10]$

try $u(x,t) = v(\frac{2\pi}{10} x, \alpha t)$ where α is to be determined.

$$u(x,t) = v\left(\frac{2\pi}{10}x, \alpha t\right)$$

$$u_t = \alpha v_{\tau} \qquad \Rightarrow v_{\tau} = \frac{1}{\alpha} u_t$$

$$u_x = \left(\frac{2\pi}{10}\right) v_{\tilde{x}} \qquad \Rightarrow v_{\tilde{x}} = \left(\frac{10}{2\pi}\right) u_x$$

$$u_{xx} = \left(\frac{2\pi}{10}\right)^2 v_{\tilde{x}\tilde{x}} \Rightarrow v_{\tilde{x}\tilde{x}} = \left(\frac{10}{2\pi}\right)^2 u_{xx}$$

If v solves

$$v_{\tau} + v v_{\tilde{x}} = D v_{\tilde{x}\tilde{x}}$$

then u solves

$$\frac{1}{\alpha} u_t + \left(\frac{10}{2\pi}\right) u u_x = D \left(\frac{10}{2\pi}\right)^2 u_{xx}$$

$$\Rightarrow u_t + \alpha\left(\frac{10}{2\pi}\right) u u_x = D\alpha\left(\frac{10}{2\pi}\right)^2 u_{xx}$$

we want to choose α and D so that

$$\alpha\left(\frac{10}{2\pi}\right) = 1$$

$$D\alpha\left(\frac{10}{2\pi}\right)^2 = 3$$

then we'll have a solution of

$$u_t + u u_x = 3 u_{xx}.$$

$$\alpha = \frac{2\pi}{10}, \quad D = \frac{3}{\alpha}\left(\frac{2\pi}{10}\right)^2 = 3\left(\frac{2\pi}{10}\right)$$

So if

$v(\tilde{x}, \tau)$ solves

$$v_\tau + v v_{\tilde{x}} = 3\left(\frac{2\pi}{10}\right) v_{\tilde{x}\tilde{x}}$$

then

$$u(x,t) := v\left(\frac{2\pi}{10}x, \frac{2\pi}{10}t\right)$$

solves

$$u_t + u u_x = 3 u_{xx}$$

A: Your supervisor wants the solution at time $t=1$. So you give the black box $v_0(\tilde{x}) = u_0\left(\frac{10}{2\pi}\tilde{x}\right)$, $D = 3\left(\frac{2\pi}{10}\right)$ and $\tau_0 = \frac{2\pi}{10}$. It then gives you $v\left(\tilde{x}, \frac{2\pi}{10}\right)$. You use this to construct $u(x,1) := v\left(\frac{2\pi}{10}x, \frac{2\pi}{10}\right)$ and plot that.

A: If $D=0$ and $v_0(\tilde{x}) = \cos(\tilde{x})$ then the solution blows up at time $\tau_0 = 1$. This means that $u(x,t)$ will blow up at time $\frac{10}{2\pi}$