Implicit Function Theorem and Lagrange Multipliers

March 4, 2014

Problem. 1. (Sept 03 #5) Suppose that $F = F(u, v)$ is a smooth function from $\mathbb{R}^2 \to \mathbb{R}$ with $F(1, 2) = 0$ and $F_v(1, 2) \neq 0$. Show that:

$$F(xy, \sqrt{x^2 + z^2}) = 0$$

defines a smooth surface $(x, y, z(x, y))$ in a n'h'd of $(1, 1, \sqrt{3})$.Find a normal vector $n \neq 0$ to this.

Problem. 2. (Jan 09 #3) Let $\Sigma = \{x \in \mathbb{R}^3 : x_1x_2 + x_1x_3 + x_2x_3 = 1\}$ and $f(x) = x_1^2 + x_2^2 + \frac{9}{2}x_3^2$. Show that Σ is a smooth surface in $\mathbb{R}^3$. Show that $\inf_{x\in\Sigma} f(x)$ is achieved. Find $\inf_{x\in\Sigma} f(x)$.

Problem. 3. (Jan 10 #3) Let x, y, z be positive real numbers. Show that the following inequality holds (Hint: Use homogeneity):

$$\frac{1}{x^3} + \frac{16}{y^3} + \frac{1}{z^3} \geq \frac{256}{(x + y + z)^3}$$

Problem. 4. (Sept 06 #4) Part 1: Explain the idea behind the method of Lagrange multipliers for finding an extremum of$f(x, y)$ subject to the constraint $g(x, y) = a$ with $a \in \mathbb{R}$ and smooth f and g. (Please use the function $\phi(x, y, \lambda) = f - \lambda(g - a)$).

Part 2: Show that $df*/da = \lambda*$ where $f*(a)$ is the value of f at the conditional extremum and $\lambda*$ is the corresponding value of the Lagrange multiplier.

Part 3: Find the minimum of $x^2 + y^2$ subject to $x - y = 1$. Without recomputing, what is your best estimate for the minimum if the constraint is changed to $x - y = 2$ or $x - y = 0$?

Problem. 5. (Jan 13 #2) Let the nonnegative real numbers x_1, x_2, x_3, x_4 satisfy $x_1 + x_2 + x_3 + x_4 = \pi$. (a) Show that $\sin(x_1)\sin(x_2)\sin(x_3)\sin(x_4) \leq \frac{1}{4}$. (b) Find all (x_1, x_2, x_3, x_4) that result in equality above.

Problem. 6. (Jan 05 #5) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear function given by $f(x) = a_1x_1 + \ldots + a_nx_n$. Find the extrema of f on the ellipsoid $\mu_1x_1^2 + \ldots + \mu_nx_n^2 = 1$.