appeared in Journal of Differential Equations 182(2002)2:377–415

ENERGY LEVELS OF STEADY STATES FOR THIN FILM TYPE EQUATIONS

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ABSTRACT. We study the phase space of the evolution equation

$$h_t = -(f(h)h_{xxx})_x - (g(h)h_x)_x$$

by means of a dissipated energy. Here $h(x,t) \geq 0$, and at $h = 0$ the coefficient functions $f > 0$ and g can either degenerate to 0, or blow up to ∞, or tend to a nonzero constant.

We first show all positive periodic steady states are saddles in the energy landscape, with respect to zero–mean perturbations, if $(g/f)'' \geq 0$ or if the perturbations are allowed to have period longer than that of the steady state.

For power law coefficients ($f(y) = y^n$ and $g(y) = \mathcal{B}y^m$ for some $\mathcal{B} > 0$) we analytically determine the relative energy levels of distinct steady states. For example, with $m-n \in [1,2)$ and for suitable choices of the period and mean value, we find three fundamentally different steady states. The first is a constant steady state that is stable and is a local minimum of the energy. The second is a positive periodic steady state that is linearly unstable and has higher energy than the constant steady state; it is a saddle point for the energy. The third is a periodic collection of 'droplet' (compactly supported) steady states having lower energy than either the positive steady state or the constant one. Since the energy must decrease along every orbit, these results significantly constrain the dynamics of the evolution equation.

1. INTRODUCTION

We study the evolution equation

$$h_t = -(f(h)h_{xxx})_x - (g(h)h_x)_x. \tag{1}$$

This is the one dimensional version of $h_t = -\nabla \cdot (f(h)\nabla\Delta h) - \nabla \cdot (g(h)\nabla h)$, which has been used to model the dynamics of a thin film of viscous liquid. The air/liquid interface is at height $z = h(x, y, t) \geq 0$ and the liquid/solid interface is at $z = 0$. The one dimensional equation (1) applies if the liquid film is uniform in the y direction.

The fourth order term in the equation reflects surface tension effects, and the second order term can reflect gravity, van der Waals interactions, thermocapillary effects or the geometry of the solid substrate. Typically $f(h) = h^3 + \beta h^p$ where $0 < p < 3, \beta \geq 0$, and $g(h) \sim \pm h^m$ as $h \to 0$, for some $m \in \mathbb{R}$. In certain applications $g(h)$ changes sign at some positive h. The extensively studied Cahn–Hilliard equation has the form (1) with $f \equiv 1$ and $g(h) = 1 - 3h^2$.

We refer to [18, 20] for reviews of the physical and modeling literature.

Date: May 7, 2001.

Background and goals. In [14] we proved linear stability and instability results for the positive *periodic* steady states of (1). Periodicity is not a constraint here if $f, g > 0$, since then positive bounded steady states are automatically periodic (or constant) by [13, Theorem B.1].

In this paper we concentrate mostly on positive periodic steady states, and on compactly supported 'droplet' steady states with zero contact angles. Our main investigative tool is the energy

$$\mathcal{E}(h(\cdot,t)) = \int_0^X \left[\frac{1}{2}h_x(x,t)^2 - H(h(x,t))\right] dx,$$

where $h(x,t)$ is a smooth solution of (1) that is X-periodic in x, and $H(y)$ satisfies $H'' = g/f$. This energy is strictly dissipated: $\frac{d}{dt}\mathcal{E}(h(\cdot,t)) \leq 0$ with equality if and only if h is a steady state (see §2.1). Incidentally, this energy dissipation means there can be no periodic travelling wave solutions.

We address two questions about the energy landscape of the evolution.

(1) Which steady states are local minima of the energy, and which are saddle points?
(2) Which steady state has the lowest energy?

Answering these questions will clarify the phase portrait of the evolution. In both questions we assume our solutions have a given spatial period and a given area (fluid volume) per period. This volume constraint is natural because the evolution (1) conserves volume: $\int h(x,t)\,dx = \int h(x,0)\,dx$ for all t. The assumption of spatial periodicity includes also the case of Neumann ('no flux') boundary conditions on an interval, as explained in Section 2.6.

A sketch of definitions and results. If there is an X-periodic zero–mean perturbation v such that $\mathcal{E}(h_{ss} + \varepsilon v) < \mathcal{E}(h_{ss})$ for all small $\epsilon > 0$, then we call the steady state h_{ss} 'energy unstable' at period X. If instead $\mathcal{E}(h_{ss} + \varepsilon v) > \mathcal{E}(h_{ss})$ for all sufficiently small $\varepsilon > 0$, for each X-periodic zero–mean perturbation v, then we call the steady state 'energy stable' at period X.

Our main stability results, in Section 2, are roughly stated as follows.

- Theorem 1. For positive periodic steady states, linear instability implies energy instability. Hence our linear instability results in [14] imply that every positive X^*-periodic steady state is energy unstable at the periods $X = 2X^*, 3X^*, \ldots$, and is also energy unstable at period X^* if g/f is a strictly convex function.
- Theorems 2–3. Further, for the 'power law' coefficients $f(y) = y^n$ and $g(y) = \mathcal{B}y^m$ with $\mathcal{B} > 0$, we completely characterize energy stability at period X^* even when g/f is not convex, that is, when $m - n \in (0,1)$.

Then in Section 3 we determine the *relative* energy levels of three different kinds of steady state: constant steady states, positive periodic steady states, and zero contact angle droplet

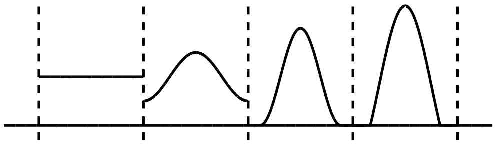

FIGURE 1. Four types of steady state.

steady states. Figure 1 illustrates these three steady states, as well as showing a nonzero contact angle droplet steady state (about which we say little in this paper). Our main energy level results for $f(y) = y^n$ and $g(y) = \mathcal{B}y^m$ are:

- Theorem 6. If $m - n \notin [0, 1)$ then a positive periodic steady state always has *higher* energy than the constant steady state with the same mean value. For $m-n \in (0, 0.75]$ our analytical and numerical work suggest the positive periodic steady state has *lower* energy than the constant steady state.
- Theorem 9. When $m - n \approx 0.77$ there can be two steady states h_{ss1} and h_{ss2} with the same period and area, and with $\min_x h_{\mathrm{ss1}}(x) < \min_x h_{\mathrm{ss2}}(x)$; we essentially prove h_{ss2} is energy unstable and has higher energy than h_{ss1}, which is energy stable.
- Theorem 7. If $m - n \in (-2, 0) \cup [1, 2)$ then a positive periodic steady state always has *higher* energy than a zero contact angle droplet steady state with the same mean value.
- Theorem 11. The constant steady state can have higher energy than the zero contact angle droplet steady state with the same mean value. When $m - n \in [1, 2)$, for example, a mountain pass scenario can occur, in which the constant steady state is a local minimum of the energy, the positive periodic steady state is an energy unstable saddle, and there is a zero contact angle droplet having lower energy than either of them.

Our energy level results suggest the existence of basins of attraction around the stable steady states, and heteroclinic connections between certain steady states. In the companion article [15, §5], we investigate such possibilities with numerical simulations. For example, for the mountain pass scenario in Theorem 11 we find that perturbing the saddle point (the periodic steady state) in one direction leads to relaxation to the constant steady state and perturbing in the opposite direction gives apparent relaxation to a droplet. A similar dichotomy was found for axisymmetric surface diffusion by Bernoff, Bertozzi and Witelski, [1, p. 744], with perturbed unduloids either relaxing to a cylinder or else pinching off in finite time.

Terminology. We write $\mathbb{T}_X$ for a circle of circumference $X > 0$. As usual, one identifies functions on $\mathbb{T}_X$ with functions on $\mathbb{R}$ that are X-periodic, calling them *even* or *odd* according to whether they are even or odd on $\mathbb{R}$.

Positive periodic steady states are assumed to satisfy the steady state equation classically. A droplet steady state $h_{\mathrm{ss}}(x)$ is by definition positive on some interval (a, b) and zero elsewhere, with $h_{\mathrm{ss}} \in C^1[a, b]$; we require h_{ss} to satisfy the steady state equation on the open interval (a, b) only, and to have equal acute contact angles: $0 \leq h'_{\mathrm{ss}}(a) = -h'_{\mathrm{ss}}(b) < \infty$. (Throughout the paper, if a function has only one independent variable then we use $'$ to denote differentiation with respect to that variable: $h'_{\mathrm{ss}} = (h_{\mathrm{ss}})_x$.)

We say a droplet steady state h_{ss} has 'zero contact angle' if $0 = h'_{\mathrm{ss}}(a) = -h'_{\mathrm{ss}}(b)$, and 'nonzero contact angle' otherwise. A 'configuration' of droplet steady states is a collection of steady droplets whose supports are disjoint. For more on the steady states and their properties, see [13].

2. Energy stability for periodic steady states

We assume throughout this section that $f(y)$ and $g(y)$ are C^2-smooth for $y > 0$, and that $f > 0$. Define

$$r = \frac{g}{f}.$$

Take $X > 0$.

We investigate stability and dynamical questions by means of a dissipated energy. A few of the theorems follow directly from our linear stability results in [14], but most are quite different and complementary.

2.1. Definition of the energy, and of energy instability.

The energy function for the evolution equation (1) is defined for $\ell \in H^1(\mathbb{T}_X)$ to be

$$\mathcal{E}(\ell) = \int_0^X \left[\frac{1}{2}(\ell')^2 - H(\ell)\right] dx, \tag{2}$$

where $H(y)$ is a function with $H'' = r = g/f$.

To verify the energy $\mathcal{E}$ is dissipated by the evolution (1), suppose $h(x,t)$ is a positive smooth solution of (1) that is X-periodic in x. Bertozzi and Pugh [2, §2] observed (generalizing [8, 9, 21]) that

$$\frac{d}{dt}\mathcal{E}(h(\cdot,t)) = -\int_0^X \frac{1}{f(h(x,t))}\left[f(h(x,t))h_{xxx}(x,t) + g(h(x,t))h_x(x,t)\right]^2 dx \leq 0.$$

This energy dissipation is strict at each time t unless $f(h)h_{xxx} + g(h)h_x \equiv 0$, which occurs only when $h(\cdot,t)$ is a steady state, by (4) below.

By the way, we can restrict to positive smooth solutions $h(x,t)$ in this paper since our perturbation arguments are localized around steady states. But solutions that are initially positive might not always remain so, and when they go to zero they can lose regularity. Thus *weak* solutions must generally be considered. See Bertozzi and Pugh [2] for existence of nonnegative weak solutions of (1) that still dissipate the energy.

Returning now to our main task, let $h_{\text{ss}} \in C^4(\mathbb{T}_X)$ be a positive periodic steady state of (1). It is easy to show that h_{ss} is a critical point for the energy $\mathcal{E}$ with respect to zero-mean perturbations (cf. formula (15)).

Definition. Call h_{ss} an *energy unstable* critical point (with respect to zero–mean perturbations at period X) if there exists a smooth X-periodic perturbation $u(x)$ with mean value zero such that

$$\mathcal{E}(h_{\text{ss}} + \varepsilon u) < \mathcal{E}(h_{\text{ss}}) \qquad \text{for all small } \varepsilon > 0.$$

That is, small perturbations in the direction u decrease the energy. (Some authors call this *formal* instability [12].)

Our requirement that the perturbation u have zero mean seems reasonable because the evolution equation (1) preserves volume: $\int h(x,t)\,dx = \int h(x,0)\,dx$ for all time t, for spatially periodic solutions. Thus zero–mean perturbations of h_{ss} permit the possibility of relaxation back to h_{ss}, while nonzero–mean perturbations do not.

Note that an energy unstable steady state is necessarily a saddle point in the energy landscape, since $\mathcal{E}$ is decreased by some perturbation u but is *increased* by the perturbation $u(x) = \varepsilon \cos(2\pi k x/X)$ for $k \gg 1$.

Further, an energy unstable steady state is not asymptotically stable in $H^1(\mathbb{T}_X)$, in the following sense: suppose $h(x,t)$ is a positive smooth solution with initial data $h_{\text{ss}} + \varepsilon u$; then $h(\cdot,t) \not\to h_{\text{ss}}(\cdot)$ in $H^1(\mathbb{T}_X)$ since for all t, $\mathcal{E}(h(\cdot,t)) \leq \mathcal{E}(h(\cdot,0)) = \mathcal{E}(h_{\text{ss}} + \varepsilon u) < \mathcal{E}(h_{\text{ss}})$ (while convergence in H^1 would imply convergence in L^∞ and hence convergence of the energy). Indeed by the same reasoning, $h(\cdot,t)$ cannot converge to any *translate* of h_{ss}.

2.2. **Energy instability results.** In [14, §2] we linearized the evolution equation (1) around the positive periodic steady state h_{ss} and then reduced the linear stability question to determining the sign of the first eigenvalue of a certain self-adjoint fourth order linear operator. We will not repeat the linearization here, or re-state the linear stability results of [14]. But we should warn readers that when we say a steady state is 'linearly stable', we are including the neutrally stable case in which the first eigenvalue of the linearized operator is zero. This is unavoidable: the operator *always* has a zero eigenvalue in its spectrum, corresponding to an infinitesimal translation of the steady state in space (the evolution equation is translation invariant).

The next theorem states that if a steady state is linearly unstable then it is energy unstable. Also, we present some particular unstable directions, when g/f is strongly convex.

Theorem 1. *Let $f, g \in C^2(0,\infty)$ with $f > 0$. Take $X > 0$ and suppose $h_{\mathrm{ss}} \in C^4(\mathbb{R})$ is an X-periodic positive steady state of (1).*

If h_{ss} is linearly unstable with respect to zero–mean perturbations at period X, then it is also energy unstable at period X.

In particular, h_{ss} is energy unstable at period X if it is non-constant and either: the least period of h_{ss} is X/j for some integer $j \geq 2$ or else $r = g/f$ is convex ($r'' \geq 0$) and non-constant on the range of h_{ss}. For example, if h_{ss} is non-constant and $r'' > 0$, then h_{ss} is energy unstable in the directions $u = \pm h'_{\mathrm{ss}}$ and $\pm h''_{\mathrm{ss}}$.

This is proved in Section 4.1. We believe that energy unstable steady states are in fact dynamically unstable under the evolution, but we have not been able to prove this.

In particular the theorem covers the van der Waals evolution

$$h_t = -(h^3 h_{xxx})_x - A(h^{-1}h_x)_x \tag{3}$$

with $A > 0$, for which $r(y) = Ay^{-1}/y^3 = Ay^{-4}$ is strongly convex. Thus all positive periodic steady states of the van der Waals evolution are energy unstable. The van der Waals equation has been studied by a number of other authors, *e.g.* [4, 22, 23, 24, 25], mostly with regard to similarity solutions and film rupture (where the solution goes to zero in finite time, at some point). In [15, §5.1] we connect these results to our own work on instability, by studying numerically how the unstable steady states of (3) can be perturbed to evolve towards rupture.

In Theorem 1 we have assumed f, g are C^2-smooth on the whole interval $(0,\infty)$, which is generally the case for the thin film equations that are our main motivation. But our arguments are all local (involving only small perturbations of the steady state), and so the theorem still holds if the coefficient functions $f(y)$ and $g(y)$ are defined and C^2-smooth merely for y-values in a neighborhood of the range of h_{ss}.

2.3. **Review of power law steady states and their rescalings.** We now turn to *power law* coefficients: $f(y) = y^n$ and $g(y) = \mathcal{B}y^m$ for some exponents $n, m \in \mathbb{R}$ and some positive constant $\mathcal{B} > 0$. Here

$$r(y) = \mathcal{B}y^{q-1}$$

where

$$\boxed{q := m - n + 1}$$

This exponent q determines many properties of the steady states, including (usually) their linear stability.

The evolution equation (1) becomes

$$h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x.$$

To state our results on energy stability for this power law evolution, we first review some properties of the steady states and explain how to rescale them to solutions of a canonical nonlinear oscillator ODE, given as equation (8–9) below.

We start with a non-constant positive periodic steady state $h_{\rm ss} \in C^4(\mathbb{T}_X)$ of the general evolution (1). The steady state condition for (1) integrates to give $f(h_{\rm ss})h_{\rm ss}''' + g(h_{\rm ss})h_{\rm ss}' = C$ for some constant C. And note the least period of $h_{\rm ss}$ is X/j for some integer $j \geq 1$.

One finds that the constant C (the flux) equals zero, by dividing $f(h_{\rm ss})h_{\rm ss}''' + g(h_{\rm ss})h_{\rm ss}' = C$ by $f(h_{\rm ss}) > 0$ and integrating over a period. Hence the steady state satisfies

$$h_{\rm ss}''' + r(h_{\rm ss})h_{\rm ss}' = 0. \tag{4}$$

[If $h_{\rm ss}$ were a droplet steady state then again $C = 0$, by [13, Theorem 2.1], and equation (4) would hold wherever $h_{\rm ss}$ is positive.]

Integrating (4), the steady states have a nonlinear oscillator formulation:

$$h_{\rm ss}'' + H'(h_{\rm ss}) = 0 \tag{5}$$

holds wherever $h_{\rm ss}$ is positive. Here $H(y)$ is a function with $H'' = r = g/f$; if we regard x as a 'time' variable then $\frac{1}{2}h_{\rm ss}'(x)^2 + H(h_{\rm ss}(x))$ is a conserved quantity.

Specializing to the power law evolution, remember $r(y) = \mathcal{B}y^{q-1}$. Thus for $q \neq 0$ we can write the steady state equation (5) as

$$h_{\rm ss}'' + \frac{\mathcal{B}h_{\rm ss}^q - D}{q} = 0 \tag{6}$$

for some constant D. For $q = 0$ the analogous equation is $h_{\rm ss}'' + \mathcal{B}\log h_{\rm ss} - D = 0$. This oscillator equation involves three constants: q, $\mathcal{B}$, and D. We remove $\mathcal{B}$ and D by rescaling: let

$$k(x) = \begin{cases} \left(\frac{\mathcal{B}}{D}\right)^{1/q} h_{\rm ss}\left(\left(\frac{D}{\mathcal{B}}\right)^{1/2q} \frac{x}{D^{1/2}}\right), & q \neq 0, \\ e^{-D/\mathcal{B}} h_{\rm ss}\left(e^{D/2\mathcal{B}} \frac{x}{\mathcal{B}^{1/2}}\right), & q = 0. \end{cases} \tag{7}$$

For $q \neq 0$ this uses that $D > 0$, by [13, §3.1]. Now the steady state equation (6) rescales to

$$k'' + \frac{k^q - 1}{q} = 0, \qquad q \neq 0, \tag{8}$$

$$k'' + \log k = 0, \qquad q = 0. \tag{9}$$

Differentiating, we find for all q that $k''' + k^{q-1}k' = 0$, and so k satisfies $(k^n k''' + k^m k')' = 0$, *i.e.* it is a steady state of $k_t = -(k^n k_{xxx})_x - (k^m k_x)_x$.

Since $h_{\rm ss}$ is non-constant, positive and periodic, we know $k''(x_0) > 0$ for some point x_0. Evaluating (8–9) at x_0 shows the minimum value of k is less than 1. Also $k'(0) = 0$ since

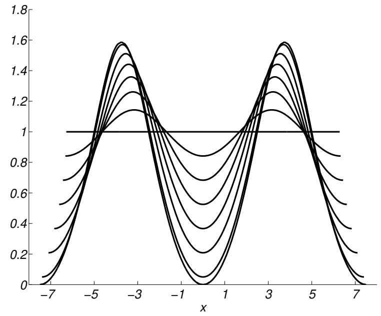

FIGURE 2. Steady states $k_\alpha(x)$, when $q = 3$. Note their differing periods and areas.

(after a suitable translation) h_{ss} has its minimum at $x = 0$. Introducing the notation k_α for the solution k that has minimum value $\alpha \in (0, 1)$, at $x = 0$, we have

$$0 < k_\alpha(0) = \alpha < 1, \qquad k'_\alpha(0) = 0. \tag{10}$$

Conclusions. Thus every steady state h_{ss} can be rescaled to a k_α, as above. Conversely, for each $q \in \mathbb{R}$ and $\alpha \in (0, 1)$ there exists a unique smooth positive periodic k_α satisfying equations (8–9) and (10) (see [13, Proposition 3.1]). The same holds for $\alpha = 0$ when $q > -1$, except that k_0 may be only C^1-smooth at $x = 0$ (see [13, Theorem 3.2]). To illustrate, Figure 2 plots the steady states k_α over two periods, for $q = 3$ and eight α-values between 0 and 1; see [13, §6.1] for details.

Note that the map $(\alpha, x) \mapsto k_\alpha(x)$ is C^∞-smooth for $(\alpha, x) \in (0, 1) \times \mathbb{R}$, by an ODE theorem giving smooth dependence on the initial data [11, Ch. V §4]. We write

$$P = P(\alpha) \qquad \text{and} \qquad A = A(\alpha) = \int_0^P k_\alpha(x)\, dx$$

for the least period of k_α and for the area under its graph, respectively. Then P and A are smooth functions of α that approach 2π as $\alpha \to 1$, by [14, Lemma 6]. It turns out that the function

$$E(\alpha) := P(\alpha)^{3-q} A(\alpha)^{q-1} = P(\alpha)^2 [A(\alpha)/P(\alpha)]^{q-1}$$

determines whether the steady states are energy unstable or stable, in several results below.

The above rescaling ideas are a useful tool throughout the paper. We hope this tool does not obscure the fact that stability and energy level properties for equations of type (1) seem to be determined by the *period map* of a family of steady states h_{ss} with fixed area and with period depending on the amplitude, or alternatively by the *area map* of a family of steady states with fixed period and with area (per period) depending on the amplitude. One should think of the function $E(\alpha)$ as such an area map; see [14, §6.3] for more on this. The same underlying idea appears in the work of Grinfeld and Novick–Cohen [10] on the Cahn–Hilliard equation.

2.4. **Energy in/stability for the power law evolution.**

Theorem 2. *Let $h_{\mathrm{ss}} \in C^4(\mathbb{R})$ be a non-constant positive X-periodic steady state of the power law evolution $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$. Translate h_{ss} to put its minimum at $x = 0$ so that h_{ss} rescales to k_α for some $\alpha \in (0,1)$, as in §2.3.*
If $q < 1$ or $q > 2$ then h_{ss} is energy unstable in the directions $u = \pm h'_{\mathrm{ss}}$ and $u = \pm h''_{\mathrm{ss}}$.
If $q = 2$, or if $q > 1$ and $E'(\alpha) > 0$, then h_{ss} is energy unstable.

We prove the theorem in Section 4.2. Its first statement follows immediately from Theorem 1, since $r(y) = \mathcal{B}y^{q-1}$ is strongly convex ($r'' > 0$) when $q < 1$ or $q > 2$. The final statement of the theorem certainly applies when $q \geq 2$, since then $E' > 0$ by [14, Theorem 11]. Computational studies [13, §6.1] suggest $E'(\alpha) > 0$ for all $\alpha \in (0,1)$ except when $q \in [1, 1.794]$ (approximately); Figure 5 plots $E(\alpha)$ for certain of the exceptional q-values.

To help explain the appearance in the last part of the theorem of the criterion $E'(\alpha) > 0$ ('monotonicity of the area map'), we refer the reader to [14, §6.3], and to [15, §3] for a 'bifurcation diagram' interpretation.

The case $1 < q < 2$ with $E'(\alpha) \leq 0$ is not covered by Theorem 2. By [14, Theorem 9] we know h_{ss} is linearly stable in this case. We have been unable to prove (nonlinear) stability, but we do prove in Theorem 3 below that if $E'(\alpha) < 0$ then small perturbations of h_{ss} in every possible direction strictly increase the energy; this is consistent with stability.

Definition. Let $h_{\mathrm{ss}} \in C^4(\mathbb{T}_X)$ be a non-constant positive periodic steady state of (1). Call h_{ss} *energy stable* (with respect to zero–mean perturbations at period X) if for each $u \in H^1(\mathbb{T}_X) \setminus \{0\}$ with $\int_0^X u\,dx = 0$ we have

$$\mathcal{E}(h_{\mathrm{ss}} + \varepsilon u) > \mathcal{E}(h_{\mathrm{ss}}) \qquad \text{for all small } \varepsilon > 0.$$

It is perhaps conceivable that an energy stable steady state might not be a local minimum of the energy. (An example from calculus is $F(x,y) = (y - x^2)(y - 3x^2)$, for which the origin is a minimum point on each straight line $y = kx$, though it is not a local minimum in $\mathbb{R}^2$.)

Another cautionary note is that the energy is insensitive to translation; in particular, a steady state and its translates have the same energy. Now, numerical simulations in our article [15, §5.4] demonstrate that perturbations of an energy stable steady state will generically evolve towards some *translate* of that steady state. This suggests that any asymptotic stability result that can be proved will hold only up to translation, unless evenness (or some other condition) is imposed on the solution.

In Section 4.3 we prove a result complementary to Theorem 2:

Theorem 3. *Let $h_{\rm ss} \in C^4(\mathbb{R})$ be a non-constant positive periodic steady state of the power law equation $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$, with $h_{\rm ss}$ having least period X. Translate $h_{\rm ss}$ to put its minimum at $x = 0$ so that $h_{\rm ss}$ rescales to k_α for some $\alpha \in (0,1)$, as in §2.3.*

If $1 < q < 2$ and $E'(\alpha) < 0$ then $h_{\rm ss}$ is energy stable.

The hypothesis $E'(\alpha) < 0$ seems numerically to hold for all $\alpha \in (0,1)$ when $1 < q \leq 1.75$, as indicated by Figure 3 in [14].

We have no energy stability result when $1 < q < 2$ and $E'(\alpha) = 0$; fortunately this case seems rare, with $E'(\alpha) = 0$ for at most *one* α-value, for each q, as shown numerically by Figures 3–5 in [14]. For such an α-value, Theorem 9 of [14] implies linear stability, but [14, Theorem 10(b)] shows the space of neutrally stable directions contains a second direction in addition to the 'translational' one, and this probably leads to instability.

We next address the $q = 1$ case.

Lemma 4. *Let $q = 1$ (i.e. $m = n$) and suppose $h_{\rm ss} \in C^4(\mathbb{R})$ is a non-constant positive periodic steady state of $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^n h_x)_x$ with least period X, and translate so that $h_{\rm ss}$ has its minimum at $x = 0$. Then $h_{\rm ss}$ is not asymptotically stable in $H^1(\mathbb{T}_X)$ with respect to even perturbations.*

[The $q = 1$ steady states are of course not asymptotically stable with respect to *general* perturbations either, since one can always perturb by translating the steady state a small distance — this remark applies for all q.]

Proof. The steady state equation (6) with $q = 1$ has general solution $h_{\rm ss}(x) = D/\mathcal{B} + c\cos(\sqrt{\mathcal{B}}x)$, where we have used that $h_{\rm ss}$ has an extremum at $x = 0$. Hence the period is $X = 2\pi/\sqrt{\mathcal{B}}$, and for all small ε we see that the perturbed function $h_{\rm ss}(x) + \varepsilon\cos(\sqrt{\mathcal{B}}x)$ is another positive periodic steady state solution. Thus $h_{\rm ss}$ is not asymptotically stable in $H^1(\mathbb{T}_X)$ with respect to even perturbations. □

To summarize, when $q = 1$ the positive periodic steady states are linearly neutrally stable with respect to zero–mean perturbations of the same period, by [9] or [14, Lemma 8], and are not even asymptotically stable with respect to 'even' perturbations, by the above lemma. Our numerical simulations in [15, §5.3] and those of [9] suggest that a wide range of small even perturbations yield solutions relaxing to nearby positive periodic steady states, suggesting steady states are stable (though not asymptotically stable).

2.5. **Odd perturbations.** Returning momentarily to general coefficients f and g, in Section 4.4 we prove the energy increases under *odd* perturbations, when r is concave.

Theorem 5. *Let $h_{\mathrm{ss}} \in C^4(\mathbb{R})$ be a non-constant positive periodic steady state of (1) with coefficient functions $f, g \in C^2(0,\infty), f > 0$. Suppose h_{ss} has least period X, and translate h_{ss} to put its minimum at $x = 0$.*

If $r = g/f$ is strongly concave ($r'' < 0$) then for every nontrivial $u \in H^1(\mathbb{T}_X)$ that is odd and is such that $h_{\mathrm{ss}} + u > 0$, we have $\mathcal{E}(h_{\mathrm{ss}} + u) > \mathcal{E}(h_{\mathrm{ss}})$.

The theorem is global since the perturbations are not required to be small, and it is consistent with asymptotic stability (although convergence to a *translate* of h_{ss} seems more likely than convergence to h_{ss} itself).

Theorem 5 applies in the power law case with $1 < q < 2, r(y) = y^{q-1}$. Another example with strongly concave r is an equation [16, eqn. (24)] describing the dynamics of a population of aphids, for which $f(y) = y, g(y) = y - c$ and $r(y) = 1 - c/y$.

2.6. **Relation between the periodic and Neumann stability problems.** Suppose h_{ss} is an even X-periodic steady state of the evolution equation (1) with extrema at $x = 0, \pm X/2, \ldots$, so that $h'_{\mathrm{ss}} = h'''_{\mathrm{ss}} = 0$ at these points. As we observed at the end of [14, §2.5], linear instability of h_{ss} with respect to periodic boundary conditions on $(-X/2, X/2)$ is equivalent to linear instability with respect to Neumann ('no flux') conditions on the half-interval $(0, X/2)$; these Neumann conditions are: $h_x = h_{xxx} = 0$ at $x = 0, X/2$.

The energy of a positive smooth solution is still dissipated in the case of Neumann boundary conditions, and obviously energy instability of h_{ss} in an *even* direction $u(x)$ for the periodic problem on $(-X/2, X/2)$ is equivalent to energy instability in the direction $u(x)$ for the Neumann problem on $(0, X/2)$. (If the perturbation $u(x)$ is even and has mean value zero on $(-X/2, X/2)$ then it has mean value zero on $(0, X/2)$ as well.) Similarly, 'periodic' energy stability in all even directions on $(-X/2, X/2)$ is equivalent to 'Neumann' energy stability in all directions on $(0, X/2)$.

In short, for the Neumann problem on $(0, X/2)$, the stability result in Theorem 3 still holds, and the instability claims involving $\pm h''_{\mathrm{ss}}$ in Theorems 1 and 2 also still hold, since these are even functions; the claims involving $\pm h'_{\mathrm{ss}}$ do not carry over, since those are odd.

3. Relative energy levels of periodic, constant and droplet steady states

In this section we investigate the phase space of the power law equation $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$ by comparing the value of the energy at positive periodic, constant and zero-angle droplet steady states. Let $X > 0$ and recall $q = m - n + 1$.

3.1. **Positive periodic vs. constant steady states.** The fluid volume $\int_0^X h(x,t)\,dx$ is conserved by the evolution, under periodic boundary conditions, and so the mean value

$$\overline{h} := \frac{1}{X}\int_0^X h\,dx$$

is constant in time. Suppose the initial data $h(\cdot,0)$ arises from a small zero-mean perturbation of h_{ss}. It is natural to ask whether h can converge (while staying positive and smooth) towards the constant steady state $\overline{h_{\text{ss}}}$, as $t \to \infty$. This *cannot* happen if $\mathcal{E}(h_{\text{ss}}) < \mathcal{E}(\overline{h_{\text{ss}}})$ and if also $h(\cdot,0)$ is so close to h_{ss} that $\mathcal{E}(h(\cdot,0)) < \mathcal{E}(\overline{h_{\text{ss}}})$.

Theorem 6. *Let $h_{\text{ss}} \in C^4(\mathbb{R})$ be a non-constant positive periodic steady state of the power law equation $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$, with least period X. Translate h_{ss} to put its minimum at $x = 0$ so that h_{ss} rescales to k_α for some $\alpha \in (0,1)$, as in §2.3.*

If $q \geq 2$ or $q < 1$ then $\mathcal{E}(h_{\text{ss}}) > \mathcal{E}(\overline{h_{\text{ss}}})$.

If $q = 1$ then $\mathcal{E}(h_{\text{ss}}) = \mathcal{E}(\overline{h_{\text{ss}}})$.

If $1 < q < 2$ and $E' > 0$ on $(\alpha,1)$ then $\mathcal{E}(h_{\text{ss}}) > \mathcal{E}(\overline{h_{\text{ss}}})$.

If $1 < q < 2$ and $E' < 0$ on $(\alpha,1)$ then $\mathcal{E}(h_{\text{ss}}) < \mathcal{E}(\overline{h_{\text{ss}}})$.

The theorem is proved in Section 5.1.

The last two statements of the theorem deserve some explanation. When $1 < q \leq 1.75$, numerical evidence in Figure 3 of [14] suggests $E'(\alpha) < 0$ for all α. If this is true, then $\mathcal{E}(h_{\text{ss}}) < \mathcal{E}(\overline{h_{\text{ss}}})$ by the theorem. When $1.75 < q \leq 1.794$, the sign of E' can be determined from Figure 5 below.

When $1.795 \leq q < 2$, Figure 5 of [14] suggests $E'(\alpha) > 0$ for all α and so $\mathcal{E}(h_{\text{ss}}) > \mathcal{E}(\overline{h_{\text{ss}}})$ by the theorem. Thus when $q \geq 1.795$ or $q < 1$, the instability result Theorem 2 and the energy level result Theorem 6 together lead us to suspect the existence of a heteroclinic connection from h_{ss} to $\overline{h_{\text{ss}}}$. In Figure 3 we present numerical simulations of such heteroclinic

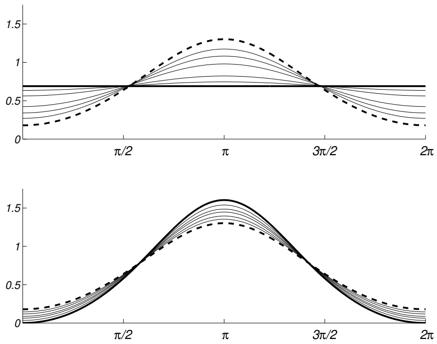

FIGURE 3. $q = 2.5$ and $n = 1$, dashed line: initial data.

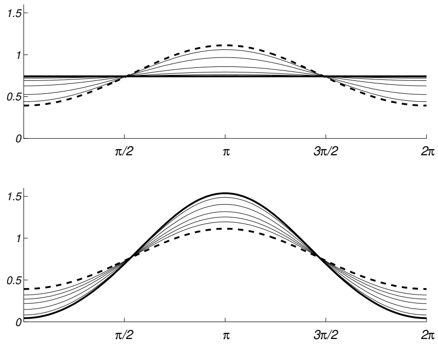

FIGURE 4. $q = 1.768$, $n = 1$, dashed line: initial data.

orbits, taken from the companion article [15, §5.6]. The top part of the figure presents an orbit connecting the positive periodic steady state to the constant steady state. The bottom part presents a solution connecting in finite time a perturbation of the same positive periodic steady state to a droplet-type profile (which presumably then continues to evolve towards a droplet steady state).

For the Cahn–Hilliard equation, the analogue of Theorem 6 (comparing energy levels of non-constant and constant steady states) can be found in Grinfeld and Novick–Cohen's work [10, Theorem 4.1]; further, [10, §7] discusses a number of results on existence of heteroclinic connections. See also [17, §3.4].

Incidentally, the van der Waals equation (3) has $q = -3$, and so by Theorem 6, the energy of any positive periodic steady state is greater than that of the constant steady state. This was observed numerically by Witelski and Bernoff [24, §3].

3.2. **Positive periodic vs. droplet steady states.** We do not yet have stability results for *droplet* steady states. But here we do show under certain conditions that the energy of a zero-angle droplet steady state must be lower than that of any positive periodic steady state h_{ss} whose period exceeds the length of the droplet. In our article [15, §5] we show numerically that in these cases, the droplet seems to be strongly attracting, as shown in the bottom part of Figure 3.

Theorem 7. *Let $h_{\text{ss}} \in C^4(\mathbb{R})$ be a non-constant positive periodic steady state of the power law equation $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$, with h_{ss} having least period X and area $A_{\text{ss}} = \int_0^X h_{\text{ss}}\, dx$.*

If $-1 < q < 1$ or $2 \le q < 3$, or if $1 < q < 2$ and $E'(\alpha) > 0$ for all $\alpha \in (0,1)$, then there exists a zero contact angle droplet steady state $\hat{h}_{\text{ss}}$ with length $\hat{X} < X$ and area $\hat{A} = A_{\text{ss}}$. Furthermore $\mathcal{E}(\hat{h}_{\text{ss}}) < \mathcal{E}(h_{\text{ss}})$.

We prove the theorem in Section 5.2.

Remarks

1. Steady states with zero contact angle can occur only for $q > -1$, by [13, §2.2], and so we do not consider $q \le -1$ in the theorem.

2. We think the theorem applies to $1.795 \le q < 2$, since it seems numerically that $E' > 0$ for those q-values. For $q \ge 3$ we think a similar theorem might hold but with $\hat{h}_{\text{ss}}$ being a *configuration* of disjoint zero–angle droplets.

3. Since $\mathcal{E}(h_{\text{ss}}) > \mathcal{E}(\hat{h}_{\text{ss}})$, there might be an orbit from h_{ss} to $\hat{h}_{\text{ss}}$. This orbit might describe a positive solution that converges to the nonnegative droplet profile as $t \to \infty$, or it might describe a positive solution that loses positivity in finite time and *then* approaches the droplet as a nonnegative weak solution. For $-1 < q < 1$ and $2 \le q < 3$, Theorems 6 and 7 are consistent with the idea that the unstable positive periodic steady state h_{ss} and

its stable manifold form a separatrix between the basin of attraction of the constant steady state and the basin of attraction of the droplet steady state. In particular, after perturbing h_{ss} in one direction one seems to find a solution that converges to the constant steady state, while perturbing in the opposite direction often yields a solution that converges to a droplet profile. We present some numerical evidence for such behavior in Figure 3, and discuss this at length in the companion article [15, §5]. See also the 'mountain pass' remark after Theorem 11.

For $q \in (1, 1.75]$, we believe there does not even *exist* a zero contact angle steady state with length less than X:

Theorem 8. *Let $1 < q \leq 1.75$ and suppose $E'(\alpha) < 0$ for all $\alpha \in (0,1)$. Let $h_{\mathrm{ss}} \in C^4(\mathbb{T}_X)$ be a non-constant positive periodic steady state of the power law equation $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$, with h_{ss} having least period X and area $A_{\mathrm{ss}} = \int_0^X h_{\mathrm{ss}}\, dx$. Assume $\hat{h}_{\mathrm{ss}}$ is nonnegative and piecewise-C^1 on $\mathbb{T}_X$, has area A_{ss}, and is smooth on the set where it is positive and satisfies there the 'nonlinear oscillator' steady state equation (6).*

Then $\hat{h}_{\mathrm{ss}}$ is either constant or is a translate of h_{ss}, or is a configuration of nonzero contact angle droplet steady states. Specifically, $\hat{h}_{\mathrm{ss}}$ cannot be a zero contact angle droplet steady state.

We prove the theorem in Section 5.3. The hypothesis $E' < 0$ seems to hold for $1 < q \leq 1.75$, by the numerical evidence in Figure 3 of [14].

Finally, for $q \in (1.75, 1.794]$ approximately, we know by the analytical and numerical work in [13, §5.1] that there can be *two* positive periodic steady states with the same period and area. The next theorem shows that the steady state with smaller minimum value (and larger amplitude) is energy stable, and has lower energy than the other one, which is energy unstable. The bottom plot in Figure 4 presents a numerical simulation of an apparent orbit connecting these two steady states [15, §5.5]. (The top plot presents an orbit connecting the unstable positive periodic steady state to the constant steady state, similar to the top plot of Figure 3.)

Theorem 9. *Assume $1 < q < 2$ and there exists $\alpha_{crit} \in (0,1)$ with $E'(\alpha) < 0$ on $(0, \alpha_{crit})$ and $E'(\alpha) > 0$ on $(\alpha_{crit}, 1)$, and assume $\alpha \mapsto \alpha P(\alpha)^{2/(q-1)}$ is strictly increasing for $\alpha \in (0,1)$.*

Suppose $h_{\mathrm{ss}1}$ and $h_{\mathrm{ss}2}$ are non-constant positive periodic steady states of the power law equation $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$, with $h_{\mathrm{ss}1}$ and $h_{\mathrm{ss}2}$ having the same least period X and same area $\int_0^X h_{\mathrm{ss}1}\, dx = \int_0^X h_{\mathrm{ss}2}\, dx$.

If $h_{\mathrm{ss}1}(x)$ has lower minimum value than $h_{\mathrm{ss}2}(x)$, then $h_{\mathrm{ss}1}$ is energy stable, $h_{\mathrm{ss}2}$ is energy unstable, and $\mathcal{E}(h_{\mathrm{ss}1}) < \mathcal{E}(h_{\mathrm{ss}2})$. Furthermore, $\mathcal{E}(h_{\mathrm{ss}2}) > \mathcal{E}(\overline{h_{\mathrm{ss}2}})$.

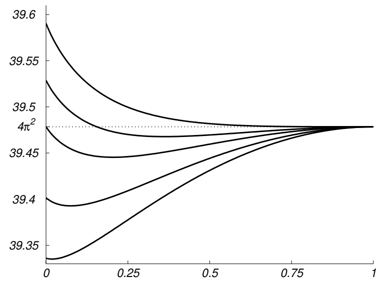

FIGURE 5. Plots of $E(\alpha)$ for $q = 1.75, 1.76, 1.768, 1.78, 1.79$. Top curve: $q = 1.75$, the curves move down as q increases.

We prove this in Section 5.4. The hypothesis about $E(\alpha)$ being first strictly decreasing and then strictly increasing is confirmed numerically for q in the interval $(1.750, 1.794]$ by [13, §6.1]; see Figure 5. Numerical work also confirms the hypothesis about $\alpha P(\alpha)^{2/(q-1)}$ being a strictly increasing function of α for all $q \in (1, 2)$.

Theorem 9 is analogous to [10, Theorem 4.1(v)] for the transitional and metastable cases of the Cahn–Hilliard equation, where Grinfeld and Novick–Cohen show the energy of a monotonic 'interface' solution is less than that of a monotonic 'spike' solution having the same length and area.

3.3. **Constant steady states: stability results.** For any number $\overline{h} > 0$ the constant function $h_{\mathrm{ss}} \equiv \overline{h}$ is a steady state of the general evolution equation (1). We now develop analogues, for this constant steady state, of our earlier stability results. Then in the following subsection we can compare the energies of the constant and droplet steady states, for power law coefficients.

We start by recalling a characterization of linear instability for steady states. In [14, Appendix A], the evolution (1) was linearized around an X-periodic positive steady state $h_{\mathrm{ss}} \in C^4(\mathbb{R})$, reducing the linear stability question to determining the lowest eigenvalue of a fourth order eigenproblem. By integrating up, we reduced to a second order self-adjoint problem. The end result was the following instability criterion: h_{ss} is linearly unstable with respect to zero–mean perturbations at period X if and only if $\tau_1(h_{\mathrm{ss}}) < 0$, where the eigenvalue τ_1 is given by the Rayleigh principle

$$\tau_1(h_{\mathrm{ss}}) = \min_u \frac{\int_0^X [(u')^2 - r(h_{\mathrm{ss}})u^2]\,dx}{\int_0^X u^2\,dx}; \tag{11}$$

the minimum here is taken over $u \in H^1(\mathbb{T}_X)$ with $\int_0^X u\,dx = 0, u \not\equiv 0$.

Theorem 10. *Let $f, g \in C^2(0,\infty)$ with $f > 0$, and write $r = g/f$. Let $\overline{h}, X > 0$. Then for the constant steady state $h_{\mathrm{ss}} \equiv \overline{h}$ of (1), the eigenvalue τ_1 in (11) is $\tau_1(\overline{h}) = (2\pi/X)^2 - r(\overline{h})$. The $\tau_1(\overline{h})$-eigenspace is spanned by $\sin(2\pi x/X)$ and $\cos(2\pi x/X)$.*

Thus with respect to zero–mean perturbations at period X, the constant steady state is

$$\begin{aligned} \textit{linearly unstable} \quad \textit{if} \quad & r(\overline{h})X^2 > 4\pi^2, \\ \textit{linearly neutrally stable} \quad \textit{if} \quad & r(\overline{h})X^2 = 4\pi^2, \\ \textit{linearly asymptotically stable} \quad \textit{if} \quad & r(\overline{h})X^2 < 4\pi^2. \end{aligned}$$

(a) If $r(\overline{h})X^2 > 4\pi^2$, or if $r(\overline{h})X^2 = 4\pi^2$ and $r''(\overline{h}) > 0$, then the constant steady state is energy unstable in the directions $\pm\sin(2\pi x/X)$ and $\pm\cos(2\pi x/X)$.

(b) If $r(\overline{h})X^2 < 4\pi^2$, or if $r(\overline{h})X^2 = 4\pi^2$ and $r''(\overline{h}) < 0$, then the constant steady state is energy stable with respect to zero–mean perturbations of period X. In fact, if $r(\overline{h})X^2 < 4\pi^2$ then the constant steady state is a strict local minimum of the energy with respect to zero–mean perturbations in $H^1(\mathbb{T}_X)$, and $\overline{h}$ is stable under the evolution (1), in an $H^1(\mathbb{T}_X)$-sense made precise in the proof.

We prove the theorem in Section 5.5. Its linear stability assertions are well-known and are included only for the sake of completeness.

Goldstein, Pesci and Shelley [9, §IIIB] used the energy to prove (nonlinear) instability of the constant steady state for the $q = 1$ case ($f(y) = y^n, g(y) = \mathcal{B}y^n, X = 2\pi$) with either $2 \leq \mathcal{B} < 4$ or $\mathcal{B} = j^2$ for some integer $j \geq 2$.

3.4. **Constant vs. droplet steady states.** Consider power law coefficients, so that $r(y) = \mathcal{B}y^{q-1}$; then the previous theorem shows that the stability of the constant steady state $\overline{h}$ is determined by whether the quantity $\mathcal{B}\overline{h}^{q-1}X^2$ is $>, =$ or $< 4\pi^2$.

Fix $X > 0$. Does a zero-angle droplet steady state exist with length at most X and with the same area $\overline{h}X$ as the constant steady state? If such a droplet steady state exists, can it have lower energy than the constant steady state?

In this direction, in Section 5.6 we prove:

Theorem 11. *Let $\overline{h}, X > 0$, and consider the constant steady state $h_{ss} \equiv \overline{h}$ of the power law evolution equation $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$.*

(a) Suppose $-1 < q < 3$. Then there exists a zero contact angle droplet steady state $\hat{h}_{ss}$ of length $\hat{X} \leq X$ and area $\overline{h}X$ if and only if

$$\mathcal{B}\overline{h}^{q-1}X^2 \geq E(0) =: E_0(q). \tag{12}$$

If such a droplet steady state exists, then $\mathcal{E}(\overline{h}) > \mathcal{E}(\hat{h}_{ss})$ if and only if

$$\mathcal{B}\overline{h}^{q-1}X^2 > A(0)^2 \left[\frac{3+q}{(3-q)(q+1)}\right]^{(3-q)/q} =: L(q) \qquad (\textit{for } -1 < q < 3, q \neq 0) \tag{13}$$

or $\mathcal{B}\overline{h}^{-1}X^2 > 4e^2\pi/3 =: L(0)$ (for $q = 0$).

(b) For $q = 3$, such a droplet steady state $\hat{h}_{\rm ss}$ exists if and only if $\mathcal{B}\overline{h}^{2} X^2 = E(0)$. For $q > 3$, $\hat{h}_{\rm ss}$ exists if and only if $\mathcal{B}\overline{h}^{q-1} X^2 \leq E(0)$. For all $q \geq 3$, if $\hat{h}_{\rm ss}$ exists then $\mathcal{E}(\overline{h}) < \mathcal{E}(\hat{h}_{\rm ss})$.

To understand conditions (12) and (13), see the plots of $E_0(q)$ and $L(q)$ in Figures 6 and 7 (the plots were constructed using the formulas for $A(0), P(0), E(0)$ in [13, §3.1.2]). The

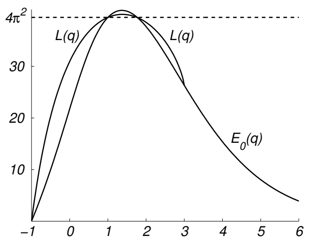

FIGURE 6. Plots of $E_0(q)$ and $L(q)$. Note $L(q)$ is only defined for $-1 < q < 3$.

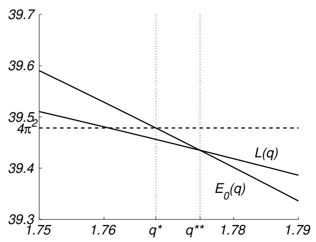

FIGURE 7. Close-up view near $q^{**} \approx 1.775$.

graphs of E_0 and L intersect at $q = -1, 1, 3$ and at $q^{**} \approx 1.775$. For $1 < q < q^{**}$ the figure suggests $E_0 > L$, and so if the droplet steady state in Theorem 11(a) exists then it certainly has lower energy than the constant steady state. On the other hand, it appears that $E_0 < L$ when $-1 < q < 1$ and when $q^{**} < q < 3$; in these cases the energy condition (13) may or may not hold when the existence condition (12) holds, so that the energy of the droplet steady state (if it exists) might be higher or lower than that of the constant steady state.

The dashed line at height $4\pi^2$ in Figures 6 and 7 intersects $E_0(q)$ at 1 and $q^* \approx 1.768$, and intersects $L(q)$ at 1 and 1.761 (approx.). This line matters because the constant steady state is a strict local minimum for the energy when $\mathcal{B}\overline{h}^{q-1} X^2 < 4\pi^2$, by Theorem 10(b). For example, suppose that $2 \leq q < 3$ and $\overline{h}$ and X are such that $E_0(q) < L(q) < \mathcal{B}\overline{h}^{q-1} X^2 < 4\pi^2$. Then the constant steady state $\overline{h}$ is a strict local minimum of the energy but is *not* a global minimum since $\mathcal{E}(\overline{h}) > \mathcal{E}(\hat{h}_{\rm ss})$ by Theorem 11(a). A mountain pass idea then suggests the energy might have a saddle point at which its value is greater than $\mathcal{E}(\overline{h})$. Such a saddle ought to be an energy unstable positive periodic steady state, and should have period X and area $\overline{h}X$. In fact we already know such a positive periodic steady state exists, by [14, Theorem 12] and Theorem 6 of this paper; we illustrate it in Figure 3. Perturbing from the saddle in one direction leads to relaxation to the constant (see the top part of Figure 3), while perturbing in the opposite direction yields apparent relaxation to a droplet (the bottom part of Figure 3).

4. PROOFS OF THEOREMS 1–5

4.1. **Proof of Theorem 1.** Recall that the positive periodic steady state h_{ss} satisfies equations (4) and (5):

$$h_{ss}''' + r(h_{ss})h_{ss}' = 0, \qquad h_{ss}'' + H'(h_{ss}) = \text{const.}, \tag{14}$$

where $H''(y) = r(y)$ is defined for $y > 0$.

We compute the first four variations of the energy (2). Let $u \in H^1(\mathbb{T}_X)$ have mean value zero. For all ε small enough that $h_{ss} + \varepsilon u > 0$ (so that $H(h_{ss} + \varepsilon u)$ makes sense), we have

$$\frac{d}{d\varepsilon}\mathcal{E}(h_{ss} + \varepsilon u)\Big|_{\varepsilon=0} = -\int_0^X \left[h_{ss}'' + H'(h_{ss})\right] u\,dx = 0 \tag{15}$$

by (14), since u has mean value zero,

$$\frac{d^2}{d\varepsilon^2}\mathcal{E}(h_{ss} + \varepsilon u)\Big|_{\varepsilon=0} = \int_0^X \left[(u')^2 - r(h_{ss})u^2\right]\,dx \tag{16}$$

$= \text{numerator of Rayleigh quotient (11) for } \tau_1(h_{ss}),$

$$\frac{d^3}{d\varepsilon^3}\mathcal{E}(h_{ss} + \varepsilon u)\Big|_{\varepsilon=0} = -\int_0^X r'(h_{ss})u^3\,dx, \tag{17}$$

$$\frac{d^4}{d\varepsilon^4}\mathcal{E}(h_{ss} + \varepsilon u)\Big|_{\varepsilon=0} = -\int_0^X r''(h_{ss})u^4\,dx. \tag{18}$$

If the steady state h_{ss} is linearly unstable with respect to zero–mean perturbations at period X, then the numerator of the Rayleigh quotient (11) is negative for some zero–mean trial function $u \in H^1(\mathbb{T}_X) \setminus \{0\}$. We can assume $u \in C^\infty(\mathbb{T}_X)$. From (16), the second variation of the energy in the direction u is negative, so that h_{ss} is energy unstable in the direction u as desired.

From Theorems 1 and 3 in [14], h_{ss} is linearly unstable with respect to zero–mean perturbations at period X if it is non-constant and: either the least period of h_{ss} is X/j for some integer $j \geq 2$, or $r = g/f$ is convex ($r'' \geq 0$) and non-constant on the range of h_{ss}. Hence h_{ss} is energy unstable in those situations.

Now assume h_{ss} is non-constant and r is strongly convex ($r'' > 0$). We consider the second variation of $\mathcal{E}$ in the direction $u = \pm h_{ss}''$:

$$\begin{aligned}
\frac{d^2}{d\varepsilon^2}\mathcal{E}(h_{ss} \pm \varepsilon h_{ss}'')\Big|_{\varepsilon=0} &= \int_0^X \left[{h_{ss}'''}^2 - r(h_{ss}){h_{ss}''}^2\right] dx \\
&= -\int_0^X r(h_{ss})\left[h_{ss}'h_{ss}''' + {h_{ss}''}^2\right] dx \qquad \text{by (14)} \\
&= -\int_0^X r(h_{ss})\left[h_{ss}'h_{ss}''\right]' dx = \int_0^X r'(h_{ss}){h_{ss}'}^2 h_{ss}''\,dx \\
&= \frac{1}{3}\int_0^X r'(h_{ss})\left[{h_{ss}'}^3\right]' dx = -\frac{1}{3}\int_0^X r''(h_{ss}){h_{ss}'}^4\,dx. \qquad (19)
\end{aligned}$$

By assumption, $r''(h_{ss}(x)) > 0$ for all x. Since h'_{ss} is not identically zero, the second variation in the direction $u = \pm h''_{ss}$ is negative, and so $\pm h''_{ss}$ is an energy unstable direction for h_{ss}.

It remains to prove h_{ss} is energy unstable in the directions $u = \pm h'_{ss}$. Here the second variation is zero, since (16) becomes

$$\int_0^X \left[(h''_{ss})^2 - r(h_{ss})(h'_{ss})^2 \right] dx = -\int_0^X \left[h'''_{ss} + r(h_{ss}) h'_{ss} \right] h'_{ss} \, dx = 0 \qquad \text{by parts and (14).}$$

The third variation is zero for $u = \pm h'_{ss}$ by (17), because h_{ss} is even about its minimum point while $(\pm h'_{ss})^3$ is odd, by uniqueness for the nonlinear oscillator equation $h''_{ss} + H'(h_{ss}) =$ const in (14). The fourth variation is negative by (18), because $r''(h_{ss}(x)) > 0$ by assumption. Hence h_{ss} is energy unstable in the directions $u = \pm h'_{ss}$, completing the proof. □

A number of other authors, working on closely related topics, have noted that translation invariance of the evolution implies the second variation of the energy in the direction h'_{ss} is zero [6, 7, 19]. Those authors then proved energy instability in the direction $u = h'_{ss} + \eta$ for some small function η. Their arguments relied either on h_{ss} having least period X/j for some $j \geq 2$ or else they did not impose a zero–mean requirement on the perturbation. Above, we have relied instead on the assumption $r'' > 0$.

4.2. **Proof of Theorem 2.**

When $q < 1$ or $q > 2$, Theorem 2 follows from the last statement of Theorem 1 since $r(y) = \mathcal{B} y^{q-1}$ is strongly convex.

When $q = 2$ and $u = -h''_{ss}$, formula (19) shows the second variation of the energy is zero, since $r(y) = \mathcal{B} y$ and $r'' \equiv 0$. The third variation is negative because

$$\begin{aligned}
\frac{d^3}{d\varepsilon^3} \mathcal{E}(h_{ss} - \varepsilon h''_{ss}) \bigg|_{\varepsilon=0} &= \mathcal{B} \int_0^X (h''_{ss})^3 \, dx && \text{by (17)} \\
&= -2\mathcal{B} \int_0^X h'_{ss} h''_{ss} h'''_{ss} \, dx && \text{by parts} \\
&= 2\mathcal{B}^2 \int_0^X h_{ss} (h'_{ss})^2 h''_{ss} \, dx && \text{since } h'''_{ss} = -\mathcal{B} h_{ss} h'_{ss} \text{ by (14)} \\
&= -\frac{2}{3} \mathcal{B}^2 \int_0^X (h'_{ss})^4 \, dx < 0.
\end{aligned}$$

Thus the steady state is energy unstable in the direction $u = -h''_{ss}$.

Now suppose $q > 1$ and $E'(\alpha) > 0$. To obtain the unstable direction u, start by rescaling k_α (defined in §2.3) to give a new function

$$K_\alpha(x) := \frac{P(\alpha)}{A(\alpha)} k_\alpha \left(P(\alpha) x \right). \tag{20}$$

By construction, K_α has period 1 and mean value 1. Then define

$$\kappa_\alpha(x) := \frac{\partial}{\partial\alpha} K_\alpha(x);$$

κ_α is well-defined and smooth because P and A depend smoothly on α while $k_\alpha(x)$ is jointly smooth in (α, x). Notice κ_α is even in x, has period 1, and has mean value zero:

$$\int_0^1 \kappa_\alpha(x)\,dx = \frac{\partial}{\partial\alpha}\int_0^1 K_\alpha(x)\,dx = \frac{\partial}{\partial\alpha}(1) = 0.$$

(See [14, §5.4] for more on κ_α.) Let $u(x) = \pm\kappa_\alpha(x/X)$.

We want to show u is an unstable direction. First we show that composing the rescalings of $h_{\rm ss}$ to k_α and then k_α to K_α yields

$$\mathcal{B}X^2 h_{\rm ss}(xX)^{q-1} = EK_\alpha(x)^{q-1}. \tag{21}$$

Indeed, from the definition (20) of K_α, the righthand side of (21) reduces to $P^2 k_\alpha(Px)^{q-1}$, and then one can substitute for k_α in terms of $h_{\rm ss}$ using (7). Next one obtains the lefthand side of (21) by using the relation $P = (\mathcal{B}/D)^{1/2q} D^{1/2} X$ which relates the periods of k_α and $h_{\rm ss}$ (P and X, respectively).

Now that we have (21), use (16) to compute the second variation of the energy in the direction $u(x) = \pm\kappa_\alpha(x/X)$ as

$$\begin{aligned}
\int_0^X \left[\kappa_\alpha'(x/X)^2/X^2 - \mathcal{B}h_{\rm ss}(x)^{q-1}\kappa_\alpha(x/X)^2\right]dx \; &= \int_0^1 \left[\kappa_\alpha'(x)^2 - \mathcal{B}X^2 h_{\rm ss}(xX)^{q-1}\kappa_\alpha(x)^2\right]dx/X \\
&= \int_0^1 \left[(\kappa_\alpha')^2 - EK_\alpha^{q-1}\kappa_\alpha^2\right]dx/X \qquad \text{by (21)} \\
&< 0
\end{aligned}$$

by the proof of [14, Prop. 14], which uses both $q > 1$ and $E'(\alpha) > 0$. Thus the steady state is energy unstable in the direction $u = \pm\kappa_\alpha(x/X)$. □

4.3. **Proof of Theorem 3.** Since $E'(\alpha) < 0$ by hypothesis, Proposition 15 of [14] implies $\mu_1(\alpha) \geq 0$, where

$$\mu_1(\alpha) := \min\left\{ \frac{\int_0^1 [(v')^2 - E(\alpha)K_\alpha^{q-1}v^2]\,dx}{\int_0^1 v^2\,dx} : v \in H^1(\mathbb{T}_1)\setminus\{0\}, \int_0^1 v(x)\,dx = 0 \right\}.$$

Notice $\mu_1(\alpha) = X^2\tau_1(h_{\rm ss})$, by letting $v(x) = u(xX)$ and using the identity (21) and the definition (11) of τ_1. Hence $\tau_1(h_{\rm ss}) \geq 0$.

Consider $u \in H^1(\mathbb{T}_X)\setminus\{0\}$ with mean value zero. The first variation of $\mathcal{E}$ in the direction u is zero by (15), and the second variation of $\mathcal{E}$ in (16) is nonnegative because it equals the numerator of the Rayleigh quotient for $\tau_1(h_{\rm ss})$.

If the second variation is positive then $h_{\rm ss}$ is energy stable in the direction $h_{\rm ss}$ and we are done. If the second variation is zero then the Rayleigh quotient of u is zero, and so

$\tau_1(h_{ss}) = 0$ and u minimizes the Rayleigh quotient for τ_1 in (11). Hence u satisfies the Euler–Lagrange condition $u'' + r(h_{ss})u = \text{const}$, and so it satisfies $\mathcal{L}u = 0$ where $\mathcal{L}$ is the linearized operator defined in [14, eq. (4)] (take $a = 0$ there). Theorem 10(a) in [14] and the hypothesis $E'(\alpha) < 0$ now imply u is a multiple of h'_{ss}, and so u is odd. Therefore the third variation (17) of $\mathcal{E}$ in the direction u is zero, because u is odd and h_{ss} is even. The fourth variation is positive by (18) because $r(y) = \mathcal{B}y^{q-1}$ and $r''(y) < 0$ for $1 < q < 2$. Thus h_{ss} is energy stable in the direction u. □

4.4. **Proof of Theorem 5.** As usual, the first variation of $\mathcal{E}$ at h_{ss} is zero by (15). We now prove non-negativity of the second variation, given by (16).

First note that h_{ss} is symmetric about every point at which $h'_{ss} = 0$, by uniqueness for the ODE $h''_{ss} + H'(h_{ss}) = \text{const}$ (see (14)); the uniqueness theorem applies here since the coefficient function H' is C^1 (even C^3) on the range of the positive bounded function h_{ss}.

Since h_{ss} has a minimum at $x = 0$ by hypothesis, we conclude h_{ss} is even and $h'_{ss} > 0$ on $(0, X/2)$ (otherwise h_{ss} would have period less than X).

Consider a minimizer of the Rayleigh quotient (11) with respect to *odd* functions u; it is a smooth odd function $\tilde{u}$ satisfying $\tilde{u}'' + r(h_{ss})\tilde{u} + \tilde{\tau}\tilde{u} = 0$ for some constant $\tilde{\tau}$. Since $\tilde{u}(0) = 0$, one must have $\tilde{u}'(0) \neq 0$ because otherwise $\tilde{u} \equiv 0$ by the uniqueness theorem for linear ODEs. Also, $\tilde{u}(X/2) = 0$ by the oddness and periodicity of $\tilde{u}$, and so there is a point $b \in (0, X/2]$ with $\tilde{u}(b) = 0$ and $\tilde{u} \neq 0$ between 0 and b. Assume $\tilde{u} > 0$ between 0 and b (otherwise consider $-\tilde{u}$). Then

$$\begin{aligned}
\tilde{\tau}\int_0^b \tilde{u}h'_{ss}\,dx &= -\int_0^b [\tilde{u}'' + r(h_{ss})\tilde{u}]\,h'_{ss}\,dx \qquad \text{since } \tilde{u}'' + r(h_{ss})\tilde{u} + \tilde{\tau}\tilde{u} = 0 \\
&= -\tilde{u}'(b)h'_{ss}(b) - \int_0^b [h'''_{ss} + r(h_{ss})h'_{ss}]\,\tilde{u}\,dx \qquad \text{by parts, since } h'_{ss}(0) = 0 \\
&= -\tilde{u}'(b)h'_{ss}(b) \qquad \text{by (14)} \\
&\geq 0,
\end{aligned}$$

because $\tilde{u}'(b) \leq 0$ and $h'_{ss}(b) \geq 0$. Since $\tilde{u}$ and h'_{ss} are positive on $(0, b)$ it follows that $\tilde{\tau} \geq 0$. Hence the second variation of $\mathcal{E}$ in (16) is nonnegative, as desired.

The third variation at $\varepsilon = 0$ is zero by (17), since u is odd and h_{ss} is even.

The fourth variation (18) is positive not just at $\varepsilon = 0$ but at all $\varepsilon \in (0, 1)$:

$$\frac{d^4}{d\varepsilon^4}\mathcal{E}(h_{ss} + \varepsilon u) = -\int_0^X r''(h_{ss} + \varepsilon u)u^4\,dx > 0$$

by the strong concavity of r, provided $u \not\equiv 0$. Taylor's theorem completes the proof, since for some $\tilde{\varepsilon} \in (0, 1)$,

$$\mathcal{E}(h_{ss} + u) = \mathcal{E}(h_{ss}) + \frac{1}{2!}\frac{d^2}{d\varepsilon^2}\mathcal{E}(h_{ss} + \varepsilon u)\bigg|_{\varepsilon=0} + \frac{1}{4!}\frac{d^4}{d\varepsilon^4}\mathcal{E}(h_{ss} + \varepsilon u)\bigg|_{\varepsilon=\tilde{\varepsilon}} > \mathcal{E}(h_{ss}).$$

□

5. Proofs of Theorems 6—11

5.1. Proof of Theorem 6. We start by relating $\mathcal{E}(h_{\rm ss})$ to $\mathcal{E}(k_\alpha)$. Take

$$H(y) := \begin{cases} \frac{1}{q}\left[\frac{y^{q+1}}{q+1} - y\right], & q \neq 0, -1, \\ y \log y - y, & q = 0, \\ y - \log y, & q = -1, \end{cases} \tag{22}$$

so that $H''(y) = y^{q-1}$, and recall from the definition (2) of the energy that

$$\mathcal{E}(h_{\rm ss}) = \int_0^X \left[\frac{1}{2}(h_{\rm ss}')^2 - \mathcal{B}H(h_{\rm ss})\right] dx \quad \text{and} \quad \mathcal{E}(k_\alpha) = \int_0^{P(\alpha)} \left[\frac{1}{2}(k_\alpha')^2 - H(k_\alpha)\right] dx. \tag{23}$$

Denote the period of $h_{\rm ss}$ by $P_{\rm ss} = X$, and the area by $A_{\rm ss} = \int_0^X h_{\rm ss}\,dx$. Writing $P = P(\alpha)$ and $A = A(\alpha)$, the rescaling (7) implies

$$P = \begin{cases} \left(\frac{\mathcal{B}}{D}\right)^{1/2q} D^{1/2} P_{\rm ss}, & q \neq 0, \\ e^{-D/2\mathcal{B}} \mathcal{B}^{1/2} P_{\rm ss}, & q = 0, \end{cases} \quad \text{and} \quad A = \begin{cases} \left(\frac{\mathcal{B}}{D}\right)^{3/2q} D^{1/2} A_{\rm ss}, & q \neq 0, \\ e^{-3D/2\mathcal{B}} \mathcal{B}^{1/2} A_{\rm ss}, & q = 0. \end{cases} \tag{24}$$

Notice that the rescaling (7) can be written as

$$h_{\rm ss}(x) = \frac{A_{\rm ss}}{A}\frac{P}{P_{\rm ss}}\, k_\alpha\left(\frac{P}{P_{\rm ss}}x\right). \tag{25}$$

From (24) we obtain the invariance relation

$$\mathcal{B}P_{\rm ss}^{3-q}A_{\rm ss}^{q-1} = P^{3-q}A^{q-1} = E(\alpha), \tag{26}$$

and this implies

$$\left(\frac{A_{\rm ss}}{A}\right)^2 \left(\frac{P}{P_{\rm ss}}\right)^3 = \mathcal{B}\left(\frac{A_{\rm ss}}{A}\right)^{q+1}\left(\frac{P}{P_{\rm ss}}\right)^q. \tag{27}$$

Using (25), (26), (27) and the definitions (23), we at last deduce a relation between $\mathcal{E}(h_{\rm ss})$ and $\mathcal{E}(k_\alpha)$:

$$\mathcal{B}^{-1}\overline{h_{\rm ss}}^{\,-(q+1)} \frac{\mathcal{E}(h_{\rm ss}) - \mathcal{E}(\overline{h_{\rm ss}})}{P_{\rm ss}} = \overline{k_\alpha}^{\,-(q+1)} \frac{\mathcal{E}(k_\alpha) - \mathcal{E}(\overline{k_\alpha})}{P(\alpha)}, \tag{28}$$

where the mean values are $\overline{h_{\rm ss}} := A_{\rm ss}/P_{\rm ss}$ and $\overline{k_\alpha} := A(\alpha)/P(\alpha)$. (When checking (28), one can omit the linear terms in $H(y)$ from the calculations, since $h_{\rm ss}$ and $\overline{h_{\rm ss}}$ have the same mean value, as do k_α and $\overline{k_\alpha}$.)

In view of (28), then, Theorem 6 follows from:

Proposition 12. *Fix $\alpha_1 \in (0,1)$.*
If $q \geq 2$ or $q < 1$ then $\mathcal{E}(k_\alpha) > \mathcal{E}(\overline{k_\alpha})$ for all $\alpha \in (0,1)$.
If $q = 1$ then $\mathcal{E}(k_\alpha) = \mathcal{E}(\overline{k_\alpha})$ for all $\alpha \in (0,1)$.
If $1 < q < 2$ and $E'(\alpha) > 0$ $\forall\alpha \in (\alpha_1, 1)$ then $\mathcal{E}(k_\alpha) > \mathcal{E}(\overline{k_\alpha})$ $\forall\alpha \in [\alpha_1, 1)$.

If $1 < q < 2$ and $E'(\alpha) < 0$ $\forall \alpha \in (\alpha_1, 1)$ then $\mathcal{E}(k_\alpha) < \mathcal{E}(\overline{k_\alpha})$ $\forall \alpha \in [\alpha_1, 1)$.

Note that if $q \geq 2$ or $q < 1$ then $E'(\alpha) > 0$ for all α by [14, Theorem 11].

Proof of Proposition 12. If $q = 1$ then $\mathcal{E}(k_\alpha) = \mathcal{E}(\overline{k_\alpha})$, as one sees directly from the formula in (23), using that $k_\alpha(x) = 1 + (\alpha - 1)\cos x$, $P(\alpha) = 2\pi$ and $\overline{k_\alpha} = 1$. So we assume $q \neq 1$ from now on, and $\alpha \in (0, 1)$.

The following proof depends on a number of elementary differential equations and inequalities from §§5.1–5.2 of [14].

First, by definition

$$\begin{aligned} \frac{\mathcal{E}(k_\alpha) - \mathcal{E}(\overline{k_\alpha})}{P} &= \frac{1}{P}\int_0^P \left[\frac{1}{2}{k'_\alpha}^2 - H(k_\alpha)\right] dx + H(A/P) \\ &= \frac{1}{P}\int_0^P {k'_\alpha}^2\, dx - H(\alpha) + H(A/P) \qquad \text{by [14, eq. (21)].} \end{aligned} \tag{29}$$

Temporarily assume $q \neq -1$; then

$$\frac{d}{d\alpha}\frac{\mathcal{E}(k_\alpha) - \mathcal{E}(\overline{k_\alpha})}{P} = -\frac{P'(\alpha)}{P^2}\int_0^P {k'_\alpha}^2\, dx + \left(1 + (q+1)H(A/P)\left(\frac{A}{P}\right)^{-1}\right)\left(\frac{A}{P}\right)'(\alpha), \tag{30}$$

where we have used [14, eq. (35)] and the identity $H'(y) = 1 + (q+1)H(y)y^{-1}$ (valid for $q \neq -1$). Differentiating the function

$$\mathcal{F}(\alpha) := \left(\frac{A}{P}\right)^{-(q+1)} \frac{\mathcal{E}(k_\alpha) - \mathcal{E}(\overline{k_\alpha})}{P} \tag{31}$$

(which is inspired by (28)) with respect to α, we find from (29) and (30) that

$$\mathcal{F}'(\alpha) = \left(\frac{A}{P}\right)^{-(q+2)} \left\{\frac{1}{P}\int_0^P {k'_\alpha}^2\, dx \cdot \left[-(q+1)\left(\frac{A}{P}\right)' - \frac{AP'}{P^2}\right] + \left[\frac{A}{P} + (q+1)H(\alpha)\right]\left(\frac{A}{P}\right)'\right\}.$$

Substituting

$$\frac{A}{P} + (q+1)H(\alpha) = \frac{q+3}{2}\frac{1}{P}\int_0^P {k'_\alpha}^2\, dx$$

from [14, eqs. (31–32)] yields

$$\mathcal{F}'(\alpha) = -\frac{1}{2}\left(\frac{A}{P}\right)^{-(q+2)} \frac{1}{P}\int_0^P {k'_\alpha}^2\, dx \cdot \left[(q-1)\left(\frac{A}{P}\right)' + 2AP^{-2}P'\right] \qquad \text{for } q \neq -1. \tag{32}$$

For $q = -1$ we obtain exactly the same formula (32) for $\mathcal{F}'(\alpha)$, as follows:

$$\begin{aligned} \mathcal{F}'(\alpha) &= \frac{d}{d\alpha}\frac{\mathcal{E}(k_\alpha) - \mathcal{E}(\overline{k_\alpha})}{P} \\ &= -\frac{P'}{P^2}\int_0^P {k'_\alpha}^2\, dx + (1 - P/A)\left(\frac{A}{P}\right)' \qquad \text{from (29) and [14, eq. (35)]} \\ &= -\frac{1}{2}\left(\frac{A}{P}\right)^{-1}\frac{1}{P}\int_0^P {k'_\alpha}^2\, dx \cdot \left[-2\left(\frac{A}{P}\right)' + 2AP^{-2}P'\right] \end{aligned}$$

using that $(1 - P/A) = \int_0^P (k'_\alpha)^2\, dx/A$ when $q = -1$, by [14, eq. (28)]. The last equation is (32) for $q = -1$.

Equation (32) simplifies to

$$\mathcal{F}'(\alpha) = -\left[\frac{1}{2}P^{-3}\left(\frac{A}{P}\right)^{-2q}\int_0^P {k'_\alpha}^2\, dx\right]E'(\alpha), \tag{33}$$

using that $E = P^2(A/P)^{q-1}$ by (26). Hence

$$E' > 0 \iff \mathcal{F}' < 0, \qquad \text{and} \qquad E' < 0 \iff \mathcal{F}' > 0. \tag{34}$$

Also

$$\mathcal{F}(\alpha) \to 0 \quad \text{as} \quad \alpha \to 1, \tag{35}$$

by the formula (29) together with the facts that $P, A \to 2\pi$ and $P/A \to 1$ as $\alpha \to 1$ (see [14, Lemma 6]) and that $k'_\alpha \to 0$ uniformly as $\alpha \to 1$, by [14, eq. (21)]. Proposition 12 now follows from (34), (35) and [14, Theorem 11] (which shows $E' > 0$ when $q \geq 2$ or $q < 1$). For example, if $E' > 0$ on $(\alpha_1, 1)$ then $\mathcal{F}' < 0$ on $(\alpha_1, 1)$; since $\mathcal{F}(1) = 0$ we deduce $\mathcal{F} > 0$ on $[\alpha_1, 1)$, and so $\mathcal{E}(k_\alpha) - \mathcal{E}(\overline{k_\alpha}) > 0$ for $\alpha \in [\alpha_1, 1)$. $\square$

5.2. **Proof of Theorem 7.** The proof relies on formulas for $A'(\alpha)$ and $E'(\alpha)$ that were derived in Lemmas 16 and 18 of [14]: for all $q \neq -1$,

$$A' = -(q+1)H(\alpha)P' - \frac{q-1}{2}H'(\alpha)P, \tag{36}$$

$$E' = -\left(\frac{A}{P}\right)^{q-2}\left\{P'\left[(q-3)A + (q-1)(q+1)H(\alpha)P\right] + \frac{1}{2}(q-1)^2H'(\alpha)P^2\right\}. \tag{37}$$

Note that $E' > 0$ if $q < 1$ or $q \geq 2$, by [14, Theorem 11]. Our assumptions therefore imply $-1 < q < 3$ and $E'(\alpha) > 0$ for all $\alpha \in (0, 1)$. Also $E(0) > 0$ because $q > -1$ [13, §3.1.2]. If we define

$$\hat{X} = \left[E(0)/(\mathcal{B}A_{\text{ss}}^{q-1})\right]^{1/(3-q)}$$

then

$$\mathcal{B}\hat{X}^{3-q}A_{\text{ss}}^{q-1} = E(0) < E(\alpha) = \mathcal{B}X^{3-q}A_{\text{ss}}^{q-1}$$

by (26); here the value $\alpha \in (0, 1)$ is determined by translating and rescaling h_{ss} to a particular k_α, as in §2.3. Hence $0 < \hat{X} < X$ (using that $q < 3$).

By rescaling the zero contact angle function k_0 on the interval $[0, P(0)]$ (as in §2.3; see [13, Claim 5.1.2] for details) we obtain a zero contact angle droplet steady state $\hat{h}_{\text{ss}}$ of $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$, with length $\hat{X}$ and area A_{ss} as desired.

It remains to prove that the energy of this droplet steady state is lower than the energy of the positive periodic steady state $h_{\rm ss}$. That is, we want to prove

$$\int_0^X \left[\frac{1}{2}(\hat{h}'_{\rm ss})^2 - \mathcal{B}G(\hat{h}_{\rm ss})\right] dx < \int_0^X \left[\frac{1}{2}(h'_{\rm ss})^2 - \mathcal{B}G(h_{\rm ss})\right] dx \tag{38}$$

where

$$G(y) = \begin{cases} \frac{y^{q+1}}{q(q+1)}, & q \neq 0, -1, \\ y\log y - y, & q = 0, \\ -\log y, & q = -1; \end{cases}$$

note that we can use G instead of H in the energy because they differ only by a linear function (cf. the definition (22) of H) and $\hat{h}_{\rm ss}$ and $h_{\rm ss}$ have the same area, $A_{\rm ss}$.

Since $\hat{h}_{\rm ss}$ is supported on $(0, \hat{X})$ and because $G(0) = 0$, we need only integrate over $(0, \hat{X})$ in (38). Next rescale $\hat{h}_{\rm ss}$ to k_0 and $h_{\rm ss}$ to k_α: from (25) and (27) (with $P_{\rm ss}$ replaced by $\hat{X}$ or X as appropriate) we deduce that (38) is equivalent to

$$\left[\frac{A_{\rm ss}}{A(0)}\right]^2 \left[\frac{P(0)}{\hat{X}}\right]^3 \int_0^{P(0)} \left[\frac{1}{2}(k_0')^2 - G(k_0)\right] dx < \left[\frac{A_{\rm ss}}{A(\alpha)}\right]^2 \left[\frac{P(\alpha)}{X}\right]^3 \int_0^{P(\alpha)} \left[\frac{1}{2}(k_\alpha')^2 - G(k_\alpha)\right] dx,$$

except that when $q = 0$ we have to subtract

$$\left[\log \frac{A_{\rm ss}}{A(0)} \frac{P(0)}{\hat{X}}\right] k_0 \quad \text{and} \quad \left[\log \frac{A_{\rm ss}}{A(\alpha)} \frac{P(\alpha)}{X}\right] k_\alpha$$

from the integrands on the left and right sides, respectively. By substituting the relations

$$\hat{X} = \left[E(0)/(\mathcal{B}A_{\rm ss}^{q-1})\right]^{1/(3-q)} \quad \text{and} \quad X = \left[E(\alpha)/(\mathcal{B}A_{\rm ss}^{q-1})\right]^{1/(3-q)}$$

into the last inequality and using the definition $E(\alpha) = P(\alpha)^{3-q}A(\alpha)^{q-1}$, we see the desired inequality reduces to $\mathcal{G}(0) < \mathcal{G}(\alpha)$ where (for $q \neq 3$)

$$\mathcal{G}(\alpha) = A(\alpha)^{(q+3)/(q-3)} \int_0^{P(\alpha)} \left[\frac{1}{2}(k_\alpha')^2 - G(k_\alpha)\right] dx \quad \left(+\frac{2}{3}\log A(\alpha), \text{ when } q = 0\right). \tag{39}$$

Thus to show the energy of $\hat{h}_{\rm ss}$ is lower than that of $h_{\rm ss}$, it suffices to show $\mathcal{G}'(\alpha) > 0$ for all $\alpha \in (0,1)$, assuming $-1 < q < 3$ and $E'(\alpha) > 0$ for all $\alpha \in (0,1)$.

To show $\mathcal{G}' > 0$, we substitute [14, eq. (32)] and [14, eq. (30)] into the definition (39) of $\mathcal{G}$, obtaining that

$$\mathcal{G} = \frac{q-3}{q(q+3)}A^{2q/(q-3)} + \frac{q-1}{q+3}H(\alpha)PA^{(q+3)/(q-3)} \qquad \text{when } q \neq 3, 0, -1, -3.$$

After differentiating this formula with respect to α and then substituting for $A'(\alpha)$ from (36), we simplify with the help of (37) to obtain

$$\mathcal{G}' = \frac{1}{q-3}H(\alpha)A^{6/(q-3)}\left(\frac{A}{P}\right)^{-q+2} E' \qquad \text{when } q \neq 3, 0, -1, -3. \tag{40}$$

When $q = 0$ we obtain $\mathcal{G} = \frac{2}{3} - \frac{1}{3}HPA^{-1} + \frac{2}{3}\log A$, by putting the $q = 0$ versions of [14, eq. (32)] and [14, eq. (30)] into (39). Hence $\mathcal{G}' = -\frac{1}{3}HP^{-2}E'$, by differentiating and using (36) and (37). That is, (40) holds when $q = 0$, also.

We conclude from (40) that $\mathcal{G}' > 0$ as desired (provided $-1 < q < 3$ and $E'(\alpha) > 0$ for all $\alpha \in (0,1)$), since for $q > -1$ we have $H(\alpha) < 0$ by the definition (22). $\square$

5.3. **Proof of Theorem 8.** The proof involves rescaling arguments rather than the energy. Write $P_{\text{ss}} = X$. Assume $\hat{h}_{\text{ss}}$ is non-constant.

Suppose that in fact $\hat{h}_{\text{ss}}$ is positive (and so smooth by hypothesis). Then the least period of $\hat{h}_{\text{ss}}$ equals P_{ss}/j for some positive integer j, with the area per period equaling A_{ss}/j. If $j = 1$ then $\hat{h}_{\text{ss}}$ must be a translate of h_{ss}, by modifying slightly the uniqueness remarks in [14, §6.2] (using the assumption that $E' < 0$ to get strict monotonicity of E).

Thus we can assume $j \geq 2$. By rescaling h_{ss} and $\hat{h}_{\text{ss}}$ to k_α and $k_{\hat{\alpha}}$ for some $\alpha, \hat{\alpha} \in (0,1)$, as in (7), we get from (26) that

$$E(\alpha) = \mathcal{B}P_{\text{ss}}^{3-q}A_{\text{ss}}^{q-1} \quad \text{and} \quad E(\hat{\alpha}) = \mathcal{B}(P_{\text{ss}}/j)^{3-q}(A_{\text{ss}}/j)^{q-1}.$$

Hence

$$\begin{aligned} 4 \leq j^2 = \frac{E(\alpha)}{E(\hat{\alpha})} &< \frac{E(0)}{E(1)} \qquad \text{since } E' < 0 \\ &= \frac{1}{4\pi^2}\frac{2}{q}(1+q)\, B\left(\frac{1}{2q}, \frac{1}{2}\right)^{3-q} B\left(\frac{3}{2q}, \frac{1}{2}\right)^{q-1} =: J(q) \text{ say,} \end{aligned}$$

by the formula for $E(0)$ in [13, eq. (3.13)] and since $E(1) = P(1)^{3-q}A(1)^{q-1} = 4\pi^2$ by [14, Lemma 6]. We will obtain a contradiction by showing $J(q) < 4$, when $1 < q \leq 1.75$; this will show $\hat{h}_{\text{ss}}$ is not positive.

For $1 < q \leq 1.5$ we have

$$J(q) \leq \frac{1}{4\pi^2}\frac{2}{1}(1+1.5)\, B\left(\frac{1}{2\cdot 1.5}, \frac{1}{2}\right)^{3-1} B\left(\frac{3}{2\cdot 1.5}, \frac{1}{2}\right)^{1.5-1} \approx 3.17 < 4,$$

where we have used that the Beta function $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$ is decreasing in its arguments, and is bigger than 1 when those arguments are less than 1. For $1.5 < q \leq 1.75$ we similarly have

$$J(q) \leq \frac{1}{4\pi^2}\frac{2}{1.5}(1+1.75)\, B\left(\frac{1}{2\cdot 1.75}, \frac{1}{2}\right)^{3-1.5} B\left(\frac{3}{2\cdot 1.75}, \frac{1}{2}\right)^{1.75-1} \approx 1.73 < 4,$$

completing the contradiction.

The above contradiction implies $\hat{h}_{\text{ss}}$ is not positive everywhere. Consider therefore one component of the set $\{x : \hat{h}_{\text{ss}}(x) > 0\}$, say an interval with length $\hat{P} \leq P_{\text{ss}}$. Write $\hat{A} \leq A_{\text{ss}}$ for the area under $\hat{h}_{\text{ss}}$ on this interval. Note the contact angles of $\hat{h}_{\text{ss}}$ must be the same at the two endpoints of the interval, as a consequence of the nonlinear oscillator equation (6) (see for example [13, §2.2]). Suppose these contact angles are zero, so that $\hat{h}_{ss}$ rescales to k_0 on the interval, using (7). Then

$$E(0) = \mathcal{B}\hat{P}^{3-q}\hat{A}^{q-1} \le \mathcal{B}P_{ss}^{3-q}A_{ss}^{q-1} = E(\alpha)$$

by applying (26) twice, but this contradicts our assumption that $E' < 0$. Thus the contact angles of $\hat{h}_{ss}$ must all be nonzero, as desired. □

5.4. **Proof of Theorem 9.** Translate h_{ss1} and h_{ss2} so that they attain their minimum values at $x = 0$, and then rescale as in §2.3 to obtain k_{α_1} and k_{α_2} respectively. Since h_{ss1} and h_{ss2} have the same period and area, for which we write $P_{ss} = X$ and A_{ss} respectively, it follows from (26) that $E(\alpha_1) = E(\alpha_2)$. Notice $h_{ss1} \neq h_{ss2} \Rightarrow \alpha_1 \neq \alpha_2$, in view of the expression (25) for h_{ss} in terms of k_α and $P_{ss}, A_{ss}, P(\alpha), A(\alpha)$. Since $E(\alpha_1) = E(\alpha_2)$ while E is strictly decreasing on $(0, \alpha_{crit})$ and strictly increasing on $(\alpha_{crit}, 1)$, we conclude α_{crit} must lie between α_1 and α_2.

We show $\alpha_1 < \alpha_2$. The hypothesis $\min h_{ss1} < \min h_{ss2}$ gives

$$\alpha_1 \left(\frac{D_1}{\mathcal{B}}\right)^{1/q} < \alpha_2 \left(\frac{D_2}{\mathcal{B}}\right)^{1/q}$$

by the rescaling (7). Next apply the first equation in (24) to solve for D_1 in terms of $P(\alpha_1), P_{ss}$ and $\mathcal{B}$, and similarly solve for D_2 in terms of $P(\alpha_2), P_{ss}$ and $\mathcal{B}$. Substituting into the above inequality gives $\alpha_1 P(\alpha_1)^{2/(q-1)} < \alpha_2 P(\alpha_2)^{2/(q-1)}$. The strict increase of $\alpha \mapsto \alpha P(\alpha)^{2/(q-1)}$ implies $\alpha_1 < \alpha_2$.

Since $\alpha_1 < \alpha_{crit} < \alpha_2$, our hypothesis on E' implies $E'(\alpha_1) < 0$ and $E'(\alpha_2) > 0$. Theorem 3 then implies that h_{ss1} is energy stable, and Theorem 2 implies h_{ss2} is energy unstable.

Next we show $\mathcal{E}(h_{ss1}) < \mathcal{E}(h_{ss2})$, or $\mathcal{E}(h_{ss1}) - \mathcal{E}(\overline{h_{ss1}}) < \mathcal{E}(h_{ss2}) - \mathcal{E}(\overline{h_{ss2}})$. In view of the rescaling relation (28) for the energy, it suffices to prove $\mathcal{F}(\alpha_1) < \mathcal{F}(\alpha_2)$, where $\mathcal{F}$ was defined in (31).

To prove $\mathcal{F}(\alpha_1) < \mathcal{F}(\alpha_2)$, we write (33) as $\mathcal{F}'(\alpha) = \delta(\alpha)(1/E)'(\alpha)$, where

$$\delta(\alpha) = \frac{1}{2}P^3 A^{-2} \int_0^P {k'_\alpha}^2 \, dx.$$

The point of this transformation is that $\delta(\alpha)$ is strictly decreasing: $P' < 0$ and $A' > 0$, by [13, Props. 7.3 and 7.4], while $\alpha \mapsto \int_0^P {k'_\alpha}^2 \, dx$ is decreasing by [14, eq. (35)]. Also $(1/E)' > 0$ on $(0, \alpha_{crit})$ and $(1/E)' < 0$ on $(\alpha_{crit}, 1)$, by assumption. Hence

$$\mathcal{F}'(\alpha) > \delta(\alpha_{crit})(1/E)'(\alpha) \quad \text{for } \alpha \in (0,1), \alpha \neq \alpha_{crit}.$$

Integrating this inequality from α_1 to α_2 and using that $E(\alpha_1) = E(\alpha_2)$ gives $\mathcal{F}(\alpha_2) > \mathcal{F}(\alpha_1)$, as desired.

Finally, (33) shows $\mathcal{F}'(\alpha) < 0$ on $(\alpha_{crit}, 1)$, and so $\mathcal{F}(\alpha_2) > \mathcal{F}(1) = 0$ by (35). Thus (28) yields $\mathcal{E}(h_{ss2}) > \mathcal{E}(\overline{h_{ss2}})$. □

5.5. **Proof of Theorem 10.** From the definition (11) we see

$$\tau_1(\overline{h}) = \min_u \frac{\int_0^X (u')^2\,dx}{\int_0^X u^2\,dx} - r(\overline{h}) = \left(\frac{2\pi}{X}\right)^2 - r(\overline{h}),$$

with the minimum being attained precisely for linear combinations of $\sin(2\pi x/X)$ and $\cos(2\pi x/X)$. The first two paragraphs of the theorem follow directly.

(a) Now suppose $r(\overline{h})X^2 > 4\pi^2$. The variational formulas (15) and (16) in the proof of Theorem 1 show the constant steady state $\overline{h}$ is energy unstable in the sine and cosine directions, since these are $\tau_1(\overline{h})$-eigenfunctions and $\tau_1(\overline{h}) < 0$. Suppose next $r(\overline{h})X^2 = 4\pi^2$ and $r''(\overline{h}) > 0$. Then the first two variations of the energy in the $\pm\sin$ directions are zero, by (15) and (16). The third variation equals $r'(\overline{h})$ times the integral of $\mp\sin^3$, by (17); thus the third variation is also zero. The fourth variation is $-r''(\overline{h})\int_0^X \sin^4(2\pi x/X)\,dx$, which is negative because we assumed $r''(\overline{h}) > 0$. Thus the constant steady state is energy unstable in the $\pm\sin$ directions. Argue similarly for the $\pm\cos$ directions.

(b) If $r(\overline{h})X^2 < 4\pi^2$, or if $r(\overline{h})X^2 = 4\pi^2$ and $r''(\overline{h}) < 0$, then $\tau_1(\overline{h}) \geq 0$ and so we get energy stability by modifying the argument of part (a) as follows. The first variation of the energy in a direction u is always zero. If the second variation is positive then we are done. Otherwise the second variation must be zero, so that $\tau_1(\overline{h}) = 0$ and $r''(\overline{h}) < 0$. Then u must be a linear combination of sines and cosines, and so the third variation is also zero. Then the fourth variation is positive.

Furthermore, for $r(\overline{h})X^2 < 4\pi^2$ we will prove $\overline{h}$ is a strict local minimum of the energy, and is stable in H^1. In doing this, we will use below a certain sufficiently small number $\delta \in (0,1)$. Then considering $u \in H^1(\mathbb{T}_X)$ with mean value zero and $\|u\|_{H^1(\mathbb{T}_X)} = 1$, we find for all $\varepsilon \in [0,\delta]$ that

$$\begin{aligned}
\frac{d^2}{d\varepsilon^2}\mathcal{E}(\overline{h}+\varepsilon u) &= \int_0^X \left[(u')^2 - r(\overline{h}+\varepsilon u)u^2\right] dx \\
&= \delta\int_0^X \left[(u')^2 + u^2\right] dx + (1-\delta)\int_0^X \left[(u')^2 - \frac{r(\overline{h}+\varepsilon u)+\delta}{1-\delta}u^2\right] dx \\
&> \delta\|u\|^2_{H^1(\mathbb{T}_X)} + (1-\delta)\int_0^X \left[(u')^2 - \frac{4\pi^2}{X^2}u^2\right] dx \geq \delta\|u\|^2_{H^1(\mathbb{T}_X)}.
\end{aligned}$$

In the second-to-last step above, we used $r(\overline{h})X^2 < 4\pi^2$ and $\|u\|_\infty \leq C\|u\|_{H^1}$ and we chose δ sufficiently small (independent of u and ε).

On the other hand, $|\varepsilon u| \leq \delta \|u\|_{L^\infty} \leq \overline{h}/2$ provided δ is chosen small enough (independent of u), and hence

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}\mathcal{E}(\overline{h}+\varepsilon u) = \int_0^X \left[(u')^2 - r(\overline{h}+\varepsilon u)u^2\right] dx \;&\leq\; C(\overline{h}) \int_0^X \left[(u')^2 + u^2\right] dx \\ &\qquad \text{for some constant } C(\overline{h}) \geq 1 \\ &=\; C(\overline{h})\|u\|^2_{H^1(\mathbb{T}_X)}.\end{aligned}$$

We deduce from the preceding estimates and the vanishing of the first variation in (15) that if $u \in H^1(\mathbb{T}_X)$ has mean value zero and $\|u\|_{H^1(\mathbb{T}_X)} \leq \delta$, then $\overline{h}+u > 0$ and the energy varies quadratically away from $\overline{h}$:

$$\frac{1}{2}\delta\|u\|^2_{H^1(\mathbb{T}_X)} \leq \mathcal{E}(\overline{h}+u) - \mathcal{E}(\overline{h}) \leq \frac{1}{2}C(\overline{h})\|u\|^2_{H^1(\mathbb{T}_X)}. \tag{41}$$

The lefthand estimate implies $\overline{h}$ is a strict local minimum of the energy, with respect to X-periodic zero-mean perturbations.

We now prove $\overline{h}$ is stable, in the sense that if $h(x,t)$ is a smooth positive solution of (1) for $x \in \mathbb{T}_X$ and $t \in [0,T]$, for some $T > 0$, and if $h(\cdot,0)$ has mean value $\overline{h}$ and $\|h(\cdot,0)-\overline{h}\|_{H^1(\mathbb{T}_X)} < \sqrt{\delta^3/4C(\overline{h})}$, then $\|h(\cdot,t)-\overline{h}\|_{H^1(\mathbb{T}_X)} < \delta/2$ for all $t \in [0,T]$. Indeed, the quadratic bounds (41) and the dissipation of the energy together imply

$$\begin{aligned}\frac{1}{2}\delta\|h(\cdot,t)-\overline{h}\|^2_{H^1(\mathbb{T}_X)} \;&\leq\; \mathcal{E}(h(\cdot,t)) - \mathcal{E}(\overline{h}) \\ &\leq\; \mathcal{E}(h(\cdot,0)) - \mathcal{E}(\overline{h}) \leq \frac{1}{2}C(\overline{h})\|h(\cdot,0)-\overline{h}\|^2_{H^1(\mathbb{T}_X)} < \frac{1}{2}\frac{\delta^3}{4},\end{aligned}$$

so that $\|h(\cdot,t)-\overline{h}\|_{H^1(\mathbb{T}_X)} < \delta/2$ for all $t \in [0,T]$. This stability result holds for all sufficiently small δ. □

This proof of stability for a linearly stable constant steady state does not carry over to linearly stable positive periodic steady states h_{ss}, because there $\tau_1(h_{\text{ss}}) = 0$ (due to translational null directions). This zero eigenvalue is absent for the constant steady state, since translation of $\overline{h}$ gives $\overline{h}$ again — a trivial perturbation. Now, imposing Neumann boundary conditions eliminates the translational perturbations and their associated zero eigenvalue. Hence, the stability proof *would* hold for a positive steady state under the Neumann boundary conditions $h_x = h_{xxx} = 0$ at $x = 0, X/2$ (see §2.6), provided the steady state is strictly linearly stable, *i.e.* the first eigenvalue is positive.

5.6. **Proof of Theorem 11.** Let $\overline{h}, X > 0$. We are first of all interested in whether or not there exists a zero contact angle droplet steady state $\hat{h}_{\text{ss}}$ that is supported on a single interval of length $\hat{X} \leq X$ and has the same area $\hat{A}_{\text{ss}} = \overline{h}X$ as the constant steady state $\overline{h}$, over the interval $(0,X)$.

Suppose first that $-1 < q < 3$. By (26) with $\alpha = 0$, the steady state $\hat{h}_{ss}$ exists if and only if for some length $\hat{X} \le X$ we have

$$\mathcal{B}\hat{X}^{3-q}\hat{A}_{ss}^{q-1} = E(0) = P(0)^{3-q}A(0)^{q-1}. \tag{42}$$

That is, if and only if $\mathcal{B}X^{3-q}(\overline{h}X)^{q-1} \ge E(0)$, which is (12).

Suppose such a steady state $\hat{h}_{ss}$ exists, supported say on the interval $(0, \hat{X})$. By above,

$$\hat{X} = \mathcal{B}^{1/(q-3)}P(0)[A(0)/\overline{h}X]^{(q-1)/(3-q)}.$$

We want to show that $\mathcal{E}(\hat{h}_{ss}) < \mathcal{E}(\overline{h})$ if and only if (13) holds. For this, compute using G like in §5.2 to find

$$\begin{aligned}
\mathcal{E}(\hat{h}_{ss}) - \mathcal{E}(\overline{h}) &= \int_0^X \left[\frac{1}{2}\left(\hat{h}'_{ss}\right)^2 - \mathcal{B}G(\hat{h}_{ss}) + \mathcal{B}G(\overline{h})\right] dx \\
&= \int_0^{\hat{X}} \left[\frac{1}{2}\left(\hat{h}'_{ss}\right)^2 - \mathcal{B}G(\hat{h}_{ss})\right] dx + \mathcal{B}G(\overline{h})X \\
&= \frac{\hat{A}_{ss}^2 P(0)^3}{A(0)^2\hat{X}^3}\int_0^{P(0)} \left[\frac{1}{2}(k_0')^2 - G(k_0)\right] dx + \mathcal{B}G(\overline{h})X \qquad (43)
\end{aligned}$$

when $q > -1, q \neq 0$, by rescaling from $\hat{h}_{ss}$ to k_0 and using (25) and (27) (with P_{ss} replaced by $\hat{X}$, and A_{ss} replaced by $\hat{A}_{ss}$). Putting $\alpha = 0$ into [14, eq. (32)] and [14, eq. (30)] and using $H(0) = 0$ enables us to evaluate $\int (k_0')^2\, dx$ and $\int G(k_0)\, dx$, and hence we deduce

$$\mathcal{E}(\hat{h}_{ss}) - \mathcal{E}(\overline{h}) = \frac{\hat{A}_{ss}^2 P(0)^3}{A(0)^2\hat{X}^3}\frac{q-3}{q(q+3)}A(0) + \mathcal{B}G(\overline{h})X \qquad \text{when } q > -1, q \neq 0. \tag{44}$$

After substituting the definitions of $\hat{X}$ and $\hat{A}_{ss} = \overline{h}X$ from above, we find

$$\mathcal{E}(\hat{h}_{ss}) - \mathcal{E}(\overline{h}) = \frac{1}{q}\mathcal{B}^{3/(3-q)}(\overline{h}X)^{(3+q)/(3-q)}\left[\frac{q-3}{q+3}A(0)^{2q/(q-3)} + \frac{1}{q+1}\left(\mathcal{B}\overline{h}^{q-1}X^2\right)^{q/(q-3)}\right]$$

when $-1 < q < 3, q \neq 0$. Plainly now $\mathcal{E}(\hat{h}_{ss}) < \mathcal{E}(\overline{h})$ if and only if (13) holds.

When $q = 0$ we find (43) has an extra term $-\mathcal{B}\hat{A}_{ss}\log\left[\hat{A}_{ss}P(0)/A(0)\hat{X}\right]$, so that after again using [14, eq. (32)] and [14, eq. (30)] and substituting for $\hat{X}$ and $\hat{A}_{ss}$, we obtain

$$\mathcal{E}(\hat{h}_{ss}) - \mathcal{E}(\overline{h}) = \frac{1}{3}\mathcal{B}\overline{h}X\log\frac{A(0)^2/e}{\mathcal{B}\overline{h}^{-1}X^2} \qquad \text{when } q = 0.$$

Remembering that $A(0) = 2e^{3/2}\sqrt{\pi/3}$ when $q = 0$, from [13, §3.1.2], we conclude $\mathcal{E}(\hat{h}_{ss}) < \mathcal{E}(\overline{h})$ if and only if $\mathcal{B}\overline{h}^{-1}X^2 > 4e^2\pi/3$. Incidentally, one can check $L(q)$ is continuous at $q = 0$.

For $q \ge 3$, simply modify the above proof from the case $-1 < q < 3$. Notice when $q = 3$ that (42) becomes $\mathcal{B}(\overline{h}X)^2 = E(0)$, which yields no formula for $\hat{X}$. And when $q \ge 3$ we get $\mathcal{E}(\hat{h}_{ss}) > \mathcal{E}(\overline{h})$ because the first term in (44) is nonnegative and the second is positive. $\square$

6. Conclusions and Future Directions

If you perturb a positive periodic (or constant) steady state h_{ss} of the evolution equation (1), without changing its area, then towards which steady states might the solution subsequently evolve? That is the broad question addressed by this paper. To answer it, we focused on three specific questions:

- Existence: Do there exist *other* steady states having the same area and same period as h_{ss}, or having period a *fraction* of the period of h_{ss}? If so, then these other steady states are plausible contenders for the long time limit. The constant steady state $\overline{h_{\mathrm{ss}}}$ obviously satisfies these requirements, but there might be another positive periodic steady state (different from h_{ss} and not just a translate of it) that does also, or perhaps a array of droplet steady states having zero or nonzero contact angles.
- Stability: Are h_{ss} or these other steady states linearly stable? energy stable? If a steady state is to be the long time limit of generic initial data, then surely it must be stable under perturbations.
- Relative energy levels: Do any of these other steady states have lower energy than h_{ss}? Obviously only steady states with lower energy are accessible, when starting from a small perturbation of h_{ss}.

The existence question was substantially answered for power law coefficients by [14, Theorem 12], Theorems 7 and 8 and [14, Figures 3–5], also using Theorem 11 when h_{ss} is the constant steady state. But the existence question remains open for droplet steady states with *nonzero* contact angles, if we wish to specify the area and the length of the support. Some information on nonzero angle droplet steady states is in our earlier paper [13, §5.2].

The stability question was resolved for positive periodic steady states in the power law case by [14, Theorems 1,3,7,9] and Theorems 1–5 here; in particular, Theorem 1 related linear instability to energy instability. Theorem 10 handled the case of constant steady states. Our numerical simulations in the companion article [15, §5] suggest that linearly unstable steady states are indeed unstable, with the linear behavior dominating near the steady state, but this observation is so far unsupported by a 'linearization theorem' for the power law evolution $h_t = -(h^n h_{xxx})_x - \mathcal{B}(h^m h_x)_x$. (A linearization theorem *is* known in the Cahn–Hilliard case $f \equiv 1$, by using semilinear operator theory; see for example [19, §6].)

The energy level question has been largely settled in the power law case by Theorems 6, 7 and 9 when h_{ss} is positive and periodic, and by Theorems 6 and 11 when h_{ss} is constant. When h_{ss} has compact support with zero contact angle, use Theorems 7 and 11.

Future directions. Many of our existence, stability and relative energy level theorems for the power law evolution would be improved if we knew $E'(\alpha) < 0$ for all α when $1 < q \leq 1.75$.

We have not been able to prove this conjecture, though numerically it is clear from [14, Figure 3].

Further, the stability question for *droplet* steady states (with zero and nonzero contact angles) is open. So is the problem of computing relative energy levels of non-zero angle droplet steady states *vis a vis* zero-angle droplets and constant and periodic steady states.

On a related note, many of the theorems in this paper together with the simulations in [15, §5] suggest the existence of heteroclinic connections between steady states. One would like to have proofs of their existence. Incidentally, simulations in [15, §4] suggest that small changes in the 'mobility' coefficient f leave these heteroclinic orbits intact, but *can* affect whether or not the solution remains positive throughout its evolution.

Finally, one would like to answer the existence, stability and relative energy level questions for *general* coefficient functions f and g. We have treated power law coefficients, and Grinfeld and Novick–Cohen [10] cover the Cahn–Hilliard equation $h_t = -h_{xxxx} - ((1-3h^2)h_x)_x$, for which $f \equiv 1$ and $g(y) = 1 - 3y^2$. See also the impressive dynamic stability results in [3, 5] for the Cahn–Hilliard equation on the whole real line. But for general coefficients, about all we know is that every positive periodic steady state is linearly and energy unstable when g/f is a convex function, by Theorem 1. Note that the earlier work of Mischaikow [17] applies to a variety of gradient-like bistable equations, provided certain hypotheses can be verified.

Acknowledgments. The authors are grateful to Andrew Bernoff for stimulating comments on the energy landscape of phase space.

Laugesen was partially supported by NSF grant number DMS-9970228, by a grant from the University of Illinois Research Board, and by a fellowship from the University of Illinois Center for Advanced Study. He is grateful for the hospitality of the Department of Mathematics at Washington University in St. Louis.

Pugh was partially supported by NSF grant number DMS-9971392, by the MRSEC Program of the NSF under Award Number DMR-9808595, by the ASCI Flash Center at the University of Chicago under DOE contract B341495, and by an Alfred P. Sloan fellowship. Some of the computations were done using a network of workstations paid for by an NSF SCREMS grant, DMS-9872029. Part of the research was conducted while enjoying the hospitality of the Mathematics Department and the James Franck Institute of the University of Chicago.

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