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Mat267: Midterm 1 February 3, 2019 5:10pm-7:00pm or 6:10-8:00pm

110 minute exam; please read all problems before starting. No calculators or other aids allowed.

The exam will be marked with the assistance of the crowdmark software. Do not write on the QR code at the top of the pages. Any page with a QR code will be scanned, uploaded, and read. Please write neatly and with a pen or dark pencil — light pencil writing doesn't get picked up well by the scanner.

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ANSWER KEY

1. $\boxed{\text{5 points}}$ Consider the linear system $X' = AX$ where A is an $n \times n$ matrix with constant entries. Prove that if $\vec{V}$ is an eigenvector of A with eigenvalue λ then $X(t) = e^{\lambda t}\vec{V}$ is a solution of $X' = AX$.

$$X(t) = e^{\lambda t}\vec{V} \Rightarrow AX(t) = A(e^{\lambda t}\vec{V})$$

$$\Downarrow \qquad = e^{\lambda t}(A\vec{V})$$

$$\frac{dX}{dt} = \lambda e^{\lambda t}\vec{V} \qquad = e^{\lambda t}(\lambda\vec{V})$$

bec. $\vec{V}$ is an eigenvector w/ eigenvalue λ

Therefore $\frac{dX}{dt} = \lambda e^{\lambda t}\vec{V} = AX(t)$

and $X(t)$ is a solution of $X' = AX$. //

2. Consider the separable ODE

$$x' = tx^2$$

a) $\boxed{\text{3 points}}$ Solve the initial value problem with initial data $x(0) = 1$.

$$\frac{x'(t)}{x(t)^2} = t \quad \text{if } x(t) \neq 0$$

$$\frac{d}{dt}\left(-\frac{1}{x(t)}\right) = t$$

$$-\frac{1}{x(t)} = \frac{t^2}{2} + C = \frac{t^2+2C}{2}$$

$$x(t) = \frac{-2}{t^2+2C} \quad \text{(in general)}$$

$$x(0) = 1 = \frac{-1}{0+C} \Rightarrow C = -1$$

$$\boxed{x(t) = \frac{-2}{t^2-2} \quad \text{is the solution of the IVP}}$$

b) $\boxed{\text{3 points}}$ Solve the initial value problem with initial data $x(0) = 0$.

In the first step above, I assumed $x(t) \neq 0$ so I'm not going to be able to solve this IVP using separation of variables.

On the other hand, by inspection I see that $x(t) = 0$ is a solution and it satisfies the initial condition.

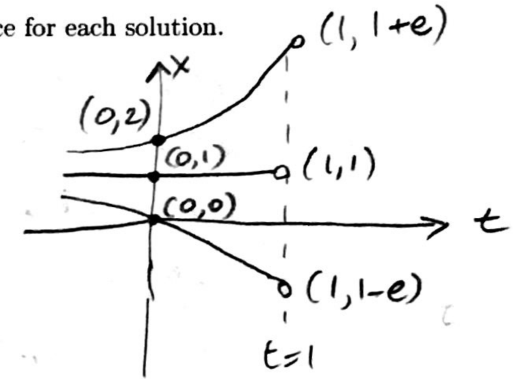
3. Consider the nonautonomous differential equation

$$x' = f(x) = \begin{cases} x-1 & \text{if } t \le 1 \\ 3-x & \text{if } t > 1 \end{cases}$$

a) $\boxed{\text{5 points}}$ Solve the three initial value problems

i. $x' = f(x)$ and $x(0) = 0$,
ii. $x' = f(x)$ and $x(0) = 1$,
iii. $x' = f(x)$ and $x(0) = 2$.

Sketch all three solutions on one graph and give the interval of existence for each solution.



$x' = x-1$

$\Rightarrow x'(t) - x(t) = -1$

$e^{-t}x'(t) - e^{-t}x(t) = -e^{-t}$

$(e^{-t}x(t))' = -e^{-t}$

$e^{-t}x(t) = e^{-t} + d$

$x(t) = 1 + ce^{t}$ ← general solution of $x' = x-1$

$x_0(t) = 1 - e^t$ satisfies $x(0) = 0$

$x_1(t) = 1$ satisfies $x(0) = 1$

$x_2(t) = 1 + e^t$ satisfies $x(0) = 2$

evaluate x' for each solution at $t=1$:

$x_0'(1) = -e^1 \neq 3-(1-e) = 3 - x_0(1)$

$x_1'(1) = 0 \neq 3-(1) = 3 - x_1(1)$

$x_2'(1) = 1+e \neq 3-(1+e) = 3 - x_2(1)$

We can build a continuous function on $(-\infty, \infty)$ that will be a solution on $(-\infty,1) \cup (1,\infty)$ and satisfy the IVP at $t=0$. But the interval of existence for each solution (x_0, x_1, x_2) is $(-\infty, 1)$.

b) $\boxed{\text{5 points}}$ Recall the the nonautonomous differential equation

$$x' = f(x) = \begin{cases} x - 1 & \text{if } t \leq 1 \\ 3 - x & \text{if } t > 1 \end{cases}$$

Solve the three initial value problems

i. $x' = f(x)$ and $x(2) = 2$,
ii. $x' = f(x)$ and $x(2) = 3$,
iii. $x' = f(x)$ and $x(2) = 4$.

Sketch all three solutions on one graph and give the interval of existence for each solution.

$$x' = 3 - x$$
$$x' + x = 3$$
$$e^t x' + e^t x = 3e^t$$
$$(e^t x)' = 3e^t$$
$$e^t x = 3e^t + C$$
$$x = 3 + Ce^{-t}$$

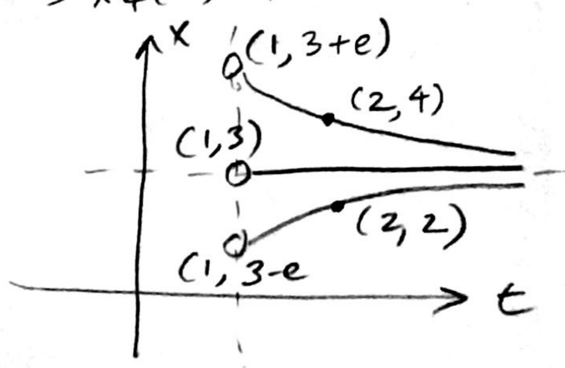
$X_2(t) = 3 - e^{-(t-2)}$ satisfies $X_2(2) = 2$

$X_3(t) = 3$ satisfies $X_3(2) = 3$

$X_4(t) = 3 + e^{-(t-2)}$ satisfies $X_4(2) = 4$

evaluate x' at $t=1$ for each solution and compare this to $x-1$. If they're not equal then we can't construct solutions on $(-\infty, \infty)$. The interval of existence for each solution will be $(1, \infty)$.

$$X_2'(1) = e \neq (3-e) - 1 = X_2(1) - 1$$
$$X_3'(1) = 0 \neq (3-1) = X_3(1) - 1$$
$$X_4'(1) = -e \neq (3+e) - 1 = X_4(1) - 1$$



c) $\boxed{\text{5 points}}$ Recall the the nonautonomous differential equation

$$x' = f(x) = \begin{cases} x-1 & \text{if } t \leq 1 \\ 3-x & \text{if } t > 1 \end{cases}$$

Find and sketch a solution whose interval of existence is $(-\infty, \infty)$.

Solve $x' = x - 1$ on $(-\infty, 1)$

$\Rightarrow x_\ell(t) = 1 + ce^t$

Solve $x' = 3 - x$ on $(1, \infty)$

$\Rightarrow x_r(t) = 3 + \hat{c}e^{-t}$

want $x_\ell'(1) = x_r'(t)$ to have a derivative at $t = 1$

i.e. $ce^1 = -\hat{c}e^{-1}$

want $x_\ell(1) = x_r(1)$ to have a continuous solution at $t=1$. i.e $1 + ce = 3 + \hat{c}e^{-1}$

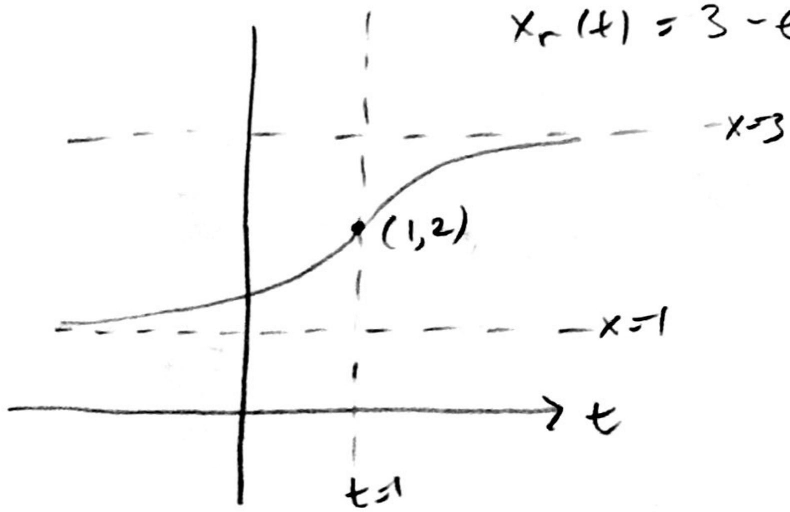
Combining: $ce = -\hat{c}\frac{1}{e} \Rightarrow 1 + ce = 1 - \hat{c}e^{-1} = 3 + \hat{c}e^{-1}$

$\Rightarrow -2 = 2\hat{c}e^{-1} \Rightarrow \hat{c} = -e$ and $c = e^{-1}$

$x_\ell(t) = 1 + e^{-1}e^t = 1 + e^{t-1}$

$x_r(t) = 3 - e e^{-t} = 3 - e^{-(t-1)}$

$$x(t) = \begin{cases} 1 + e^{t-1} & t \leq 1 \\ 3 - e^{-(t-1)} & t > 1 \end{cases}$$



alt:

Need
$x - 1 = 3 - x$
$\Rightarrow 2x = 4$
$\Rightarrow x = 2$

$x_\ell = 1 + e^{(t-1)}$ $t \leq 1$

$x_r(t) = 3 - e^{-(t-1)}$ $t > 1$

$$x(t) = \begin{cases} 1 + e^{t-1} & t \leq 1 \\ 3 - e^{-(t-1)} & t > 1 \end{cases}$$

4. Consider the initial value problem

$$X' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} X \quad \text{with} \quad X(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

a) [7 points] Find the solution of the initial value problem and sketch it in the the plane.

b) [3 points] Find the magnitude of your solution $\|X(t)\|$ and plot it as a function of t.

general solution of $X' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} X$ is

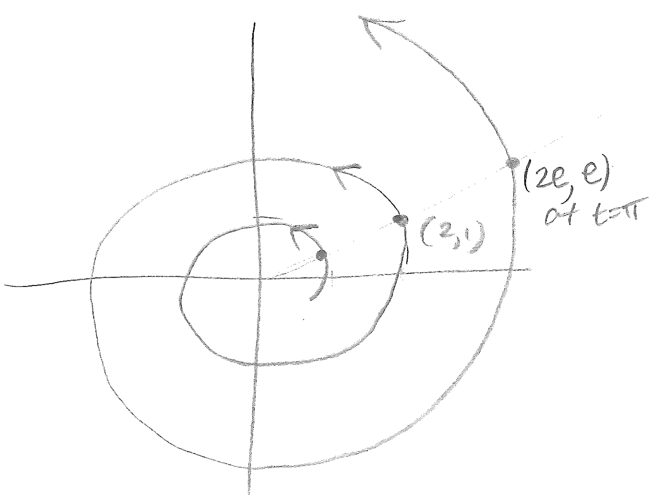
$$X(t) = c_1 e^{\alpha t}\begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + c_2 e^{\alpha t}\begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}$$

$$X(0) = c_1 \cdot 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \cdot 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow X(t) = 2e^{\alpha t}\begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + e^{\alpha t}\begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}$$

use $\alpha = 1$ $\beta = -2$

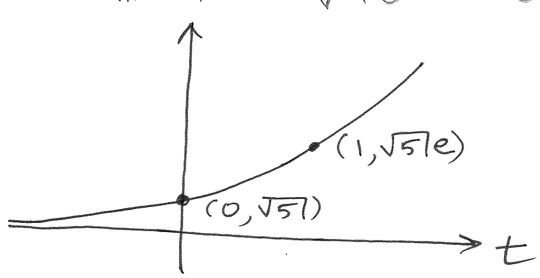
$$\boxed{X(t) = 2e^{t}\begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix} + e^{t}\begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix}}$$



$$\|X(t)\| = \sqrt{(2e^t\cos - e^t\sin)^2 + (2e^t\sin + e^t\cos)^2}$$

$$= \sqrt{4e^{2t}\cos^2 + e^{2t}\sin^2 - 4e^{2t}\sin\cos + 4e^{2t}\sin^2 + e^{2t}\cos^2 + 4e^{2t}\sin\cos}$$

$$\|x(t)\| = \sqrt{4e^{2t} + e^{2t}} = \boxed{e^t\sqrt{5} = \|X(t)\|}$$



c) $\boxed{\text{5 points}}$ Assume $X(t)$ is a solution of $X' = AX$ where A is given on the previous page. Find an ODE that's satisfied by $\|X(t)\|^2$. Solve this ODE.

$$\|X(t)\|^2 = x(t)^2 + y(t)^2$$

$$\Rightarrow \frac{d}{dt}\|X(t)\|^2 = 2xx' + 2yy'$$

$$'' \quad = 2x[x-2y] + 2y[2x+y] \quad \text{since } X' = AX$$

$$'' \quad = 2x^2 - 4xy + 4xy + 2y^2$$

$$'' \quad = 2x^2 + 2y^2$$

$$\frac{d}{dt}\|X(t)\|^2 = 2\|X(t)\|^2$$

$$\|X(t)\|^2 = \|X(0)\|^2 e^{2t} \quad \Rightarrow \|X(t)\| = \|X(0)\| e^{t}$$

5. 15 points Consider the initial value problem $x' = f(x)$ with $x(0) = x_0$ where $f : \mathbb{R} \to \mathbb{R}$ is C^1. Assume 0 is an equilibrium point: $f(0) = 0$.

Prove that if $f(x) \le -\mu x$ for all $x > 0$ and $x_0 > 0$ then $x(t) \le x_0 e^{-\mu t}$ and $x(t) \to 0$ as $t \to \infty$. (Assume $\mu > 0$.)

We know $x=0$ is an equilibrium solution because $f(0)=0$.

We know $x(0)>0 \Rightarrow x(t)>0$ on the interval of existence of the solution. (otherwise the solution would equal 0 in finite time, contradicting uniqueness of solutions.)

$\Rightarrow$ We have a solution $x(t)$ on (a,b) where $a<0$. We know

$$x'(t) = f(x(t)) \le -\mu x(t)$$

for all $t \in (a,b)$. Hence

$$x'(t) + \mu(x(t)) \le 0$$

$$e^{\mu t} x'(t) + \mu e^{\mu t} x(t) \le 0$$

$$(e^{\mu t} x(t))' \le 0$$

Hence $e^{\mu t} x(t) \le x_0$ for all $t \in [0,b)$

This proves $x(t) \le x_0 e^{-\mu t}$ for all $t \in [0,b)$

It turns out that we'll also be able to prove $b=\infty$ (you have to make this leap of faith) hence

$$0 < x(t) \le x_0 e^{-\mu t} \text{ for all } t \in [0,\infty)$$

and $\lim_{t\to\infty} x(t) = 0$ by "Sandwich theorem".

6. Consider the ODE

$$x' = ax + x^3, \quad \text{where} \quad a \in \mathbb{R}.$$

a) $\boxed{\text{3 points}}$ Fix the parameter a and find all equilibrium solutions. How does the number of equilibriums solutions depend on a?

b) $\boxed{\text{7 points}}$ Draw a bifurcation diagram for this ODE. Make sure to include representative phase lines so that it's clear what the stability of each the equilibrium solutions are.

$a > 0$ $\quad 0 = ax + x^3 = x(a + x^2)$

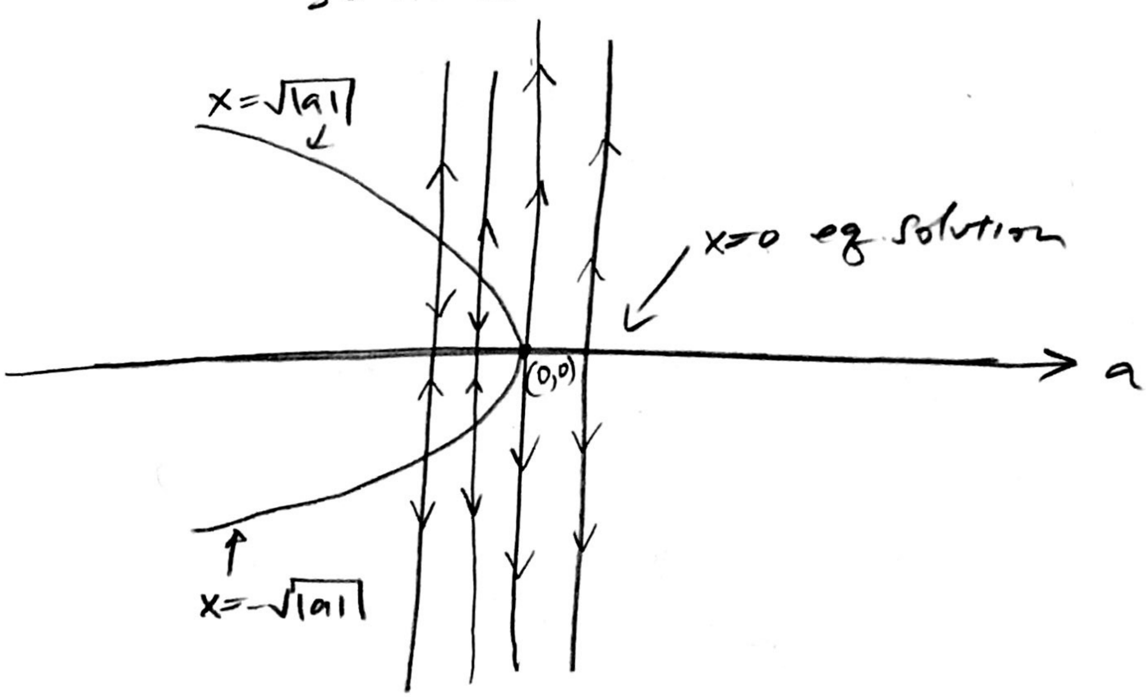
So $x = 0$ is the only equilibrium solution

$a = 0$ $\quad 0 = x^3$

So $x = 0$ is the only eq. solution

$a < 0$ $\quad 0 = x(a + x^2) = x(-|a| + x^2)$

$x = 0$, $x = \pm\sqrt{|a|}$ are the eq. solutions



7. Consider the linear system

$$X' = AX = \begin{pmatrix} a & b \\ b & a \end{pmatrix} X$$

where $a, b \in \mathbb{R}$.

a) 2 points Find the eigenvalues, $\lambda_\pm$ of A.

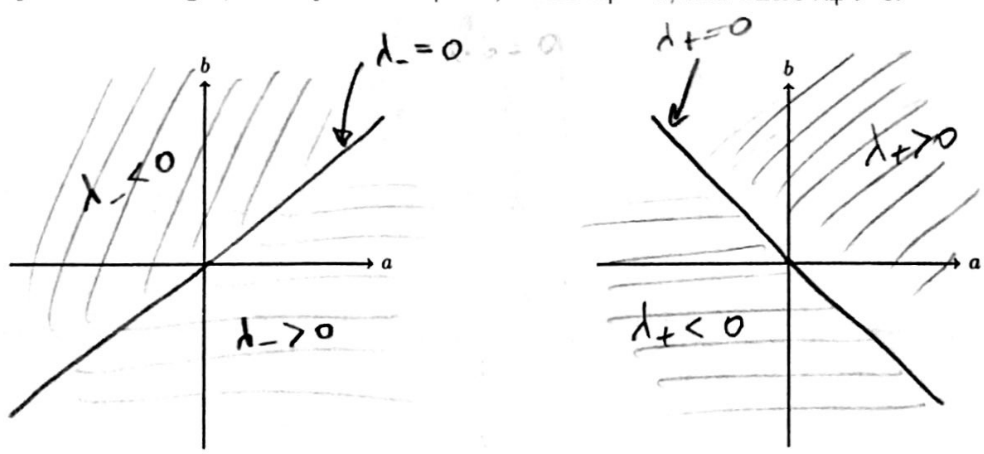
$$\det\begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix} = (a-\lambda)^2 - b^2 = 0$$

$$\Rightarrow (a-\lambda)^2 = b^2$$

$$a - \lambda = \pm b$$

$$\boxed{\lambda_\pm = a \pm b}$$

b) 4 points In the ab-plane to the left, identify where $\lambda_- < 0$, where $\lambda_- = 0$, and where $\lambda_- > 0$. In the ab-plane to the right, identify where $\lambda_+ < 0$, where $\lambda_+ = 0$, and where $\lambda_+ > 0$.



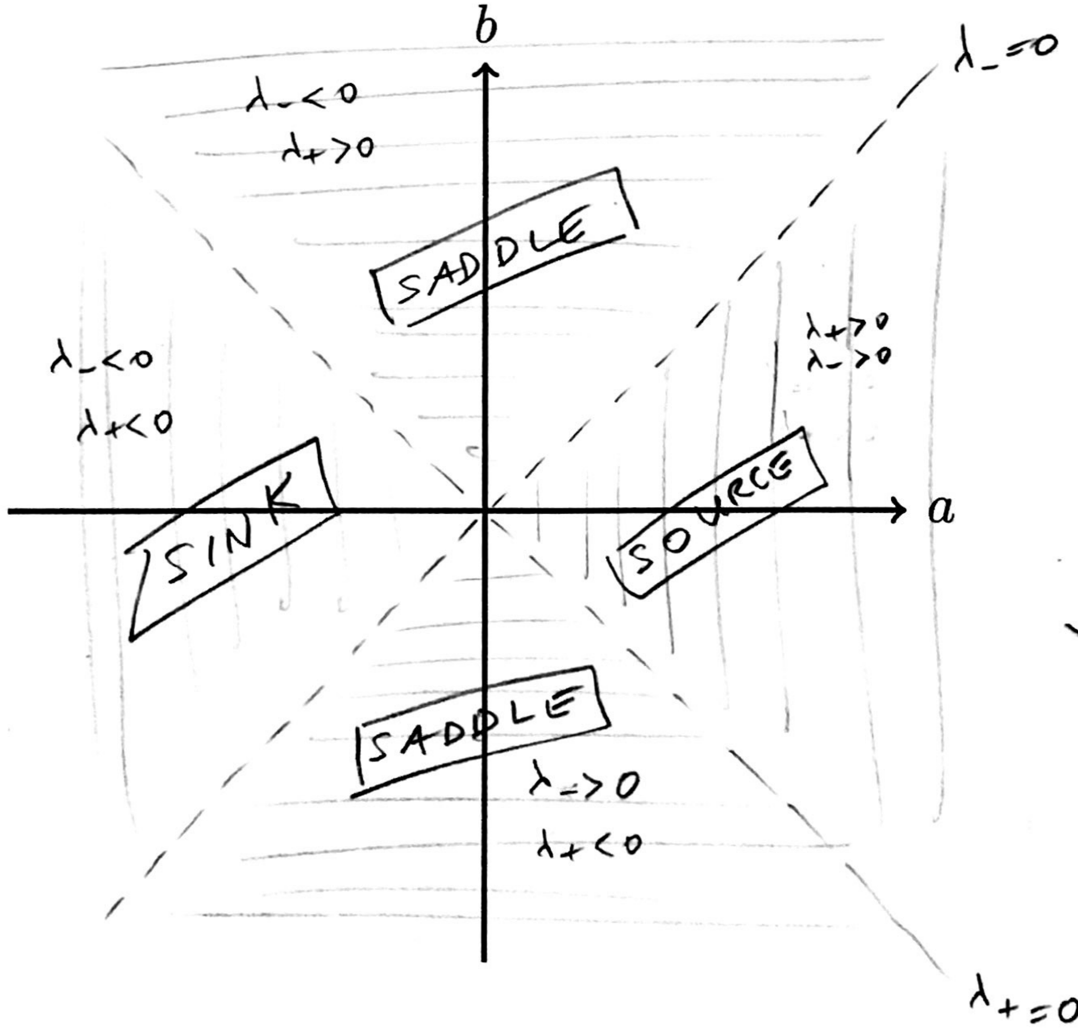
(a) properties of λ_- eigenvalue

(b) properties of λ_+ eigenvalue

$\lambda_- = a - b$

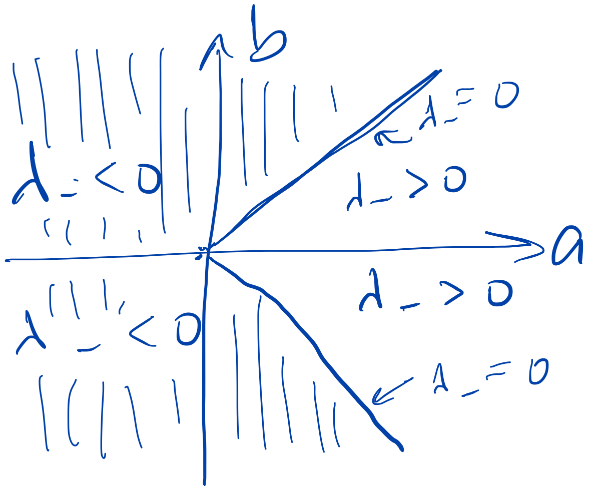
$\lambda_+ = a + b$

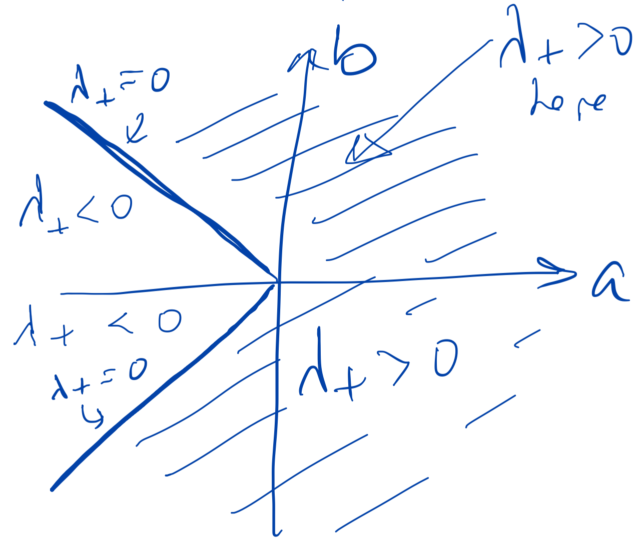
c) $\boxed{\text{4 points}}$ Informed by the work you just did, divide the ab-plane below into regions and label each region with the appropriate word from the zoology list on the aid sheet. (i.e. label with things like "spiral sink" and so forth.) *Your regions should be open sets. Don't worry about their boundaries (where $\lambda_- = 0$ or $\lambda_+ = 0$ or both).*



What if you labelled your eigenvalues differently? i.e.

$$\lambda_+ = a + |b| \quad \text{and} \quad \lambda_- = a - |b|$$





$$\lambda_- = a - |b|$$

$$\lambda_- = \begin{cases} a - b & \text{if } b \geq 0 \\ a + b & \text{if } b < 0 \end{cases}$$

$$\lambda_+ = a + |b|$$

$$\lambda_+ = \begin{cases} a + b & \text{if } b \geq 0 \\ a - b & \text{if } b < 0 \end{cases}$$