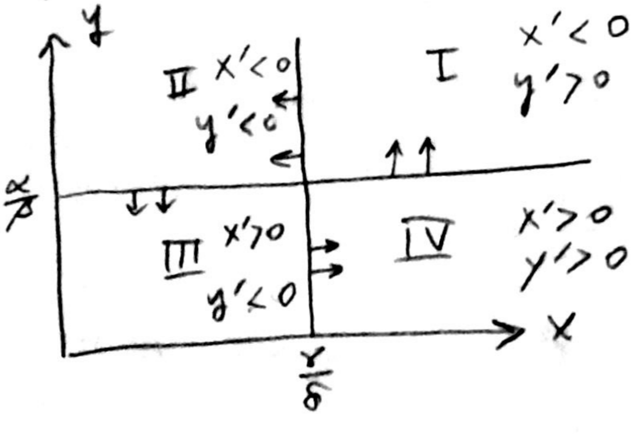
HW 7 Solutions

#1

$$\begin{cases} x' = \alpha x - \beta x y = x(\alpha - \beta y) \\ y' = -\gamma y + \delta x y = y(-\gamma + \delta x) \end{cases}$$



a) $A_x = \{(x,0) \mid x > 0\}$ is invariant under the flow? Let $(x_0, 0) \in A_x$. Then the solution is $X(t) = (x_0 e^{\alpha t}, 0) \in A_x \ \forall t \in \mathbb{R}$. By uniqueness of solutions this is the only solution through $(x_0, 0)$. Since $(x_0, 0)$ was an arbitrary point in A_x, this shows that the solution through any point in A_x stays in A_x. Also, as $t \to -\infty$ $X(t) \to (0,0)$ and as $t \to \infty$, $|X(t)| \to \infty$.

b) $A_y = \{(0,y) \mid y > 0\}$. Let $(0, y_0) \in A_y$. Then the solution is $X(t) = (0, y_0 e^{-\gamma t}) \in A_y$ $\forall t \in \mathbb{R}$. A_y is invariant by the same argument as for A_x. As $t \to \infty$, $X(t) \to (0,0)$ and as $t \to -\infty$, $|X(t)| \to \infty$.

c) Let $S_{I,II} = \{(\gamma/\delta, y) \mid y > \alpha/\beta\}$. Prove that if $X(t)$ is a solution and if $\exists$ t_0 so that $X(t_0) \in S_{I,II}$ then $\exists$ some $\tilde{\delta} > 0$ so that $X(t) \in$ quadrant I $\forall$ $t \in (t_0 - \tilde{\delta}, t_0)$ and $X(t) \in$ quadrant II $\forall t \in (t_0, t_0 + \tilde{\delta})$.

Fix $X_0 = (\gamma/\delta, y_0) \in S_{I,II}$. Choose $\varepsilon > 0$ so that $\overline{B_\varepsilon(X_0)}$ does not contain the equilibrium solution $(\gamma/\delta, \alpha/\beta)$

<u>lemma</u>: Assume F is C^1 on $\mathbb{R}^2$ and $F(X) \neq \vec{0}$ for every $X \in \overline{B_\varepsilon(X_0)}$. Then $\exists$ $\tau_1 > 0$ and $\tau_2 > 0$ so that the solution to the IVP $X' = F(X)$ and $X(t_0) = X_0$ satisfies $X(t) \in B_\varepsilon(X_0)$ for every $t \in (t_0 - \tau_1, t_0 + \tau_2)$ and $|X(t_0 - \tau_1) - X_0| = \varepsilon$ and $|X(t_0 + \tau_2) - X_0| = \varepsilon$. (That is, we can define "first exit times" going forward and backward in time.)

<u>corollary</u>: If ε is chosen small enough that $\overline{B_\varepsilon(X_0)} \subset (I \cup II)$ and $\tilde{\delta} := \min\{\tau_1, \tau_2\}$ then $X(t) \in I$ for all $t \in (t_0 - \tilde{\delta}, t_0)$ and $X(t) \in II$ for all $t \in (t_0, t_0 + \tilde{\delta})$.

<Once the lemma + corollary are proved, part 1c is done.>

Proof of lemma:

Let $(\tilde{\alpha}, \tilde{\beta})$ be the maximal interval of existence of the solution $X(t)$.

- If $\tilde{\beta} < \infty$ then by the theorem on page 398, the solution must leave $\overline{B_\varepsilon(X_0)}$. $\exists$ some t_1 with $t_0 < t_1 < \tilde{\beta}$ and $X(t_1) \notin \overline{B_\varepsilon(X_0)}$. Let

$$T_{forward} = \{ \tau \mid X(t) \in B_\varepsilon(X_0) \ \forall t \in [t_0, t_0 + \tau) \}$$

Define $\tau_2 = \sup\{T_{forward}\}$. By construction, $t_0 + \tau_2 < t_1 < \tilde{\beta}$ and so we must have $X(t_0 + \tau_2)$ on the boundary of $B_\varepsilon(X_0)$.

- If $\tilde{\beta} = \infty$ then there are two possibilities. Either $X(t) \in \overline{B_\varepsilon(X_0)}$ $\forall$ $t \in [t_0, \infty)$ or $\exists$ some $t_1 > t_0$ that $X(t_1) \notin \overline{B_\varepsilon(X_0)}$ and we can construct a "first exit time" $t_0 + \tau_2$ as above.

Assume $X(t) \in \overline{B_\varepsilon(X_0)}$ $\forall$ $t \in [t_0, \infty)$. Let $\{t_n\}$ be a sequence with $t_n \geq t_0$ $\forall n$ and $t_n \to \infty$. $\overline{B_\varepsilon(X_0)}$ is compact and so there's a convergent subsequence of $\{X(t_n)\}$. So $X(t_{n_k}) \to X^*$ where $X^* \in \overline{B_\varepsilon(X_0)}$. The sequence $\{X(t_{n_k})\}$ is Cauchy and given $\hat{\varepsilon} > 0$ $\exists$ $K_{\hat{\varepsilon}}$ so that $K \geq K_{\hat{\varepsilon}} \Rightarrow |X(t_{n_k}) - X(t_{n_\ell})| < \hat{\varepsilon}$ for $k, \ell \geq K_{\hat{\varepsilon}}$.

By the MVT $\exists$ c between t_{n_k} and t_{n_ℓ} so that $|X'(c)|\,|t_{n_k} - t_{n_\ell}| < \hat{\varepsilon}$. That is,

$|F(X(c))|\,|t_{n_k}-t_{n_\ell}|<\hat{\varepsilon}$ for some c between t_{n_k} & t_{n_ℓ}.

We know F is continuous on $\overline{B_\varepsilon(X_0)}$ and that F never vanishes on this compact set. $\Rightarrow$ $\exists$ some lower bound m so that $m>0$ and

$$|F(X)| \geq m \quad \forall\, X \in \overline{B_\varepsilon(X_0)}.$$

Therefore $m\,|t_{n_k}-t_{n_\ell}| \leq |F(X(c))|\,|t_{n_k}-t_{n_\ell}| < \hat{\varepsilon}$ for all $k,\ell \geq K_{\hat{\varepsilon}}$. This is impossible (just fix $k \geq K_{\hat{\varepsilon}}$ and take ℓ to infinity.)

This finishes the proof that there is a first exit time $t_0+\tau_2$. To construct the first exit time (in reverse time) $t_0-\tau_1$ repeat the arguments, mutatis mutandis, with $\vec{\alpha}$ instead of $\vec{\beta}$. //

Proof of Corollary

$X_0=(\gamma/\delta, y)$ where $y>\alpha/\beta$. Therefore $F(X_0)=\begin{pmatrix}\text{neg }\#\\0\end{pmatrix}$ as a result, for $t>t_0$, t close to t_0, $X(t)$ has first component less than γ/δ. That is for $t>t_0$, t close to t_0, $X(t)$ is in the left hemisphere of $B_\varepsilon(X_0)$. $X(t)$ cannot exit the hemisphere through the line segment $\{x=\gamma/\delta\}\cap\overline{B_\varepsilon(X_0)}$

because $F(X)$ has negative first component on that line segment. Therefore $X(t) \in$ left hemisphere for <u>all</u> $t \in (t_0, t_0+\tau_2)$, and not just for t close to t_0. Therefore $X(t) \in \text{II}$ for all t in $(t_0, t_0+\tau_2)$.

Arguing similarly for times before t_0, but close to t_0, we get $X(t) \in$ right hemisphere for all t in $(t_0-\tau_1, t_0)$. Therefore $X(t) \in \text{I}$ for all t in $(t_0-\tau_1, t_0)$. The desired result follows since $\tau_1 \geq \tilde{\delta}$ and $\tau_2 \geq \tilde{\delta}$. //

1d) $S_{\text{II},\text{III}} = \{(x, \alpha/\beta) \mid 0 < x < \gamma/\delta\}$. Prove that if $X(t_0) \in S_{\text{II},\text{III}}$ then $\exists\, \tilde{\delta} > 0$ so that $X(t) \in \text{II}$ $\forall t \in (t_0-\tilde{\delta}, t_0)$ and $X(t) \in \text{III}$ $\forall t \in (t_0+\tilde{\delta})$.

- Choose ε so that $\overline{B_\varepsilon(X_0)}$ excluded both equilibrium solutions $(0,0)$ and $(\gamma/\delta, \alpha/\beta)$ and so that $\overline{B_\varepsilon(X_0)} \subset (\text{II} \cup \text{III})$.

  By lemma in 1c, $\exists\, \tau_1$ and τ_2 so that $t_0-\tau_1$ and $t_0+\tau_2$ are the first exit times. Let $\tilde{\delta} = \min\{\tau_1, \tau_2\}$ and prove that $X(t) \in \text{II}$ for all $t \in (t_0-\tilde{\delta}, t_0)$ and $X(t) \in \text{III}$ for all $t \in (t_0, t_0+\tilde{\delta})$ by looking at the upper and lower halves of $\overline{B_\varepsilon(X_0)}$.

1e) and 1f) are done similarly. ☺

1g) Let $X(t)$ be a solution w/ maximal interval of existence $(\tilde{\alpha}, \tilde{\beta})$. Assume $\exists\, t_0 \in (\tilde{\alpha}, \tilde{\beta})$ so that $X(t_0) \in$ quadrant I. Prove that either $\tilde{\beta} < \infty$ and $y(t) \to \infty$ or the solution crosses over into quadrant II at some time $t_1 > t_0$.

proof: Assume $\tilde{\beta} = \infty$. As long as $X(t)$ is in quadrant I, it has $y'(t) > 0$. $\Rightarrow$
As long as $X(t)$ is in quadrant I, $X(t) = (x(t), y(t))$ where $y(t) \geq y(t_0)$ if $t \geq t_0$ and $X(s) \in I$ for all $s \in [t_0, t)$. < That is as long as $X(t)$ hasn't exited I yet, we know that $y(t) \geq y(t_0)$. > Therefore, as long as $X(t)$ hasn't exited I yet, we know

$\alpha - \beta y(t) \leq \alpha - \beta y(t_0) \Rightarrow x(t)(\alpha - \beta y(t))$ is less than or equal to $x(t)(\alpha - \beta y(t_0))$

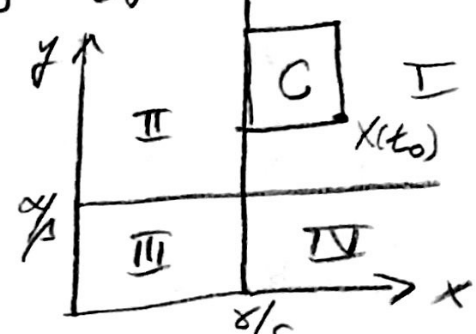
And so $x'(t) \leq \underbrace{(\alpha - \beta y(t_0))}_{\text{negative}} x(t)$

By Grönwall, as long as $X(t)$ hasn't exited I yet, $x(t) \leq x_0 e^{(\alpha - \beta y(t_0))(t-t_0)}$.

This forces $x(t)$ to reach $x = \gamma/\delta$ in finite time ... $X(t)$ must exit I by time t^* where $x_0 e^{(\alpha-\beta y(t_0))(t^*-t_0)} = \gamma/\delta$.

The above is what happens if $\tilde{\beta} = \infty$.

- Assume $\tilde{\beta} < \infty$ and that $\exists$ some upper bound M on $X(t)$ as long as $X(t)$ is still in the I quadrant. <That is if τ_I is the maximal time so that $X(t) \in I$ for all $t \in (t_0, t_0 + \tau_I)$ then $y(t) \leq M$ for all $t \in (t_0, t_0+\tau_I)$.> We know $X(t)$ leaves every compact set at some time because $\tilde{\beta} < \infty$. Let $C = [\gamma/\delta, x(t_0)] \times [y(t_0), M]$.

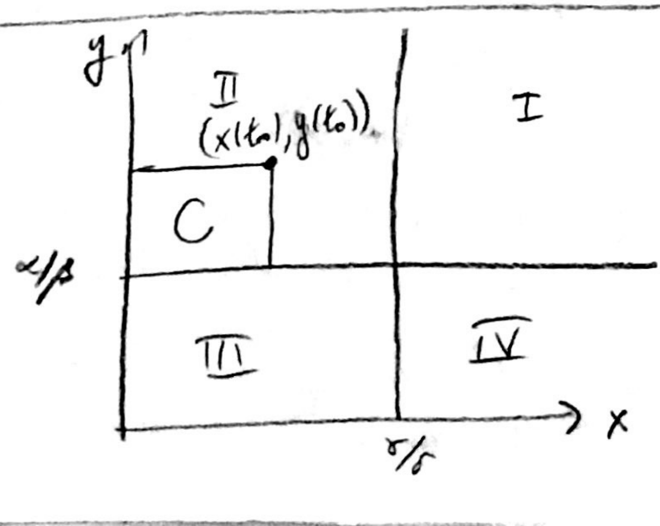


$X(t)$ cannot leave through the R side because $x' < 0$ in I. It cannot leave through the bottom because $y' > 0$ in I. It cannot leave through the top because of the upper bound M. So $X(t)$ must leave through the left side and enter quadrant II in finite time.

- Assume $\tilde{\beta} < \infty$ and there is no uniform upper bound on $X(t)$ and that $X(t)$ never enters into the quadrant II. Need to show that $y(t) \uparrow \infty$ as $t \uparrow \tilde{\beta}$.

For each n, let C_n be the rectangle $[\gamma/\delta, x(t_0)] \times [y(t_0), n]$. Because $\tilde{\beta} < \infty$ $\exists$ some time t_n so that $X(t_n) \notin C_n$. As before $X(t)$ cannot exit C_n through the right side or through the bottom. By assumption, it doesn't exit through the left side because we assumed there's no time at which the solution enters into the quadrant II. Therefore, $X(t)$ must have exited C_n through the top: $y(t_n) \geq n$. And $y'(t) > 0$ for all $t \in (t_0, \tilde{\beta})$ So we know $y(t) \geq y(t_n) \geq n$ For all $t \geq t_n$. It follows that $y(t) \uparrow \infty$ as $t \uparrow \tilde{\beta}$.

1h.



Let $X(t)$ be a solution and assume $\exists\, t_0$ so that $X(t_0) \in \mathrm{II}$. Prove that the solution crosses into region III at some t_1 where $t_1 > t_0$.

Consider the compact set $C = [0, x(t_0)] \times [\alpha/\beta, y(t_0)]$. Let $(\tilde{\alpha}, \tilde{\beta})$ be the maximal interval of existence.

- If $\tilde{\beta} < \infty$ then $\exists$ some time $\hat{t} > t_0$ so that $X(\hat{t}) \notin C$. Let $t_1 \le \hat{t}$ be the first exit time $t_1 = \sup\{\tau \mid X(t) \in C$ for all $t \in [t_0, \tau]\}$. $X(t) \in \mathrm{II}$ for all $t \in [t_0, t_1)$ and so $y'(t) < 0$ for all $t \in [t_0, t_1) \Rightarrow X(t)$ can't exit through the top of C. Similarly, $x'(t) < 0$ for all $t \in [t_0, t_1) \Rightarrow X(t)$ can't exit through the right side of C. $X(t)$ can't exit through the left side of C without violating uniqueness of solutions (see part 1b) and so $X(t)$ must exit through the bottom of C and then enter into quadrant III (see part 1d).
- If $\beta = \infty$ then use Grönwall. $x'(t) < 0 \Rightarrow x(t) < x(t_0)$ as long as $X(t) \in \mathrm{II}$

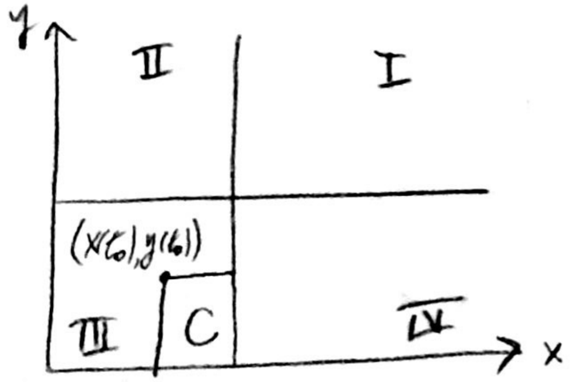
$$y'(t) \le y(t)\,[-\gamma + \delta\, x(t_0)]$$ as long as $X(t) \in \mathrm{II}$

and so $y(t) \leq y(t_0)\, e^{[-\gamma + \delta x(t_0)](t-t_0)}$ as long as $X(t) \in \mathrm{II}$

$\Rightarrow X(t)$ must exit II by time t^* where

$$y(t_0)\, e^{[-\gamma + \delta x(t_0)](t^*-t_0)} = \alpha/\beta$$

1i) Assume $X(t_0) \in \mathrm{III}$. Show that $\exists\, t_1 > t_0$ so that $X(t)$ crosses into region IV at time t_1.

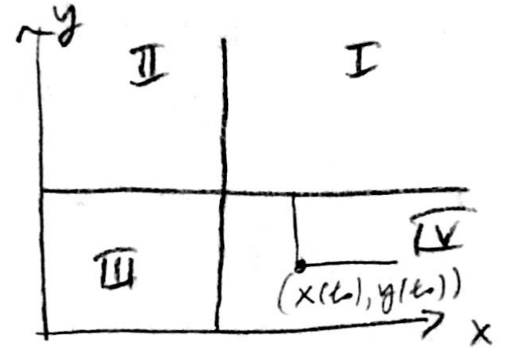


Consider the compact set

$$C = [x(t_0), \gamma/\delta] \times [0, y(t_0)]$$

Let $(\tilde{\alpha}, \tilde{\beta})$ be the maximal interval of existence of the solution $X(t)$. As for 1h, if $\tilde{\beta} < \infty$ then $X(t)$ must exit the set C, and so there's a first exit time t_1. $\Rightarrow X(t) \in \mathrm{III}$ for all $t \in [t_0, t_1]$. The solution can't exit through the top of C because $y' < 0$ in III. It can't exit through the left side of C because $x' > 0$ in III. It can't exit through the bottom of C without violating uniqueness of solutions. Therefore $X(t_1) \in S_{\mathrm{III},\mathrm{IV}}$ and the solution is in IV for times slightly larger than t_1 (by problem 1e.)

1j) Assume $X(t) = (x(t), y(t))$ is a solution w/ maximal interval of existence $(\tilde{\alpha}, \tilde{\beta})$. Assume $\exists\, t_0$ so that $X(t_0) \in \text{IV}$. Show that either $\tilde{\beta} < \infty$ and $x(t) \to \infty$ or the solution crosses into the region I in finite time. t_1 where $t_1 > t_0$.



Case 1: $\tilde{\beta} = \infty$.

Let $t_1 = \sup \{ \tau \mid X(t) \in \text{IV} \text{ for all } t \in [t_0, \tau) \}$.
For all $t \in [t_0, t_1)$ we have $X(t) \in \text{IV}$
and therefore $x'(t) > 0$ and $y'(t) > 0$. So
for all $t \in [t_0, t_1)$ we have $x(t) \geq x(t_0)$.
Hence $\forall t \in [t_0, t_1)$ we have $-\gamma + \delta x(t) \geq -\gamma + \delta x(t_0)$.
And so for all $t \in [t_0, t_1)$ we have

$$y'(t) = y(t)(-\gamma + \delta x(t)) \geq y(t)[-\gamma + \delta x(t_0)]$$

By Grönwall's inequality, for all $t \in [t_0, t_1)$ we have

$$y(t) \geq y(t_0) e^{[-\gamma + \delta x(t_0)](t - t_0)}$$

Can $t_1 = \infty$? No because the lower bound on $y(t)$ forces $X(t)$ into I in finite time. So t_1 is finite and the the solution reaches $S_{\text{IV},\text{I}}$ in finite time (recall $x' > 0$ and so the y-axis line is crossed to the right of $x = \gamma/\delta$).

Case 2: $\tilde{\beta} < \infty$. We have to worry about whether the solution could have $x(t) \to \infty$ before $y(t)$ has succeeded increasing to the value α/β (and the solution crosses into I).

Define t_1 as before $t_1 = \sup\{\tau \mid X(t) \in \text{IV} \ \forall t \in [t_0, \tau)\}$
If $t_1 < \tilde{\beta}$ then we're done by the previous argument: the solution reaches $S_{\text{IV},\text{I}}$ and crosses into region I before it stops existing.

If $t_1 = \tilde{\beta}$ then this means that the solution never leaves I before it stops existing
All we need to do is prove this means that $x(t) \to \infty$ as $t \uparrow \tilde{\beta}$.

Let $C_M = [x(t_0), M] \times [y(t_0), \alpha/\beta]$. for $M \in \mathbb{N}$, $M > x(t_0)$
Because $\tilde{\beta} < \infty$ we know $\exists$ a time t_M so that $X(t) \notin C_M$.
We know $X(t) \in \text{IV}$ $\forall t \in [t_0, \tilde{\beta})$ by assumption and so $x' > 0$ and $y' > 0$ $\Rightarrow$ cannot exit C_M through the bottom or left side. By assumption, the solution isn't exiting through the top of C_M. And so we must have exited C_M through the right side: $x(t_M) \geq M$.
And $x'(t) > 0$ for $\forall t \in [t_0, \tilde{\beta})$ $\Rightarrow$ $x(t) \geq x(t_M)$ for all $t \geq t_M$. Taking $M \to \infty$, we have $x(t) \to \infty$ as $t \to \tilde{\beta}$. //

1k) $\frac{dy}{dx} = \frac{-\gamma y + \delta x y}{\alpha x - \beta x y} = \frac{y(-\gamma + \delta x)}{x(\alpha - \beta y)}$

separable ODE

$$\frac{\alpha - \beta y}{y} \frac{dy}{dx} = \frac{-\gamma + \delta x}{x}$$

$$\left(\frac{\alpha}{y} - \beta\right)\frac{dy}{dx} = -\frac{\gamma}{x} + \delta$$

$$\frac{d}{dx}\left[\alpha \ln(y(x)) - \beta y(x)\right] = \frac{d}{dx}\left[-\gamma \ln(x) + \delta x + C\right]$$

$$\alpha \ln(y(x)) - \beta y(x) = -\gamma \ln(x) + \delta x + C$$

$$\mathcal{L}(x,y) := -\alpha \ln(y) + \beta y - \gamma \ln(x) + \delta x$$

1l.) Assume $x(t), y(t)$ is a solution of the Lotka-Volterra system

$$\frac{d}{dt}\mathcal{L}(x(t), y(t)) = \nabla \mathcal{L}(x(t), y(t)) \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

$$= \begin{pmatrix} -\gamma/x(t) + \delta \\ -\alpha/y(t) + \beta \end{pmatrix} \cdot \begin{pmatrix} x(t)(\alpha - \beta y(t)) \\ y(t)(-\gamma + \delta x(t)) \end{pmatrix}$$

$$= (-\gamma + \delta x(t))(\alpha - \beta y(t)) + (-\alpha + \beta y(t))(-\gamma + \delta x(t)) = 0$$

So $\mathcal{L}(x,y)$ is conserved along solutions.

For it to be a Liapunov function we need an open set U upon which $\mathcal{L}(X) > 0$ except at $X = X^*$ and $\mathcal{L}(X^*) = 0$ and X^* is an equilibrium solution we know $\mathcal{L}(x,y) \to \infty$ if $x \to 0$ or $y \to 0$.

Similarly, $\mathcal{L}(x,y) \to \infty$ if $x \to \infty$ or $y \to \infty$.

compute the Hessian of $\mathcal{L}$.

$$H = \begin{pmatrix} \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{pmatrix} = \begin{pmatrix} \gamma/x^2 & 0 \\ 0 & \alpha/y^2 \end{pmatrix}$$

Because $\gamma > 0$ and $\alpha > 0$, the Hessian is positive definite in $(0,\infty)\times(0,\infty)$ (the 1st quadrant) and so $\mathcal{L}(x,y)$ is a Liapunov function if we add $\mathcal{L}(\gamma/\delta, \alpha/\beta)$ to it.

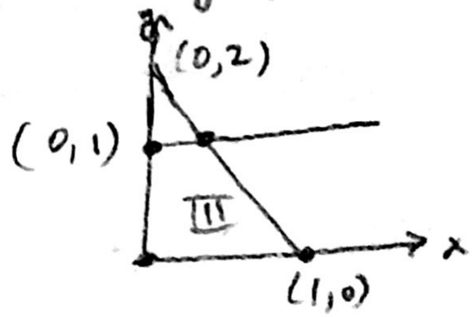
$$\mathcal{L} = -\alpha \ln(y) + \beta y - \gamma \ln(x) + \delta x - \alpha \ln(\alpha/\beta) + \beta(\alpha/\beta) - \gamma \ln(\gamma/\delta) + \delta(\gamma/\delta)$$

$$\boxed{\mathcal{L}(x,y) = -\alpha \ln(y) + \beta y - \gamma \ln(x) + \delta x - \alpha \ln(\alpha/\beta) + \alpha - \gamma \ln(\gamma/\delta) + \delta}$$

The level sets of $\mathcal{L}$ in the first quadrant are bounded closed curves. Therefore if $X(t)$ is a solution then $X(t)$ is contained in the compact set $\mathcal{L}(x,y) = \mathcal{L}(X(t_0))$. Hence the interval of existence is $(-\infty, \infty)$. The solution moves counterclockwise in the level set, crossing from I to II to III to IV in finite time. So $\exists\ T < \infty$ so that $X(t_0) = X(t_0 + T)$. Let T be the first such nonzero T. This is the period of the solution.

#2. Consider $\begin{cases} x' = x(y+2x-2) \\ y' = y(y-1) \end{cases}$

Let $\mathrm{III} = \{(x,y) \mid 0 < x < 1 - y/2,\ 0 < y < 1\}$



Prove that if $X_0 \in \mathrm{III}$ then the solution of the IVP w/ $X(t_0) = X_0$ has maximal interval of existence (α, ∞), where $\alpha < t_0$, and that $X(t) \in \mathrm{III}$ for all $t \geq t_0$, and that $X(t) \to \vec{0}$ as $t \to \infty$.

Left side of III $\{(0,y) \mid 0 < y < 1\}$ No solution can cross this side because $\exists$ a solution $X(t) = (0, y(t))$ so that $X(t) \to (0,1)$ as $t \to -\infty$ and $X(t) \to (0,0)$ as $t \to +\infty$. There's no "opening for a solution* to cross through the side and it can't cross through any of the 4 corners because they're all equilibrium solutions.

$y(t) = \dfrac{1}{1+e^t}$

* that starts in III

bottom of III $\{(x,0) \mid 0<x<1\}$.

$\exists$ solution $X(t)=(x(t),0)$ with $X(t)\to(0,0)$ as $t\to+\infty$ and $X(t)\to(1,0)$ as $t\to-\infty$ and so it's impossible for a solution that starts in III to cross through the bottom of III

$$x(t)=\frac{1}{1+e^{2t}}$$

not relevant but too lazy to erase

top of III $\{(x,1) \mid 0<x<1/2\}$

$\exists$ solution $X(t)=(x(t),0)$ with $X(t)\to(0,1)$ as $t\to\infty$ and $X(t)\to(1/2,1)$ as $t\to-\infty$. So it's impossible for a solution that starts in III to cross through the top of III

$$x(t)=\frac{1}{2+2e^{t}}$$

Okay, let's look at the solution that starts in III : $X(t_0)=(x(t_0),y(t_0))$. Define

$$C=(0,x(t_0))\times(0,y(t_0))$$

if $X\in C$ then $F(X)=\begin{pmatrix}\text{neg}\\ \text{neg}\end{pmatrix}$

It follows immediately that if (α,β) is the maximal interval of existence then for $t\in[t_0,\beta)$ $X(t)\in C$. Because it can't cross any of the sides or corners of C.

Hence $X(t) \in \overline{C}$ for all $t \in [t_0, \beta)$ where $\overline{C}$ is a compact set. $\Rightarrow \beta = \infty$ and the solution exists for all $t > t_0$ and is in C (hence in III) for all $t > t_0$.

Now to show that $X(t) \to \vec{0}$ as $t \to \infty$. Let $\{t_n\}$ be a sequence of times with $t_n \to \infty$. Then $\exists$ a subsequence $\{t_{n_k}\}$ so that $X(t_{n_k}) \to X^*$ where $X^* \in \overline{C}$. Assume $X^* \neq \vec{0}$. Choose $\hat{\varepsilon} > 0$ sufficiently small so that $\overline{B_{\hat{\varepsilon}}(X^*)}$ excludes all four equilibrium solutions: $(0,0), (0,1), (1,0)$, and $(1, 1/2)$. It follows that the vector field $\vec{F}$ is nonzero on $\overline{B_{\hat{\varepsilon}}(X^*)}$ and $\exists\, m > 0$ so that $|\vec{F}(X)| \geq m$ for all $X \in \overline{B_{\hat{\varepsilon}}(X^*)}$.

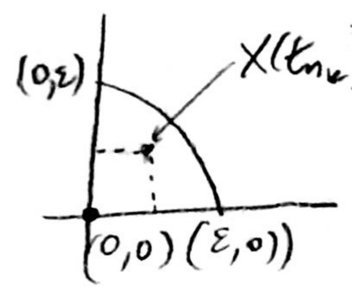
Because $X(t_{n_k}) \to X^*$, $\exists\, K_{\hat{\varepsilon}}$ so that $X(t_{n_k}) \in B_{\hat{\varepsilon}}(X^*)$ for all $k \geq K_{\varepsilon}$. The sequence $\{X(t_{n_k})\}$ is Cauchy and so if $k, l \geq K_{\varepsilon}$ then $|X(t_{n_k}) - X(t_{n_l})| < 2\hat{\varepsilon}$.

By the MVT for any pair t_{n_k}, t_{n_l}

$$2\hat{\varepsilon} > |X(t_{n_k}) - X(t_{n_l})| = |t_{n_k} - t_{n_l}|\,|X'(c)| \geq m\,|t_{n_k} - t_{n_l}|$$

where c is a point on the segment connecting $X(t_{n_k})$ to $X(t_{n_l})$. This shows that for all $k, l \geq K_{\hat{\varepsilon}}$ we have $|t_{n_k} - t_{n_l}| \leq \frac{2\hat{\varepsilon}}{m}$ which is impossible.

This proves that if X^* is a limit point of $\{X(t_{n_k})\}$ then $X^* = \vec{0}$. It remains to show that $X(t) \to \vec{0}$ as $t \to \infty$. Let $\varepsilon > 0$. Then $\exists K_\varepsilon$ so that if $k \geq K_\varepsilon$ then $X(t_{n_k}) \in B_\varepsilon(\vec{0})$.



Let $C_k = (0, x(t_{n_k})) \times (0, y(t_{n_k}))$. $C_k \subset \text{III}$ and so $F(X) = \begin{pmatrix} \text{neg} \\ \text{neg} \end{pmatrix}$ for every $X \in C_k$. It follows that $X(t) \in C_k$ for all $t > t_{n_k}$. And $C_k \subset B_\varepsilon(\vec{0})$ hence $X(t) \in B_\varepsilon(\vec{0})$ $\forall t \geq t_{n_k}$, proving $X(t) \to \vec{0}$ as $t \to \infty$.

#3)

a) <u>Prove $\exists$ a sequence $\{t_n\}$ w/ $t_n \to \infty$ so that $X(t_n) \to Z_0$ where Z_0 is some point in $B_\delta(X^*)$</u>

From the proof $X(t)$ is a solution that starts in $U - X^*$ where $U = B_\delta(X^*) \cap \{X \mid L(X) < \alpha\}$. $X_0 = X(0) \in U \subset \overline{U}$. The solution through X_0 has maximal interval of existence (α, β). As t increases $X(t)$ cannot leave U because $L(X_0) < \alpha$ and leaving U would require leaving $\{L < \alpha\}$ or leaving $B_\delta(X^*)$. The

$X(t)$ can't leave $\{L < \alpha\}$ as t increases because $L(X(t)) \le L(X_0) < \alpha$ for all $t \ge 0$.

$X(t)$ can't leave $B_\delta(X^*)$ because $L(X) \ge \alpha$ on $\partial B_\delta(X^*)$ and so $L(X(t))$ can't reach $\partial B_\delta(X^*)$ for any $t > 0$. Because $X(t) \in U \subseteq \overline{U}$ for all $t > 0$ and $\overline{U}$ is compact, the maximal interval of existence is (α, ∞) and any sequence $\{t_n\}$ with $t_n \to \infty$ will have a convergent subsequence. This proves there exists a sequence $\{X(t_{n_k})\} \subset \overline{U}$ that converges to a point $Z_0 \in \overline{U}$. Because L is continuous $L(X(t_{n_k})) \to L(Z_0)$.
$\Rightarrow L(Z_0) \le L(X_0) < \alpha \Rightarrow Z_0 \in U$ as desired. //

3b) Prove that if $X(t_n) \to X^*$ as $t_n \to \infty$ then $X(t) \to X^*$ as $t \to \infty$.

From the proof of the Liapunov theorem in the book, the only possible limit of a sequence $\{X(t_n)\}$ is X^*. We want to show that $X(t) \to X^*$ as $t \to \infty$. Assume not. Then $\exists\, \hat{\varepsilon} > 0$ so that for every $N \in \mathbb{N}$ $\exists\, t_n > N$ so that $X(t_n) \notin B_{\hat{\varepsilon}}(X^*)$. Construct an increasing sequence of times $\{t_n\}$ using this so that $t_n \to \infty$ and $X(t_n) \notin B_{\hat{\varepsilon}}(X^*)$. By the argument in part a) this sequence has a convergent subsequence which converges to some $z_0 \in U$, $z_0 \neq x^*$. But this contradicts x^* being the only possible limit of a sequence of $\{x(t_n)\}$.

4)

$$\begin{cases} x' = y \\ y' = -x - 3\varepsilon x^2 = -x(1+3\varepsilon x) \end{cases}$$

fixed points $(0,0)$ and $(-1/3\varepsilon, 0)$

$$DF(x,y) = \begin{pmatrix} 0 & 1 \\ -1-6\varepsilon x & 0 \end{pmatrix}$$

$$DF(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

eigenvalues are $\pm i$

behavior of linearized system is 

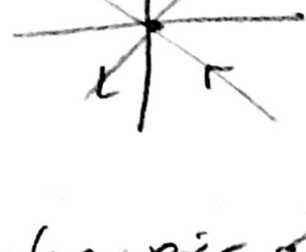
$$DF(-\tfrac{1}{3\varepsilon}, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

eigenvalues are ± 1

$+1$ has eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

-1 has eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

behavior of linearized system is



cannot predict behavior of nonlinear system near $(0,0)$ because it's not a hyperbolic eq. solution.

4b) $L(x,y) = y^2 + x^2 + 2\varepsilon x^3$

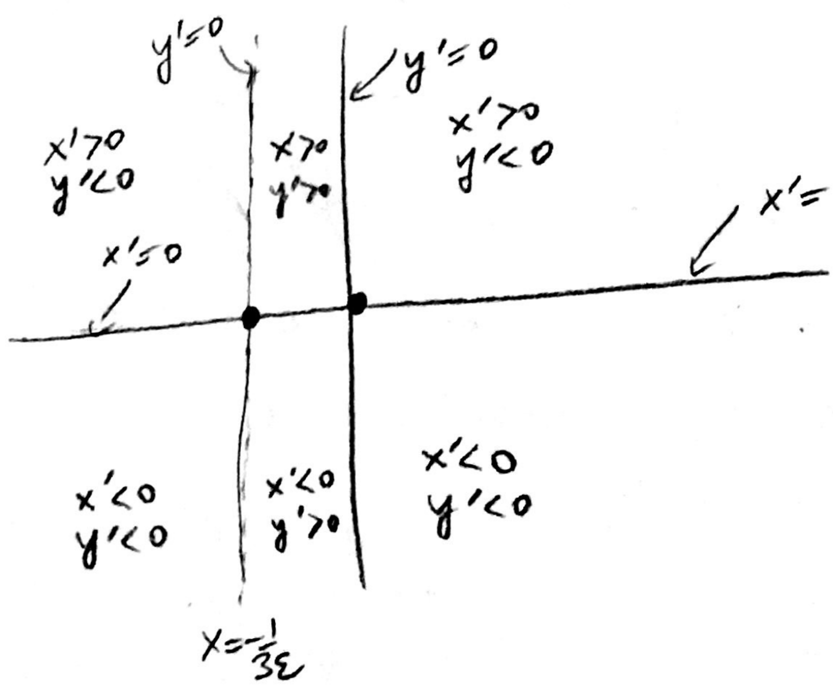
$$\Rightarrow \nabla L = \begin{pmatrix} 2x + 6\varepsilon x^2 \\ 2y \end{pmatrix}$$

$$\nabla L \cdot F = (2x + 6\varepsilon x^2)y + 2y(-x - 3\varepsilon x^2)$$
$$= (x + 3\varepsilon x^2)2y - 2y(x + 3\varepsilon x^2) \overset{\checkmark}{=} 0$$

Therefore if $X(t)$ is a solution of the system then $L(X(t)) = L(X_0)$ for all $t \in (\alpha, \beta)$ ← maximal interval of existence

4c) Nullclines of the system?

$x' = 0 \Leftrightarrow y = 0$ $\qquad$ $y' = 0 \Leftrightarrow x = 0$ or $1 + 3\varepsilon x = 0$



Level sets of the system?

$L(0,0)=0$
$L(\frac{-1}{3\varepsilon},0)=\frac{1}{27\varepsilon^2}$

$L(x,y)=y^2+x^2+2\varepsilon x^3=C$

given C seek solutions of $\nearrow$ $y^2=-2\varepsilon x^3-x^2+C$

if RHS >0 then $\exists$ 2 values of y for that RHS.

For what values of x is the RHS >0?

$RHS(x)=-2\varepsilon x^3-x^2+C$

$\frac{d}{dx}RHS(x)=-6\varepsilon x^2-2x=0 \iff x=0$ or $x=-\frac{1}{3\varepsilon}$

$\frac{d^2}{dx^2}RHS(x)=-12\varepsilon x-2$ $\quad RHS''(0)=-2<0$

$RHS''(-\frac{1}{3\varepsilon})=2>0$

RHS(x) has local max at $x=0$
local min at $x=-1/3\varepsilon$

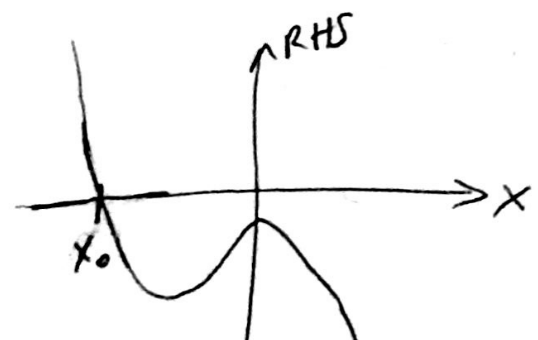
$RHS(0)=C$

$RHS(\frac{-1}{3\varepsilon})=C-\frac{1}{27\varepsilon^2}$

So we have to consider five regimes for C:

$C<0,\ C=0,\ 0<C<\frac{1}{27\varepsilon^2},\ C=\frac{1}{27\varepsilon^2},\ C>\frac{1}{27\varepsilon^2}$

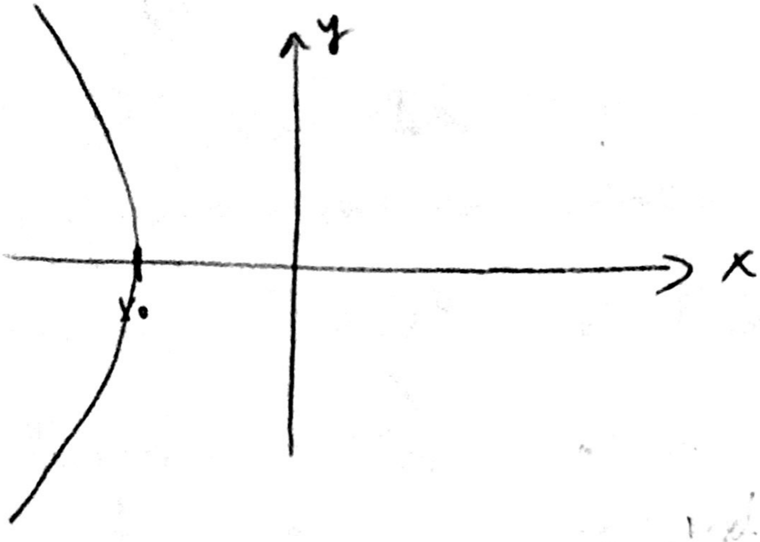
$C<0$



if $x<x_0$ then there are two points (x,y_+) and (x,y_-) so that

$L(x,y_\pm)=C$

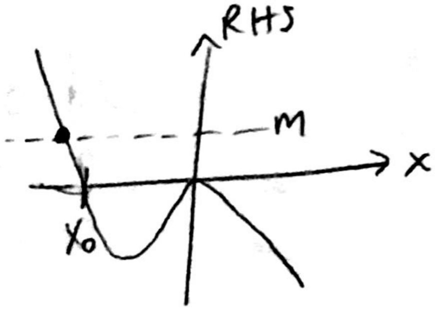
$y_\pm=\pm\sqrt{C-2\varepsilon x^3-x^2}$

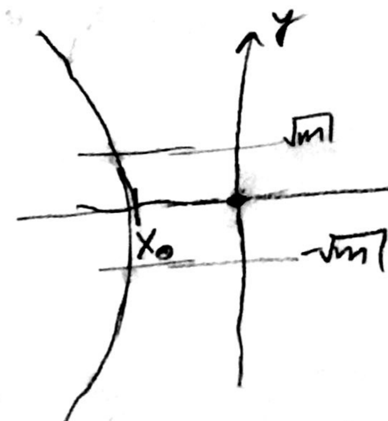


1 level set if $C<0$

for $x \ll 1$,
$y_{\pm} \sim \pm\sqrt{2\varepsilon}\,|x|^{3/2}$

$C=0$

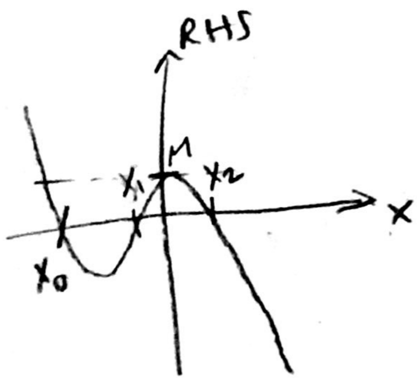


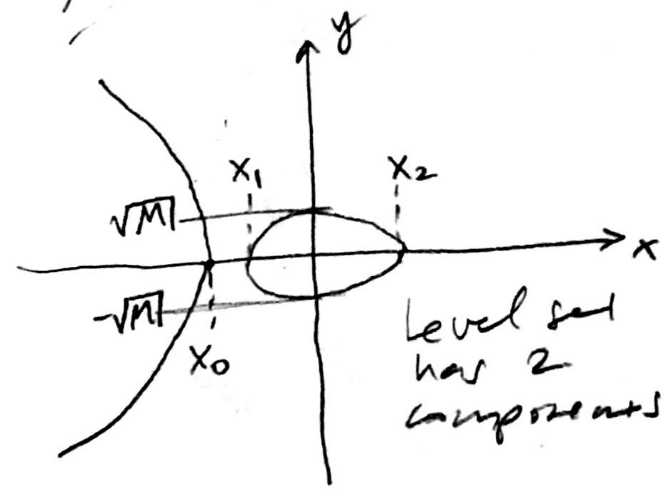


Level set has 2 components

$\boxed{L(0,0)}$

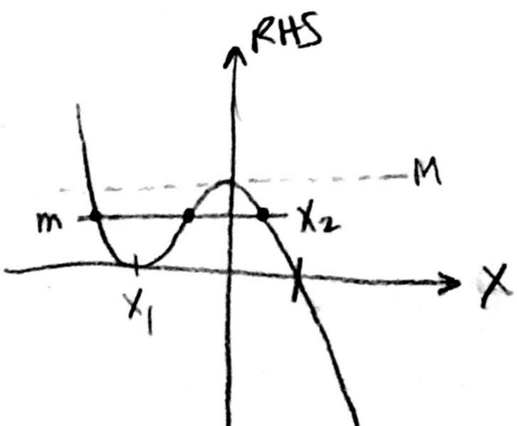
$0 < C < \frac{1}{27\varepsilon^2}$

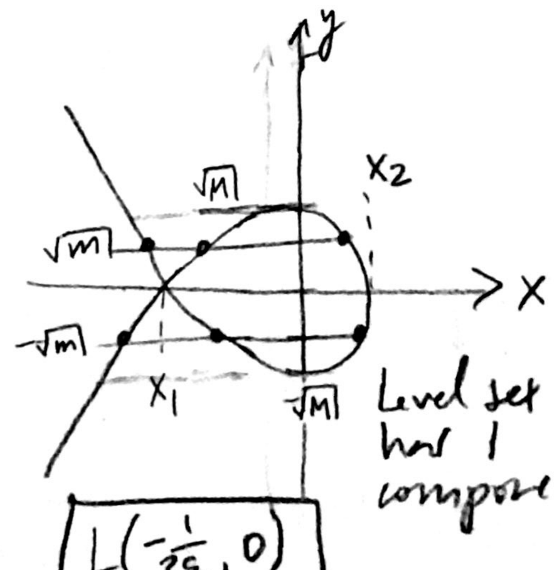




Level set has 2 components

$C = \frac{1}{27\varepsilon^2}$

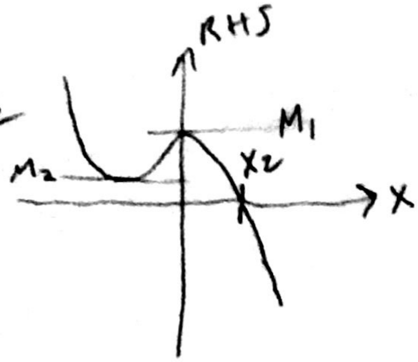


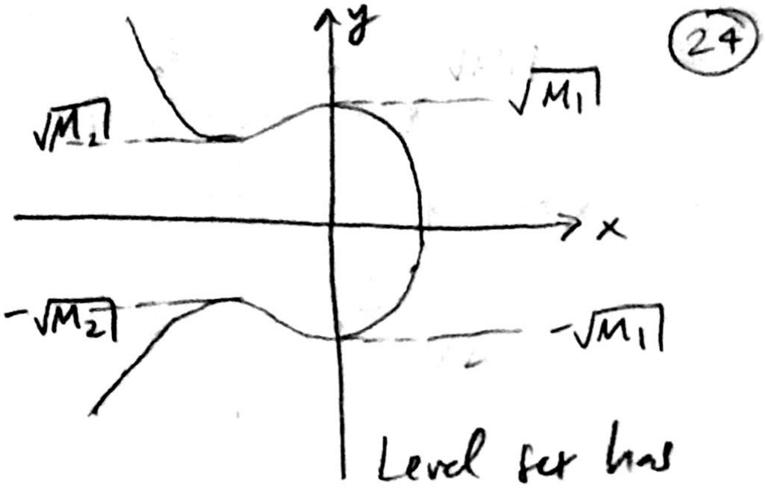


Level set has 1 component

$\boxed{L(-\frac{1}{3\varepsilon},0)}$

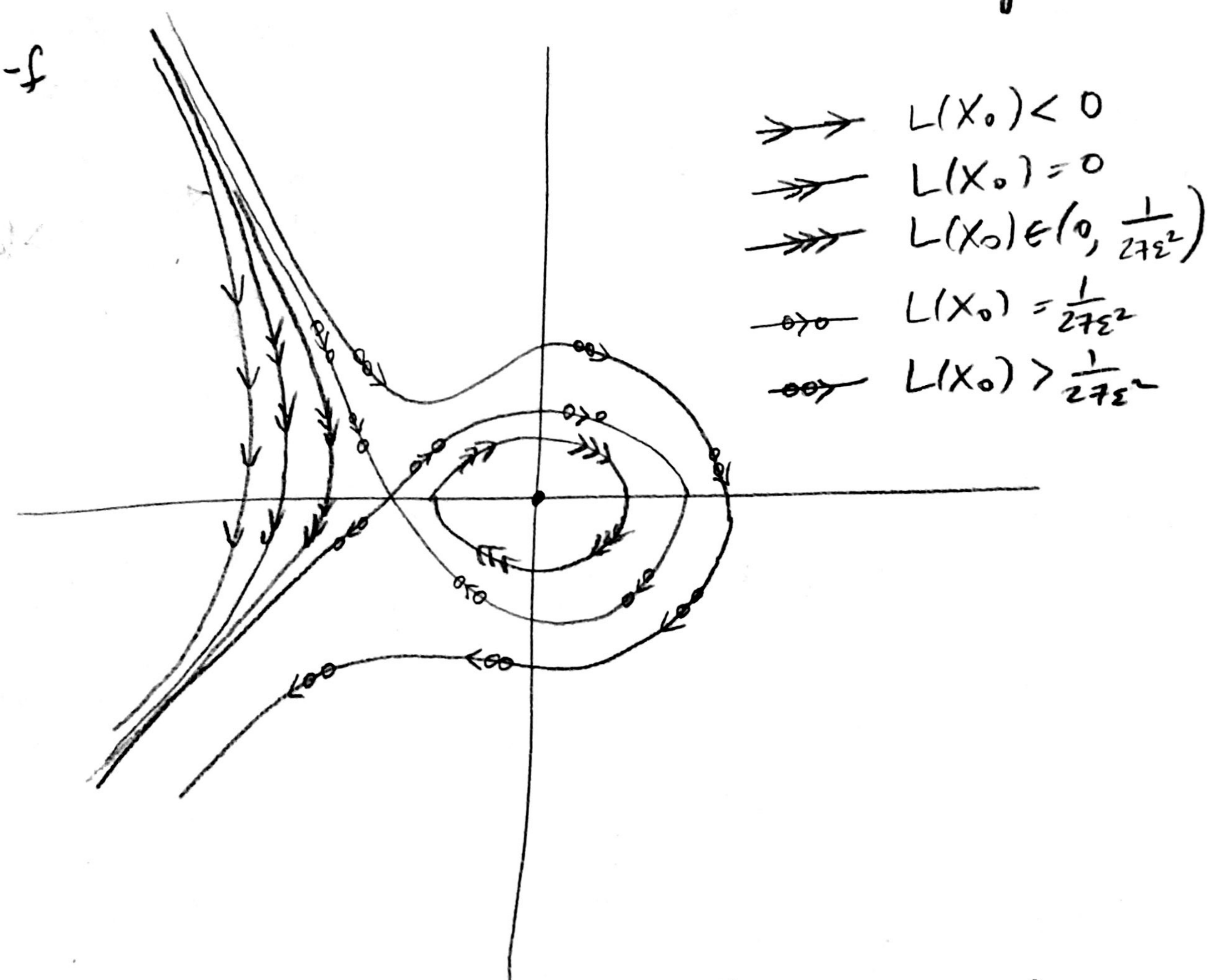
$C > \frac{1}{27\varepsilon^2}$





Level set has 1 component

4d-f



if $L(X_0) < 0$ or $L(X_0) > 1/27\varepsilon^2$ then the level set contains only one solution

if $L(X_0) = 0$ then the level set has 2 solutions, one of which is $(0,0)$ the other is unbounded

If $L(X_0) \in \left(0, \frac{1}{27\varepsilon^2}\right)$ then the level set has 2 solutions: one periodic and one unbounded

If $L(X_0) = \frac{1}{27\varepsilon^2}$ then the level set has 4 solutions:

- $\left(-\frac{1}{3\varepsilon}, 0\right) = X_{eq}$
- an unbounded one in the 2nd quadrant that goes to X_{eq} as $t \to \infty$
- an unbounded solution in the 3rd quadrant that goes to X_{eq} as $t \to -\infty$
- a bounded solution that goes to X_{eq} as $t \to \infty$ and as $t \to -\infty$.

#5a) Consider two flows

$$X' = -\nabla L(X) \quad \text{and} \quad Y' = G(Y)$$

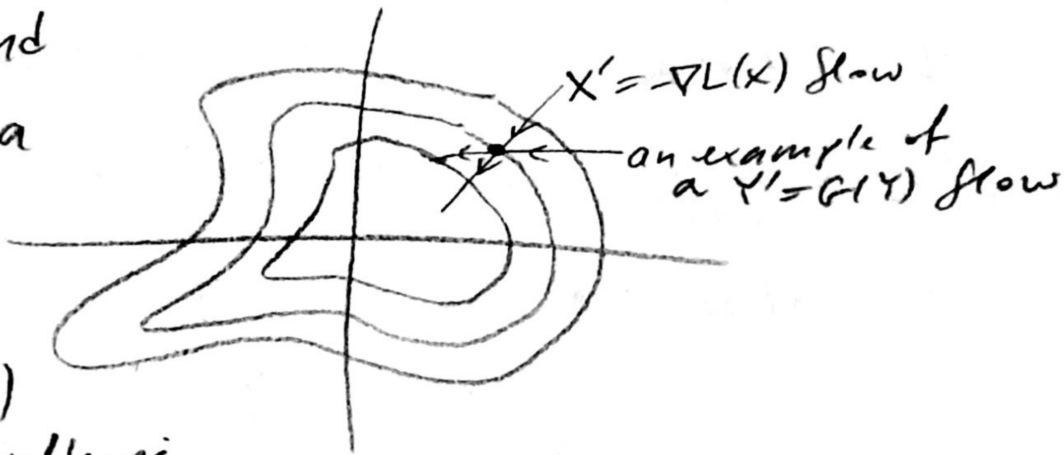
where $\|G(X)\| = \|-\nabla L(X)\|$. Show that L decreases more quickly along a solution $X(t)$ than along a solution $Y(t)$.

To show this, it suffices to show that

$$\frac{d}{dt} L(X(t)) < \frac{d}{dt} L(Y(t))$$

when evaluated at a time t_0 so that $X(t_0) = Y(t_0)$.

Pictorially, if these are level sets of L and you're at a point X_0 then the $X' = -\nabla L(X)$ flow is walking straight downhill while the $Y' = G(Y)$ flow can be going some other, gentler route. (Indeed it might not even be going downhill at all!)



$$\frac{d}{dt} L(X(t)) = \nabla L(X(t)) \cdot \frac{dX}{dt}(t) = -\nabla L(X(t)) \cdot \nabla L(X(t))$$
$$= -\|\nabla L(X(t))\|^2$$

Similarly, $\frac{d}{dt} L(Y(t)) = \nabla L(Y(t)) \cdot G(Y(t))$

Now assume we're at the initial location X_0, which flow will immediately reduce L more? i.e. is it true that

$$-\|\nabla L(X_0)\|^2 < \nabla L(X_0) \cdot G(X_0)$$

if $G(X_0) \neq -\nabla L(X_0)$ and $\|G(X_0)\| = \|-\nabla L(X_0)\|$?

This will follow from the Cauchy Schwartz inequality

$$|\vec{v} \cdot \vec{v}| \leq \|\vec{v}\| \, \|\vec{v}\|$$

and equality holds only if one of $\vec{v}$ or $\vec{v}$ is a multiple of the other.

We're told that $G(X) \neq -\nabla L(X)$ and that $\|G(X)\| = \|\nabla L(X)\|$. So if $G(X)$ is a multiple of $\nabla L(X)$ then it must be that $G(X) = \nabla L(X)$

$$\Rightarrow \quad \nabla L(X_0) \cdot G(X_0) = \|\nabla L(X_0)\|^2 > -\|\nabla L(X_0)\|^2$$

what we wanted to show.

If $G(X_0)$ isn't a multiple of $\nabla L(X_0)$ then

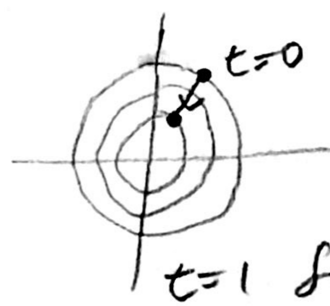
$$\begin{aligned} |\nabla L(X_0) \cdot G(X_0)| &< \|\nabla L(X_0)\| \, \|G(X_0)\| \\ &= \|\nabla L(X_0)\| \, \|\nabla L(X_0)\| \\ &= \|\nabla L(X_0)\|^2 \end{aligned}$$

$$\Rightarrow \; -\|\nabla L(X_0)\|^2 < \nabla L(X_0) \cdot G(X_0) < \|\nabla L(X_0)\|^2$$

what we wanted to show

NB: only considering $G(X)$ so that $\|G(X)\| = \|\nabla L(X)\|$ is vital to the argument. Otherwise you're not comparing apples to apples. Without this constraint you could reparametrize time and

t=1 for non-gradient descent flow that has $\|G(X)\| > \|\nabla L(X)\|$



t=1 for gradient descent flow

$L(Y(1)) < L(X(1))$.

5b)

$$L(x,y) = 132 + 16\sqrt{3} - 10x - 4\sqrt{3}x + \frac{5}{4}x^2 + 28y + 2\sqrt{3}y + \frac{1}{2}\sqrt{3}xy + \frac{7}{4}y^2$$

$$\nabla L(X) = \begin{matrix} -10 - 4\sqrt{3} + \frac{5}{2}x + \frac{1}{2}\sqrt{3}y \\ 28 + 2\sqrt{3} + \frac{\sqrt{3}}{2}x + \frac{7}{2}y \end{matrix}$$

$$\cancel{X'} = \begin{pmatrix} 5/2 & \sqrt{3}/2 \\ \frac{\sqrt{3}}{2} & 7/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -10 - 4\sqrt{3} \\ 28 + 2\sqrt{3} \end{pmatrix}$$

$$X' = \begin{pmatrix} -5/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -7/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 10 + 4\sqrt{3} \\ -28 - 2\sqrt{3} \end{pmatrix}$$

$X' = AX + B$ equilibrium solution

$$X_{eq} = -A^{-1}B = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$$

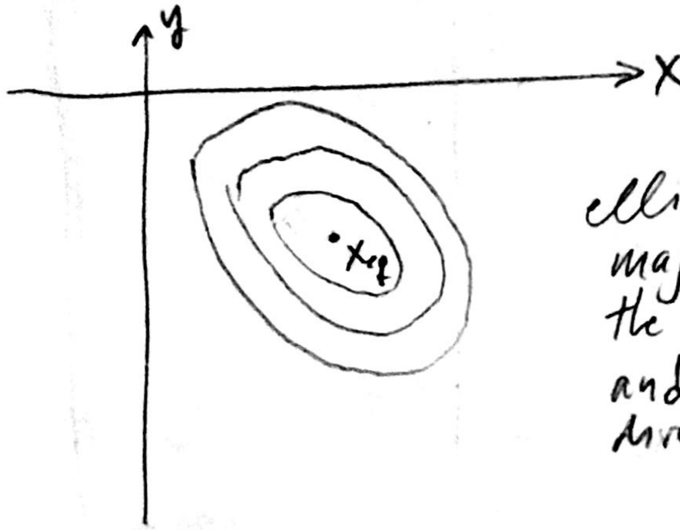
$$A \sim \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}^{-1}$$

$\Rightarrow$ solutions of $X' = -\nabla L(X)$ are

$$X(t) = \begin{pmatrix} 4 \\ -8 \end{pmatrix} + c_1\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}e^{-4t} + c_2\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}e^{-2t}$$

as $t \to \infty$ $X(t) \to X_{eq}$

Initial conditions that are on the line $l_2(s) = X_{eq} + s\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$ yield solutions that converge to X_{eq} the slowest. Initial conditions that are on the line $l_{-4} = X_{eq} + s\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$ yield solutions that converge to X_{eq} the fastest.



ellipse with major axis in the direction $\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$ and minor axis in direction $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$

5c) Seek A and $\begin{pmatrix} a \\ b \end{pmatrix}$ so that

$$L(x,y) = (x-a, y-b)\begin{pmatrix} & A & \end{pmatrix}\begin{pmatrix} x-a \\ y-b \end{pmatrix}$$

then $\nabla L(x,y) = (A + A^T)\begin{pmatrix} x-a \\ y-b \end{pmatrix}$

and $H(x,y) = A + A^T$

so let's compute ... $\nabla L(x,y) = \begin{matrix} -10 + 4\sqrt{3} + \frac{5}{2}x + \frac{1}{2}\sqrt{3}\,y \\ 28 + 2\sqrt{3} + \frac{\sqrt{3}}{2}x + \frac{7}{2}y \end{matrix}$

$$\Rightarrow H(x,y) = \begin{pmatrix} 5/2 & \frac{\sqrt{3}}{2} \\ \sqrt{3}/2 & 7/2 \end{pmatrix} = A + A^T \Rightarrow A = \begin{pmatrix} 5/4 & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & 7/4 \end{pmatrix}$$

now to solve for $\begin{pmatrix} a \\ b \end{pmatrix}$.

we know $\nabla L(x,y) = (A+A^T)\left(X - \begin{pmatrix} a \\ b \end{pmatrix}\right)$

$$= (A+A^T)X - (A+A^T)\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -10-4\sqrt{3} \\ 28+2\sqrt{3} \end{pmatrix} = -\begin{pmatrix} 5/2 & \sqrt{3}/2 \\ \frac{\sqrt{3}}{2} & 7/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$$

Putting it all together,

$$L(x,y) = (X - X_{eq})^T \begin{pmatrix} 5/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 7/4 \end{pmatrix}(X - X_{eq})$$

where $X_{eq} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$

to find out if X_{eq} is a global minimizer, I would diagonalize the matrix (doable because it's symmetric and $A = PDP^{-1}$

A is symmetric $\Rightarrow$ it can be orthogonally diagonalized so $\exists$ Q such that $Q^T = Q^{-1}$ and $A = QDQ^T$ then

$$L(X) = (X - X_{eq})^T QDQ^T (X - X_{eq})$$

$$= \left(Q^T(X - X_{eq})\right)^T D \left(Q^T(X - X_{eq})\right)$$

$$= Z^T D Z = \lambda_1 z_1^2 + \lambda_2 z_2^2$$

if $\lambda_1, \lambda_2 > 0$ then global min.

$$A = \begin{pmatrix} 5/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 7/4 \end{pmatrix} = \frac{1}{2}\underbrace{\begin{pmatrix} 5/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 7/2 \end{pmatrix}}$$

we've already diagonalized the negative of this

$$= \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}^{T}$$

$$= QDQ^T$$

where Q is an orthogonal matrix. Because 2 and 1 are positive, X_{eq} is a global minimizer of L.

#6

a) $L(x,y) = x^2y^2$ isn't a Liapunov function for $\begin{cases} x' = -2xy^2 \\ y' = -2yx^2 \end{cases}$

because the equilibrium solutions are $\{(x,0) \mid x \in \mathbb{R}\} \cup \{(0,y) \mid y \in \mathbb{R}\}$ and to be a Liapunov function you need to have an open set around an equilibrium solution so that $L(X) > 0$ if $X \in U$, $X \neq X_{eq}$.

This is impossible for this system because the equilibrium solutions aren't isolated.

6b) $\begin{cases} x' = -2xy^2 \\ y' = -2yx^2 \end{cases}$

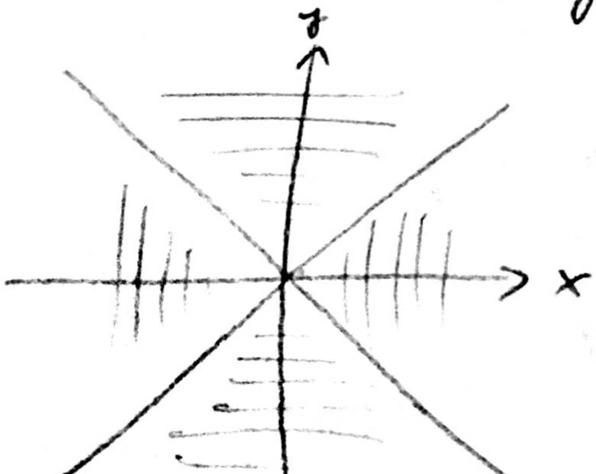
seek curves of the form $(x, y(x))$ so that solutions lie in those curves.

$$\frac{dy}{dx} = \frac{-2yx^2}{-2xy^2} = \frac{x}{y}$$

$$y\frac{dy}{dx} = x \Rightarrow \frac{d}{dx}\left(\frac{1}{2}y(x)^2\right) = x$$

$$\Rightarrow \frac{1}{2}y(x)^2 = \frac{1}{2}x^2 + C$$

$$\Rightarrow y(x)^2 = x^2 + C \qquad C \in \mathbb{R}.$$



3 cases to consider

$|y_0| > |x_0|$ (region)

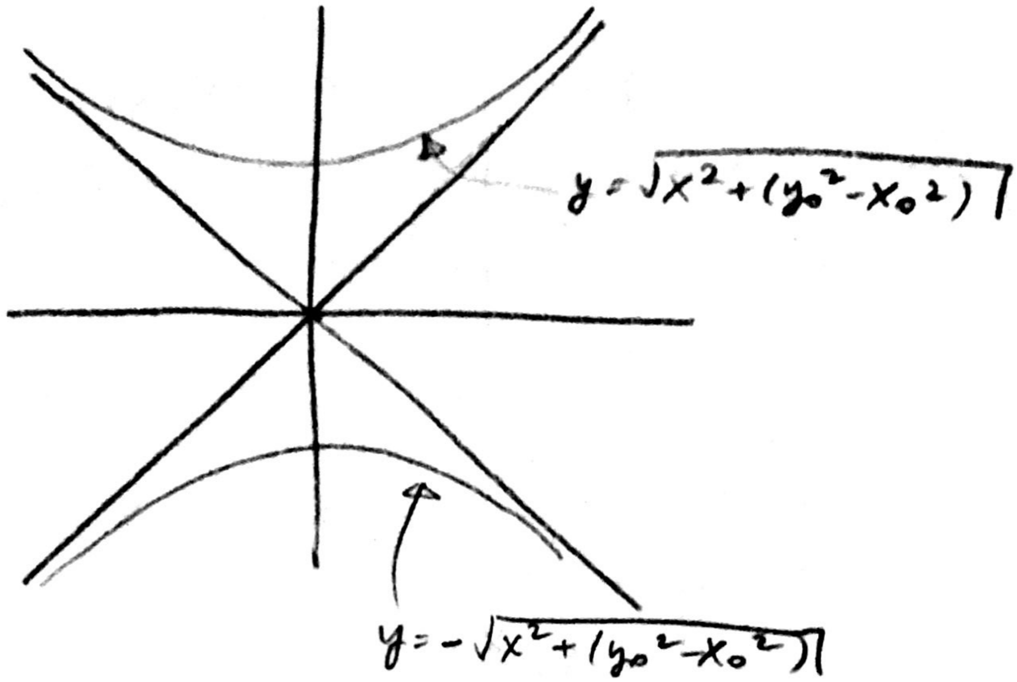
$|y_0| = |x_0|$ (curves)

$|y_0| < |x_0|$ (regions)

case 1: $|y_0| > |x_0| \Rightarrow C = y_0^2 - x_0^2 > 0$

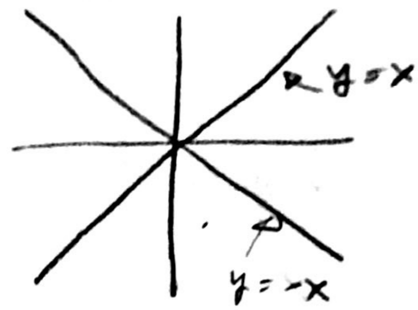
$$y(x)^2 = x^2 + (y_0^2 - x_0^2)$$

$$y(x) = \pm\sqrt{x^2 + (y_0^2 - x_0^2)}$$



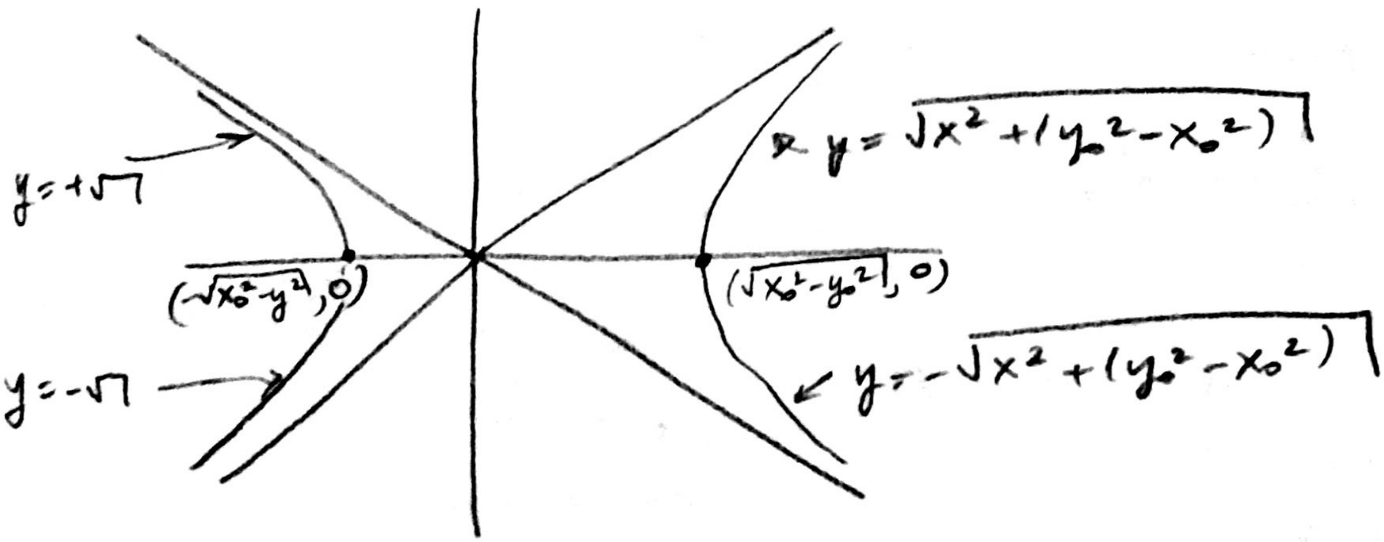
Case 2: $|y_0| = |x_0| \Rightarrow y_0^2 - x_0^2 = 0$

$\Rightarrow y(x)^2 = x^2 \Rightarrow |y(x)| = |x|$



Case 3: $|y_0| < |x_0| \Rightarrow y_0^2 - x_0^2 < 0$

$$\Rightarrow y(x) = \pm \sqrt{x^2 + \underbrace{(y_0^2 - x_0^2)}_{\text{negative}}}$$

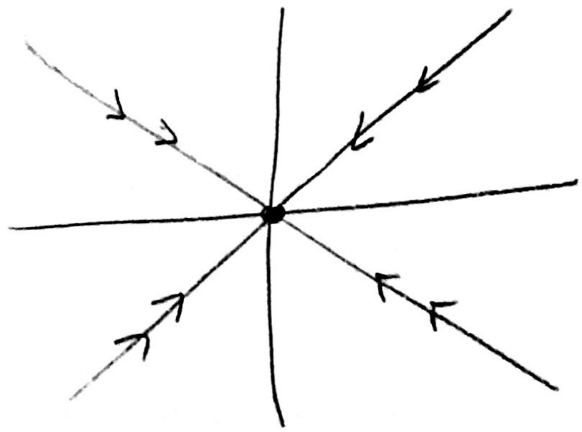


$$\begin{cases} x' = -2xy^2 \\ y' = -2yx^2 \end{cases}$$

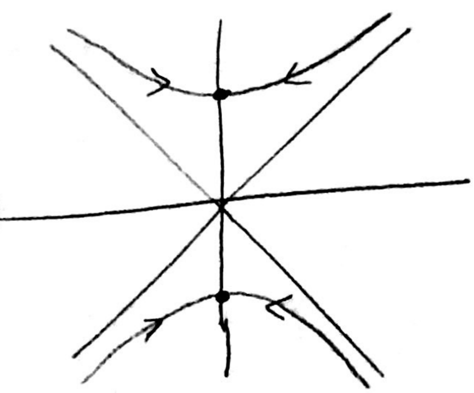
$x'>0$, $y'<0$	$x'<0$, $y'<0$
$x'>0$, $y'>0$	$x'<0$, $y'>0$

so

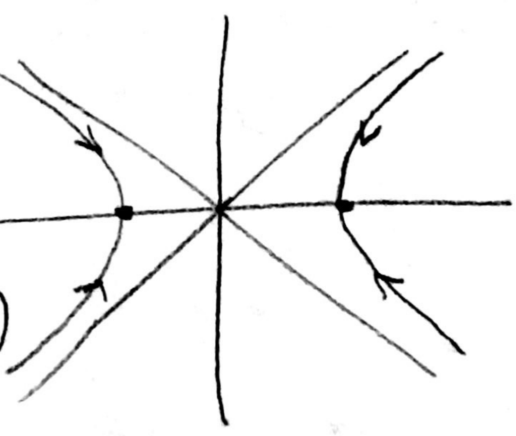
$|y_0| = |x_0|$ then $X(t) \to \vec{0}$ as $t \to \infty$



$|y_0| > |x_0|$ as $t \to \infty$ $X(t) \to \left(0, \pm\sqrt{y_0^2 - x_0^2}\right)$



$|y_0| < |x_0|$ as $t \to \infty$ $X(t) \to \left(\pm\sqrt{x_0^2 - y_0^2}, 0\right)$



So while there are infinitely many equilibrium solutions, we know which one a particular solution will converge to as $t \to \infty$

6c) Reduce the 2d problem to a 1d problem using $y(x) = \pm\sqrt{x^2+C}$

$$x' = -2xy^2 = -2x(x^2+C)$$

$$L(x,y) = x^2y^2 \rightsquigarrow \tilde{L}(x) = x^2(x^2+C)$$

So look at
$$x' = -2x(x^2+C)$$
$$\tilde{L}(x) = x^2(x^2+C) = x^4+x^2C$$

$$\frac{d}{dt}\tilde{L}(x(t)) = 4x(t)^3x'(t) + 2Cx(t)x'(t)$$
$$= \left[4x(t)^3 + 2Cx(t)\right]x'(t)$$
$$= \left[4x(t)^3+2Cx(t)\right]\left[-2x(t)(x(t)^2+C)\right]$$
$$= 2x(t)\left[2x(t)^2+C\right]\left[-2x(t)(x(t)^2+C)\right]$$
$$= -4x(t)^2\left[2x(t)^2+C\right]\left[x(t)^2+C\right]$$

Recall that it's possible for C to be negative. And so we have to worry about whether $x(t)^2+C<0$ or $2x(t)^2+C<0$.
But if $2x(t)^2+C > x(t)^2+C \geq 0$ because $y(x)$ isn't defined for $x<C$.

Given initial data x_0 and a specific value of C, this determines the ODE for $x(t)$ and the Liapunov function. If $C \geq 0$ then $\tilde{L}(x)$ is minimized at $x=0$. If $C<0$ then $\tilde{L}(x)$ is minimized at $x = \pm\sqrt{C}$.

The solution $x(t)$ has maximal interval of existence (α, β). But for $t \geq 0$ $x(t)$ is with the compact set $\{x \mid \tilde{L}(x) \leq \tilde{L}(x_0)\}$. Therefore $\beta = \infty$ and the solution exists for all $t > 0$. The proof that $x(t) \to x_{eq}$ (where $x_{eq} = 0$ if $c \geq 0$ and $x_{eq} = \pm\sqrt{-c}$ if $c < 0$) is the same as the proof of the Liapunov function theorem.