Complex Analysis Oral Exam study notes

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Abstract. These are some study notes that I made while studying for my oral exams on the topic of Complex Analysis. I took these notes from parts of the textbook by Joseph Bak and Donald J. Newman [**1**] and also a real life course taught by Fengbo Hang in Fall 2012 at Courant. Please be extremely caution with these notes: they are rough notes and were originally only for me to help me study. They are not complete and likely have errors. I have made them available to help other students on their oral exams.

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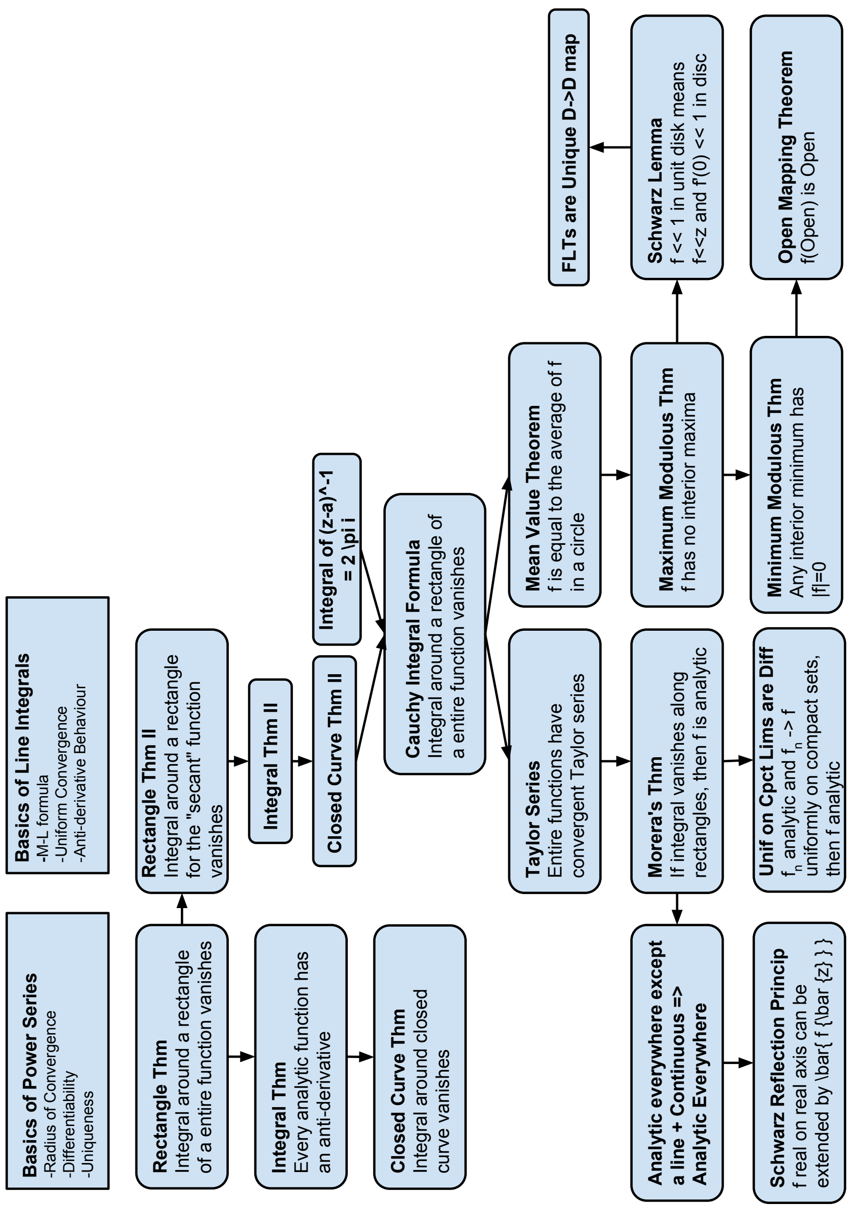
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Introductory Stuff

From real life course by Fengbo Hang.

1.1. Fundamental Theorem of Algebra

We can motivate the study of complex analysis by the fundamental theorem of algebra. This theorem says that, unlike real numbers, every $n-$th degree complex valued polynomial has n roots. Already this shows us that some things are much nicer in complex numbers than real numbers. Let us begin by defining the complex numbers as a 2-d vector space over $\mathbb{R}$:

Definition. Let i so that $i^2 = -1$ and define $\mathbb{C} = \{a + ib : a, b \in \mathbb{C}\}$. This is a vector space over $\mathbb{R}$ with basis $\{1, i\}$. We also equip ourselves with the natural norm, $|a + ib| = \sqrt{a^2 + b^2}$.

Remark. To really be precise, one can check that the set of matrices $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is a commutative 2-dimensional field and make the identification $1 \leftrightarrow Id$ and $i \leftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ to precisely define the complex numbers. Notice that this matrix arises as the matrix for the operator $\phi_z : \mathbb{C} \to \mathbb{C}$ by $\phi_z(w) = zw$. When $z = a + ib$, the matrix for ϕ_z in the basis $\{1, i\}$ is precisely $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

Lemma. *Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be any polynomial of degree $n \geq 1$. Then $p(z)$ has at least one root in $\mathbb{C}$.*

Proof. Suppose by contradiction p has no zeros. Let $z_0 \in \mathbb{C}$ be such that $|p(z_0)| > 0$ is minimal. By translating, we may suppose without loss of generality that $z_0 = 0$. We have then the expression for p:

$$\begin{aligned} p(z) &= a_0 + a_m z^m + O(z^{m+1}) \\ &= a_0 \left(1 + \frac{a_m}{a_0} z^m + O(z^{m+1})\right) \end{aligned}$$

Write $\frac{a_m}{a_0} = Re^{i\phi}$ in polar notation, and then for a parameter $t > 0$, plugin $z^\star = te^{i\frac{\pi-\phi}{m}}$ to get:

$$\begin{aligned} |p(z^\star)| &= \left| a_0 \left(1 + \frac{a_m}{a_0} z^m + O(z^{m+1})\right) \right| \\ &= |a_0| \left(\left|1 + Re^{\phi} t^m e^{\pi-\phi}\right| + O(t^{m+1}) \right) \\ &= |a_0| \left(1 - Rt^m + O(t^{m+1})\right) \end{aligned}$$

But if we take t sufficiently small, we will get $|p(z^\star)| < |a_0| = |p(0)|$ which contradicts the fact that $z = 0$ has minimal modulus. $\square$

LEMMA. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be any polynomial. Then* p *has* n *complex roots.*

PROOF. (By induction on n) The base case $n = 1$ is clear. The above lemma, along with the factoring algorithm to factor out roots reduces the degree by one. $\square$

1.2. Power Series

DEFINITION. A *power series* is a function $f : \Omega \to \mathbb{C}$ where $\Omega \subset \mathbb{C}$ is a function defined as an infinite sum:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

REMARK. The disadvantage of power series is that, because of the way they are defined, there can be problems with convergence of the infinite sum. For example the function $\frac{1}{1-z}$ is defined when $z \neq 1$ but it happens that $\frac{1}{1-z} = \sum z^n$ for $|z| < 1$.

DEFINITION. We say that a function $f : \Omega \to \mathbb{C}$ is $\mathbb{C}-$differentiable at a point z if the following limit exists:

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

Here h is a complex number, and the limit is to be interpreted with the topology on $\mathbb{C}$ induced by the norm $|\cdot|$. (This is the same topology as on $\mathbb{R}^2$)

REMARK. It turns out that $\mathbb{C}-$differentiable functions and power series are actually the same! We will see what this is later.

EXAMPLE. The usual tricks for differentiation in $\mathbb{R}$ will work for $\mathbb{C}$ too. For example, all polynomials are differentiable, and one can prove the quotient rule works too, so that rational functions are differentiable too. The following special class of rational functions will be of interest to us, so we define it below:

DEFINITION. A *fractional linear transformation* is a rational function of the form:

$$f(z) = \frac{az + b}{cz + d}$$

Where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

We will now prove some results about power series.

LEMMA. *Let* $f(z) = \sum c_n z^n$ *be a power series and suppose that* $z_1 \in \mathbb{C}$ *is so that the power series converges at* z_1 *(i.e.* $f(z_1) \in \mathbb{C}$ *makes sense). Then* $f(z)$ *is absolutely convergent for all* z *with* $|z| < |z_1|$.

PROOF. The proof is a simple comparison to a geometric series. Since $f(z_1)$ is convergent, we know that $|c_n z_1^n| \to 0$ as $n \to \infty$. In particular, this sequence must be bounded then, i.e. we have a constant C so that $|c_n z_1^n| < C$. But then $|c_n z^n| < C \left|\frac{z}{z_1}\right|^n$ and the convergence follows by comparison to a geometric series. $\square$

COROLLARY. *For* $f(z) = \sum c_n z^n$, there exists an $R \in \mathbb{R}$ so that $f(z)$ is absolutely convergent for all z with $|z| < R$ and divergent for all z with $|z| > R$.

PROOF. Just let $R = \sup\{|z| : f(z) \text{ is convergent}\}$and the result follows from the above lemma. $\square$

DEFINITION. The R above is often called the *radius of convergence.*

COROLLARY. *Power series are continuous inside their radius of convergence.*

PROOF. Recall the Weirstrass M-Test says if $|f_n(x)| < M_n$ and if $\sum M_n < \infty$, then $\sum f_n(x)$ is converging uniformly. Using this convergence test, along with the fact that a uniform limit of continuous functions is continuous we get the result. $\square$

THEOREM. *[Power series are differentiable] For a power series $f(z) = \sum c_n z^n$, we have that f is differentiable with:*

$$f'(z) = \sum n c_n z^{n-1}$$

PROOF. We introduce the notation: $f_n(z) = \sum_{k=1}^{n} c_k z^k$, $g_n(z) = \sum_{k=1}^{n} k c_k z^k$, and $g(z) = \sum_{k=1}^{\infty} k c_k z^{k-1}$. The problem becomes showing that f is differentiable and $f' = g$. We first remark that g has the same (or larger) radius of convergence as f, indeed following the idea of the lemma 1.2.7 above, let $f(z_1)$ be convergent we get the comparison that $|k c_k z^{k-1}| \leq Ck \left|\frac{z}{z_1}\right|^{k-1}$, so the series for $g(z)$ converges by comparison to an arithmetico-geometric sequence whenever $|z| < |z_1|$. Next, using the Weirestrass M-Test in a similar fashion to Corollary 1.2.10, we know that f_n converges uniformly to f and g_n converges uniformly to g inside the radius of convergence. Now, using the Fundemental Theorem of Calculus applied to the Polynomials $f_n$and $g_n$we have for any h that:

$$f_n(z+h) - f_n(z) = \int_0^1 g_n(z+th)dt$$

Now, by our uniform convergence, we can take the limit $n \to \infty$ to establish that:

$$f(z+h) - f(z) = \int_0^1 g(z+th)dt$$

Since this holds for any h, we can now divide by h and take the limit $h \to 0$. Noticing that $\lim_{h\to 0} h^{-1} \int_0^1 g(z+th)dt = g(z)$ by the mean value theorem gives the result. $\square$

DEFINITION. A *Laurent series* is a two-sided power series of the form $f(z) = \sum_n c_n z^n + \sum_n c_{-n} z^{-n}$.

REMARK. One can see that Laurent series convergence inside an annulus (i.e. a donut) because the first sum convergences inside a circle of radius R_1 while the second will converge outside a circle of radius R_2 (since $\left|z^{-1}\right| < R \iff |z| > R^{-1}$).

EXAMPLE. A branch of the logarithm is another example of a differentiable function. This can be defined by removing a ray from $\mathbb{C}$so that we can have in polar coordinates that $-\pi < \theta < \pi$ and then define $\log\left(re^{i\theta}\right) = \log r + i\theta$.One can verify (using a change of variables to polar coordinates) that $\frac{d}{dz}\log z = \frac{1}{z}$.

1.3. Fractional Linear Transformations

Recall that a fractional linear transformation is one of the form $\phi(z) = \frac{az+b}{cz+d}$ where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. The relationship between fractional linear transformations and matrices is made by viewing $\mathbb{C}$ in so called *homogenous coordinates.*

Definition. Consider $\mathbb{C} \times \mathbb{C}$ with the equivalence relationship $z_1 \times z_2 \sim w_1 \times w_2 \iff \exists \lambda \in \mathbb{C} : w_1 = \lambda z_1, w_2 = \lambda z_2$. It is an easy exercise to verify that the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ can be written in these coordinates, namely:$\hat{\mathbb{C}} \cong {}^{\mathbb{C}\times\mathbb{C}}/_{\sim}$ with the identifiction $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} := \frac{z_1}{z_2}$. These coordinates are called "homogenous coordinates"

Remark. These coordinates show exactly how fractional linear transformations behaive like matrices, for if $w = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, then:

$$\frac{aw+b}{cw+d} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

From this, we can easily see the rules for how fractional linear transformations compose/invert etc. i.e. if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\phi_A(z) = \frac{az+b}{cz+d}$ then $\phi_A \circ \phi_B = \phi_{AB}$ etc. We can also see, since we are only looking at invertible matrices, that these maps are bijections from $\hat{\mathbb{C}}$ to itself.

Proposition. *Let ϕ_A be a fractional linear transformtion. Then $\phi_A(z) = z$ has at most two solutions, unless $A = Id$.*

Proof. Write out $\phi_A(z) = \frac{az+b}{cz+d}$, so that $\phi_A(z) = z$ gives $(cz+d)z = az+b$. This is a quadratic equation! If $c \neq 0$ then it is non-degenerate and the quadratic formula gives exactly two solutions. If $c = 0$ then it is really a linear transformation, and there are NO solutions unless. $\square$

Proposition. *Fractional linear transformations are uniquely specified by the action on 3 points.*

Proof. Suppose M, N are two fractional linear transformations that have $M(z_i) = w_i$ for $i = 1, 2, 3$. Then $M \circ N^{-1}$ is a fractional linear transformation with 3 fixed points. By the above proposition, $M \circ N^{-1}$ must be the identity! Hence $M = N$. This shows uniqueness. The existence of such a map is easyily verified by expicitly writing down the map that sends $z_1 \to 0$, $z_1 \to 1$, and $z_3 \to \infty$:

$$\phi(z) = \frac{z-z_2}{z-z_3} \Big/ \frac{z_1-z_2}{z_1-z_3}$$

$\square$

Proposition. *The image of any circle or line in the complex plane through a fractional linear transformations is again a circle or line.*

Proof. It suffices to verify that inversion $z \to \frac{1}{z}$ has this property, as every FLT can be written as a composition of dilation $z \to az$, translation $z \to z + b$ and inversion, and the first two clearly preserve circles and lines. This can be verified directly with a not-so-difficult analytic computation, or can be seen by looking at the stereographic projection. $z \to \frac{1}{\bar{z}}$ is circle inversion, which is a reflection of the sphere through the plane, while $z \to \bar{z}$ is a reflection of the sphere through the vericle plane passing through the real axis. Hence $z \to \frac{1}{z}$ is the composition of these two operations, and can be done in one shot as a rotation of the sphere! □

Remark. Fractional linear transformations also preserve symmetric points. This can be verified analytically.

Example. Find the fractional linear transformation that takes the point z_0 in the unit disk D to the origin, and also takes the unit disk to the unit disk.

Proof. Firstly, we see why such an FLT must exist. One way is to map the unit disk to the upper half plane (take any three points on the boundary to 0,1,∞ to do this). Then z_0 will be some point in the upper half plane. From here we can take $z_0$to i with a dilation and a translation, this will also ensure that the points of the upper half plane don't change. Finally, we map back to the unit disk in such a way so that i goes to the origin.

To find the solution, we want ϕ with $z_0 \to 0$. Since FLTs preserve symetric points we must have $\frac{1}{\bar{z_0}} \to \infty$ too. This suggests a FLT of the form:

$$\phi(z) = C\frac{z - z_0}{z - \frac{1}{\bar{z_0}}}$$

To ensure this keeps the unit disk in place, we impost that $|\phi(1)| = 1$ and get the condition $|C| = \left|\frac{1-\frac{1}{\bar{z_0}}}{1-z_0}\right|$ so the most general solution is:

$$\phi(z) = e^{i\theta}\left|\frac{1 - \frac{1}{\bar{z_0}}}{1 - z_0}\right|\frac{z - z_0}{z - \frac{1}{\bar{z_0}}}$$

Notice there is one parameter of freedom here! We have specified the image of two points $(z_0, \frac{1}{\bar{z_0}})$ and we know there is a one parameter family for the image of the point 1 (has to be on the unit circle). Hence, since 3 points uniquely determine a FLT, there is a one parameter family of FLTs here. □

Definition. The *cross-ratio* of four points (z_1, z_2, z_3, z_4) is defined by:

$$(z_1, z_2, z_3, z_4) = \tfrac{z_1-z_3}{z_1-z_4}/\tfrac{z_2-z_3}{z_2-z_4}$$

Alternatively, it is the image of z_1 in the unique FLT that takes $z_2 \to 1$, $z_3 \to 0$ and $z_4 \to \infty$.

Proposition. *For an FLT ϕwe have that* $(z_1, z_2, z_3, z_4) = (\phi z_1, \phi z_2, \phi z_3, \phi z_4)$.

Proof. Can be checked directly. Alternatively, let ψ_{z_2, z_3, z_4} be the unique FLT that takes $z_2 \to 1, z_3 \to 0, z_4 \to \infty$. Then check that $\psi_{z_2,z_3,z_4}(z) = \psi_{\phi z_2, \phi z_3, \phi z_4}(\phi z)$ since both have the property that $z_2 \to 1, z_3 \to 0, z_4 \to \infty$. Now the result follows by the alternative definition of the cross-ratio: the cross ration is the image of z_1 through this map. □

1.4. Rational Functions

A rational function is a function of the form $R(z) = \frac{P(z)}{Q(z)}$ with P, Q polynomials. For convenience, we will label the roots of Q by $\beta_1 \ldots \beta_n$. By the quotient rule, one can see that R is holomorphic on $\mathbb{C}\backslash\{\beta_1, \ldots, \beta_n\}$. Notice that $\lim_{z\to\beta_j} R(z) = \infty$

. We call such points *poles* of the function. The following theorem is a useful decompostion for rational functions:

THEOREM. *Suppose that* $R(z) = \frac{P(z)}{Q(z)}$ *is a rational function with poles* $\beta_1 \dots \beta_n$. *By using the division algorithm, we can write* $R(z) = G(z) + H(z)$ *where* $G(z)$ *is a polynomial without a constant term, and* $H(z)$ *is a rational function with the degree of the numerator not more than the degree of the denominator. In the same vein, we can consider the rational function* $R(\beta_j + \frac{1}{\zeta})$ *and write* $R(\beta_j + \frac{1}{\zeta}) = G_j(\zeta) + H_j(\zeta)$, *where* G_j *and* H_j *are akin to* G, H *respectively. Then we have:*

$$R(z) = G(z) + \sum_{j=1}^{n} G_j\left(\frac{1}{z-\beta_j}\right)$$

PROOF. Doing a change of variable $z = \beta_j + \frac{1}{\zeta}$ on $R(\beta_j + \frac{1}{\zeta}) = G_j(\zeta) + H_j(\zeta)$, we have $R(z) = G_j\left(\frac{1}{z-\beta_j}\right) + H_j\left(\frac{1}{z-\beta_j}\right)$. Since G_j is a polynomial, $G_j\left(\frac{1}{z-\beta_j}\right)$ is a rational function and it can only have a pole at $z = \beta_j$. Notice also that $H_j\left(\frac{1}{z-\beta_j}\right)$ is finite at $z = \beta_j$, because $z = \beta_j$ corresponds to $\zeta = \infty$, and since $H_j(\zeta)$ is a rational function with the degree of the denominator $\geq$degree of numerator, the limit $\zeta \to \infty$ is finite.

Now consider the expression:

$$R(z) - \left(G(z) + \sum_{j=1}^{n} G_j\left(\frac{1}{z-\beta_j}\right)\right)$$

This is a sum of rational functions, and it can only possibly have poles at $z = \beta_1, \dots, \beta_n$. For a fixed j, the only terms with poles at $z = \beta_j$ are $R(z)$ and $G_j\left(\frac{1}{z-\beta_j}\right)$. However, since $R(z) - G_j\left(\frac{1}{z-\beta_j}\right) = H_j\left(\frac{1}{z-\beta_j}\right)$, and since $\lim_{z\to\beta_j} H_j\left(\frac{1}{z-\beta_j}\right)$ is finite, there is in fact NO pole at $z = \beta_j$. That is to say $\lim_{z\to\beta_j} R(z) - \left(G(z) + \sum_{j=1}^{n} G_j\left(\frac{1}{z-\beta_j}\right)\right) = G(\beta_j) + \sum_{k\neq j} G_k\left(\frac{1}{\beta_j-\beta_k}\right) + \lim_{z\to\beta_j} H_j\left(\frac{1}{z-\beta_j}\right)$ is finite. Since this works for every choice of j, we see that this expression is a rational function with no poles at all. But the only such rational function is a constant. Absorbing the constant into $G(z)$ gives the result. □

1.5. The Cauchy-Riemann Equations

Suppose that $f : \Omega \to \Omega$ is holomorphic. We can think of f as a function $f : \mathbb{R}^2 \to \mathbb{C}$ by the identification $f(x, y) = f(x + iy)$. Notice that:

$$\frac{\partial f}{\partial x} = \lim_{\delta\to 0} \frac{f((x+\delta)+iy) - f(x+iy)}{\delta} = f'(z)$$

The $x-$derivative is the directional derivative in the complex plane in the real-axis-direction! Similarly:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \lim_{\delta\to 0} \frac{f(x+i(y+\delta)) - f(x+iy)}{\delta} \\ &= \lim_{\delta\to 0} i\frac{f(x+iy+i\delta) - f(x+iy)}{i\delta} = if'(z) \end{aligned}$$

Hence it must be the case that $f_y = if_x$! This is known as the Cauchy-Riemann equation. If we write the real and imaginary part of f separately, say $f(z) = u(z)+iv(z)$ so that in our identification $z = x+iy$ we have $f(x,y) = u(x,y)+iv(x,y)$ then by matching the real and imaginary parts the C-R equation is saying that:

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

Conversely, if we have a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is $\mathbb{R}^2-$differentiable, and f satisfies the Cauchy Riemmann equations above, then one can show that f is indeed a holomorphic function. To see this, we suppose without loss of generality by translations that $f(0) = 0$ and that we have only to prove differentialbility at 0. By Taylors theorem, we know that in a n'h'd of 0 we can write $f(x,y) = u(x,y) + iv(x,y) = ax + by + \eta(z)z$ with $\eta(z) \to 0$ as $z \to 0$ and $a = f_x$, $b = f_y$. Rewriting, this is:

$$\begin{aligned} f(z) &= ax + by + \eta(z)z \\ &= \frac{a+ib}{2}z + \frac{a-ib}{2}\bar{z} + \eta(z)z \end{aligned}$$

But, the C-R equations are exactly saying that $a - ib = 0$, so the $\bar{z}$ term vanishes, and we get:

$$\frac{f(z)}{z} = \frac{a+ib}{2} + \eta(z)$$

Since the limit exists as $z \to 0$, we see that f is indeed differentiable here. Notice that if $f = u + iv$ satisfies the C-R equations, then $\Delta u = u_{xx} + u_{yy} = 0$ and $\Delta v = 0$ (this is immediate by equality of mixed partials when the function is twice differentiable; we will see later that all holomorphic functions are twice differentiable!) The converse is also true, given u with $\Delta u = 0$, there exists a holomorphic function f so that $f = u+iv$. This boils down to an existence theorem for PDE;s in practice.

The Complex Numbers

These are notes from Chapter 1 of [**1**]. Basic properties of complex numbers and so on go here.

DEFINITION. An open connected set is called a **region**. All regions are <u>polygonally connected</u>, which means that there is a path from one point to any other point which consists of a finite number of lines. Every region is polygonally connected; let $U = \{\text{points reachable by a polygonal line}\}$ and since the unit ball is polygonally connected, U is open. Similarly, the set $S \backslash U$ is open. By connectedness, $U = S$.

THEOREM. *(1.9) (M-test) Suppose f_k is continuous in D, if $|f_k(z)| \leq M_k$ throughout D and if $\sum_{k=1}^{\infty} M_k < \infty$ then $\sum_{k=1}^{\infty} f_k(z)$ converges (uniformly) to a function f which is continuous in D.*

PROOF. It is clear that at each point, $|\sum_{k=1}^{\infty} f_k(z)| \leq \sum_{k=1}^{\infty} M_k < \infty$ so we can define its pointwise limit f. Moreover, the convergence is uniform, because for any $\epsilon > 0$ choose N so large so that $\sum_{k=N}^{\infty} M_k < \epsilon$ and we will have $\left| f - \sum_{k \geq N} f_k \right| < \epsilon$. A uniform limit of continuous functions is continuous by a classic $\epsilon/3$ argument: Given $\epsilon > 0$ choose N so large so that $\left| f - \sum_{k \geq N} f_k \right| < \epsilon/3$ and then choose δ for $\epsilon/3$-continuity of f_N and have $|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3$. $\square$

THEOREM. *(1.10) Suppose $u(x, y)$ has partial derivatives u_x and u_y that vanish at every point in a region D. Then u is constant on D.*

PROOF. Let (x_1, y_1) and (x_2, y_2) be any two points in D. They are connected by a polygonal path. On each segment of the polygonal path, we have $u(y) - u(x) = \int v \cdot u(z) \mathrm{d}z = 0$ for some direction v. Hence u is constant on each segment. $\square$

(Section on stereographic projection here)

DEFINITION. (1.11) We say $z_k \to \infty$ if $|z_k| \to \infty$.

PROPOSITION. *Stereographic projection takes circles/lines to circles/lines*

Functions of the Complex Variable z

These are notes from Chapter 2 of [**1**].

3.6. Analytic Polynomials

Definition. A polynomial $P(x, y)$ is called **analytic polynomial** if there exists complex constants α_k such that:

$$P(x, y) = \sum \alpha_k (x + iy)^k$$

Notice that "most" polynomials of two variables are NOT analytic, only very special ones are.

Proposition. *A polynomial is analytic if and only if $P_y = iP_x$*

Proof. ($\Rightarrow$) is clear, since it holds for each term $(x + iy)^k$ individually.

($\Leftarrow$) We will show this folds for any polynomial where the degree of any term is n (i.e. the degree of the x power plus the degree of the y power) by linearity this it will then hold for all polynomials. Suppose $Q_y = iQ_x$. then write out $Q = \sum C_k x^{n-k} y^k$. $Q_y = iQ_x$ gives us a linear system of equations, which can be solved one at a time inorder and leads to $C_k = i^k \binom{n}{k} C_0$. But then $Q = C_0(x + iy)^n$ by the binomial theorem. □

Remark. $f_y = if_x$ as complex variables is the same as :

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

Where $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$. These are called the **Cauchy Riemann Equations**

Definition. A complex valued fuction is called differentiable at z if:

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

exist. Here we mean $h \to 0$ w.r.t. the topology on $\mathbb{R}^2$. The sum rule, product rule, and quotient rule all work for differentiable functions.

3.7. Power Series

Definition. A **power series** in z is an infinite series of the form $\sum_{k=0}^{\infty} C_k z^k$

Theorem. *(2.8.) Suppose that* $\limsup |C_k|^{1/k} = L$ *then:*
i) If $L = 0$ then $\sum C_k z^k$ converges for all z
ii) If $L = \infty$ then $\sum C_k z^k$ converges for $z = 0$ only
iii) If $0 < L < \infty$ then let $R = \frac{1}{L}$ and we will have that $\sum C_k z^k$ converges for $|z| < R$ and diverges for $|z| > R$.

*R here is called the **radius of convergence** of the power series.*

PROOF. All the proofs go by comparison to the geometric power series and the definition of limsup. □

3.8. Differentiability and Uniqueness of Power Series

THEOREM. *(2.9.) If $f(z) = \sum C_n z^n$ converges for $|z| < R$ then $f'(z)$ exists and equals $\sum nC_n z^{n-1}$ throughout $|z| < R$*

PROOF. (Can do this by hand by looking at $\frac{(z+h)^n - z^n}{h}$, here is a slicker way) First see that the sequence $g(z) = \sum nC_n z^{n-1}$ has the same radius of convergence. Then let $f_n = \sum_{k=1}^n C_k z^k$ and $g_n = \sum_{k=1}^n kC_k z^{k-1}$ Now $f_n \to f$ and $g_n \to g$ uniformly on compact subsets inside the radius of convergence. Notice that for every $g_n = f_n'$ and so we have for any h that $\frac{f_n(z+h) - f_n(z)}{h} = \int_0^1 g_n(z + ht)\mathrm{d}t$ (think of this as a real integral by looking at the imaginary and real parts seperatly) Since $f_n \to f$ uniformly and $g_n \to g$ uniformly, have the same holding for f and g. But then taking $h \to 0$ (use LDCT) we get that $f' = g$ as desired. □

COROLLARY. *Power series are infinetly differentiable within their domain of convergence*

COROLLARY. *If $f(z) = \sum_{n=0}^\infty C_n z^n$ has a non-zero radius of convergence then:*

$$C_n = \frac{f^{(n)}(0)}{n!} \text{ for all } n$$

THEOREM. *(Uniqueness Theorem for Power Series)*
Suppose $\sum C_n z^n$ vanishes for all points of a non-zero sequence z_k and $z_k \to 0$ as $k \to \infty$. Then $\sum C_n z^n \equiv 0$

PROOF. Have $C_0 = \lim_{k\to\infty} f(z_k) = 0$. Now let $f_1(z) = f(z)/z$, since $C_0 = 0$ $f_1 = \sum_{k\geq 1} C_k z^k$. Since $z_k \neq 0$ for all k and $f(z_k) = 0$ we have that $f_1(z_k) = 0$ for every k too, now $C_1 = \lim_{k\to\infty} f_1(z_k) = 0$ too! Repeating this argument shows that every coefficient is zero. □

COROLLARY. *If a power series equals zero at all the points of a set with an accumulation point at the origin, the power series is identically zero.*

Analytic Functions

These are notes from Chapter 3 of [**1**].

4.9. Analyticity and the Cauchy-Riemann Equations

PROPOSITION. *(3.1) If f is differentiable then $f_y = i f_x$.*

PROOF. $f_x = \lim_{h\to 0} \frac{\Delta f}{h}$ for h real while $\frac{f_y}{i} = \lim_{h\to 0} \frac{\Delta f}{h}$ for h imaginary. Since the limit exists, these are equal. $\square$

PROPOSITION. *(3.2.) If f_x and f_y eixsts in a n'h'd and if f_x and f_y are continuous at z and $f_y = i f_x$ there, then f is differentiable at z.*

PROOF. Can show that $\lim_{h\to 0} \frac{\Delta f}{h} = f_x$ by using the mean value theorem to find, for any h, a real number $0 < a < \mathrm{Re}(h)$ and $0 < b < \mathrm{Im}(h)$ so that

$$\begin{aligned} f(x+h) - f(x) &= f(x+h) - f(x+\mathrm{Re}(h)) + f(x+\mathrm{Re}(h)) - f(x) \\ &= f_y(x+\mathrm{Re}(h)+ib)\mathrm{Im}(h) + f_x(x+a)\mathrm{Re}(h) \end{aligned}$$

Now we are almost done, since we see that by continuiy of f_x and f_y the RHS, loosely speaking, looks like $f_y(x)\mathrm{Im}(h) + f_x(x)\mathrm{Re}(h) = f_x(x)\left(i\mathrm{Im}(h) + \mathrm{Re}(h)\right) = h f_x(x)$ and this will prove the result. This can be made more precise, but I'll stop here. $\square$

DEFINITION. We say f is analytic at z if it is differntiable in a n'h'd of z and analytic in a set if it is differntiable in that set.

PROPOSITION. *(3.6) If $f = u + iv$ and u is constant in some region, then f is constant in that region too.*

PROOF. Since u is constant, $u_x = u_y = 0$ so by the C-R equations, $v_x = v_y \equiv 0$ in the region too. Now by the mean value theorem, both u and v are constant. $\square$

PROPOSITION. *(3.7) If $|f|$ is constant in a region then f is constant in the region*

PROOF. Write out $u^2 + v^2 = |f|^2 = cons;t$ so taking partials and mixing them up a bit using the C.R. equations leads us to the result. $\square$

4.10. The functions e^z, $\sin z$ and $\cos z$

We wish to find an analytic function $f(z)$ so that:

$$\begin{aligned} f(z_1 + z_2) &= f(z_1) f(z_2) \\ f(x) &= e^x \text{ for all real } x \end{aligned}$$

By putting in $z_1 = x$ and $z_2 = iy$ we arrive at:

$$f(x+iy) = e^x A(y) + ie^x B(y)$$

where A, B are real valued. By the Cauchy Riemann equations, it must be that $A(y) = B'(y)$ and $A^{'}(y) = -B(y)$ from these differential equations, we find that A and B have to linear combinations of sin and cos. But since $f(x) = e^x$ is enforced, we get $A = \cos$ and $B = \sin$ so finally we have:

$$f(z) = e^x \cos(y) + ie^x \sin(y)$$

Then define:

$$\begin{aligned} \sin(z) &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\ \cos(z) &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \end{aligned}$$

Line Integrals and Entire Functions

These are notes from Chapter 4 of [**1**].

5.11. Properties of the Line Integral

DEFINITION. (4.1.) For any curve C parametrized by $z(t)$ we define:

$$\int_C f(z)\mathrm{d}z = \int_a^b f(z(t))\dot{z}(t)\mathrm{d}t$$

By the change of variable rule (i.e. in $\mathbb{R}^2$) this is invarient under the parametrization of C (expcept for chaning direction)
.... I skip some stuff about this here....

LEMMA. *(4.9.)*

$$\left|\int_a^b G(t)\,dt\right| \leq \int_a^b |G(t)|\,dt$$

PROOF. Write $\int G(t)\mathrm{d}t = Re^{i\theta}$ for some R and θ, then multiply out by $e^{-i\theta}$, take Re part, and use the inequality $\mathrm{Re}(z) \leq |z|$. □

PROPOSITION. *(4.10) (The M-L formula)*
Suppose that C is a smooth curve of length L and f is continuous on C and that $|f| \leq M$ trhoughout C then:

$$\left|\int_C f(z)\,dz\right| \leq ML$$

PROOF. By the previous lemma, $\left|\int_C f(z)\mathrm{d}z\right| \leq \int_a^b |f(z(t))|\,|\dot{z}(t)|\,\mathrm{d}t$ If you assume a unit speed parametrization here you can finish the result, otherwise you can use the mean value theorem for integrals to get that this $= |f(z(t_0))| \int_a^b |\dot{z}(t)|\,\mathrm{d}t$ for some t_0 and then get the result from $\int |\dot{z}(t)|\,\mathrm{d}t = L$. □

PROPOSITION. *(4.11) If f_n are continuous and $f_n \to f$ uniformly on C then:*

$$\int_C f(z)\,dz = \lim_{n\to\infty} \int_C f_n(z)\,dz$$

PROOF. Take n so large so that $|f - f_n| < \epsilon$ everyewher on C and then apply the ML theorem to get that the integrals are no more than ϵ·length of C appart. □

PROPOSITION. *(4.12) If $f(z) = F'(z)$ then:*

$$\int_C f(z)\,dz = F(z(b)) - F(z(a))$$

PROOF. Take any parametrization z for the curve C and let $\gamma(t) = F(z(t))$ with a bit of work, we can show that $\dot{\gamma}(t) = f(z(t))\dot{z}(t)$ then will have:

$$\begin{aligned}\int_C f(z)\mathrm{d}z &= \int f(z(t))\dot{z}(t)\\ &= \int \dot{\gamma}(t)\mathrm{d}t\\ &= \gamma(b) - \gamma(a)\\ &= F(z(b)) - F(z(a))\end{aligned}$$

□

5.12. The Closed Curve Theorem for Entire Functions

5.12.1. Important!

DEFINITION. (4.13) A curve C is **closed** if its intial and terminal poitns coincide.

THEOREM. *(4.14) (The Rectangle Thereom)*
Suppose f is entire and Γ is the boundary of a rectangle R. Then:

$$\int_\Gamma f(z)dz = 0$$

PROOF. Suppose $\int_\Gamma f(z)\mathrm{d}z = I$. The proof goes by subdividing into smaller and smaller rectangles so that $\int_{R_k} f(z)\mathrm{d}z \gg \frac{I}{4^k}$. These rectangles converge to a point z_0 around which we have the estimate $f(z) = f(z_0)+(z-z_0)f'(z_0)+\epsilon_z(z-z_0)$ with $\epsilon_z \to 0$ as $z \to z_0$. The linear term $f(z_0) + (z - z_0)f'(z_0)$ integrates to zero around a rectangle. For any $\epsilon > 0$, choose a rectangle R_k small enough so that $\epsilon_z \ll \epsilon$ for all z on the rectangles, then use the M-L to control the size of $\int_{R(k)} f(z) \ll \epsilon\frac{c}{4^k}$. Combining the upper and lower bound on R_k gives that $I \ll c\epsilon$ and since this works for every $\epsilon > 0$ we conclude that $I = 0$. □

THEOREM. *(4.15) (Integral Theorem)*
If f is entire, then f is everywhere the derivative of an analytic function. That is there exists an entrie function F so that $F'(z) = f(z)$

PROOF. Define $F(z)$ as $\int_{[0,\mathrm{Re}z]+[\mathrm{Re}z,z]} f(\zeta)\mathrm{d}\zeta$. Since the integral around a rectangle is zero, we get that $F(z + h) - F(z) = \int_{[z,z+\mathrm{Re}h]+[z+\mathrm{Re}h,z+h]} f(\zeta)\mathrm{d}\zeta$ and hence $\frac{F(z+h)-F(z)}{h} - f(z) = \int_z^{z+h} (f(\zeta) - f(z))\,\mathrm{d}\zeta \to 0$ as $h \to 0$ by the $M - L$ theorem since f is continuous here. □

THEOREM. *(4.16.) (Closed Curve Theorem)*
If f is entire and if C is a closed curve then:

$$\int_C f(z)dz = 0$$

PROOF. Write $f(z) = F'(z)$ for some analytic F. Then $\int_C f(z)\mathrm{d}z = F(z(b)) - F(z(a)) = 0$ since C is a closed curve. □

REMARK. The closed curve theorem works as long as $f = F'$ for some anaytic F. The integral theorem tells us that every analytic function is of this form, but there are additionally some other non-analytic functions that are of this form. For example z^{-2} is not anayltic at 0, but it has $z^{-2} = \left(-z^{-1}\right)'$ on the curve C so the integral would still be zero.

THEOREM. *(Morera's Theorem)*

(This comes later in the book and uses the fact that analytic functions are infinetly differentiable...but I will put it here too)

If f is continuous on an open set D and if $\int_\Gamma f(z)dz = 0$ for every rectangle Γ, then f is analytic on D

PROOF. Write $F(z) = \int_{[0,\mathrm{Re}z]+[\mathrm{Re}z,z]} f(\zeta)\mathrm{d}\zeta$. As in the proof of the Integral theorem, (using the hypotheiss that $\int_\Gamma f(z)\mathrm{d}z = 0$ for rectangles) we have that $F'(z) = f(z)$. Now, since F is analytic it is twice differentiable (this fact will be proven later...can be thought of as a consequence of Cauchy integral formula), so $f'(z) = F''(z)$ is once differentiable. □

Properties of Entire Functions

These are notes from Chapter 5 of [**1**].

6.13. The Cauchy Integral Formula and Taylor Expansion for Entire Functions

6.13.1. Important! A common technique/tool in complex analysis is to start with a function f and define a function g by:

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$$

We will show that the rectangle theorem still applies to g.

THEOREM. *(5.1.) (Rectangle Theorem II)*
g defined as above has $\int_\Gamma g(\zeta)d\zeta = 0$ for all rectangle Γ.

PROOF. We consider three cases:
i) If $a \in \text{ext } R$. In this case, by the quotient rule, g is analytic trhoughout the rectangle so the proof is exactly the same as in the proof of the rectangle theorem.
ii) If $a \in \Gamma$ or $\in \text{int } R$. Divide up the rectangle into smaller rectangles so that the point a is isolated to a very small rectangle. On the rectangles that don't touch a, we are reduced to case i. For the rectangle that touches a, use the ML theorem (g is continuous and hence bounded by some M on the rectangle) to see that the integral $\to 0$ as the size of the rectangle shrinks. □

COROLLARY. *The integral theorem and the closed curve theorem both apply to g.*

PROOF. g is continuous and has the rectangle theorem, so the proofs go as they did before. □

THEOREM. *(5.3.) (Cauchy Integral Formula)*
If f is entire and a is a fixed complex number and C is a circle centered at the origin a (i.e. $R > |a|$) then:

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

PROOF. By the closed curve theorem for g we have that:

$$\int_C \frac{f(z)-f(a)}{z-a}\mathrm{d}z = 0$$

Rearranging and using that $\int \frac{\mathrm{d}z}{z-a} = 2\pi i$ gives the result (pf of this is in the next lemma). □

LEMMA. *(5.4.) If a circle C_ρ of radius ρ centered at α contains the point a then:*

$$\int_{C_\rho} \frac{dz}{z-a} = 2\pi i$$

PROOF. Check by bare hands definition using the parametrization $\rho e^{i\theta} + \alpha$ that:

$$\int_{C_\rho} \frac{\mathrm{d}z}{z-\alpha} = 2\pi i$$

Now for any k we see that:

$$\int_{C_\rho} \frac{\mathrm{d}z}{(z-\alpha)^{k+1}} = 0$$

This can be seen in two different ways:
i) bare hands again (works out to the same calculation as before)
ii) by the remark to the integral formula, its ok because $(z-\alpha)^{-(k+1)}$ is an antiderivative on C.
Now with these two facts in hand, we do an expansion:

$$\begin{aligned} \frac{1}{z-a} &= \frac{1}{(z-\alpha)+(\alpha-a)} \\ &= \frac{1}{z-\alpha}\frac{1}{1-\frac{a-\alpha}{z-\alpha}} \\ &= \frac{1}{z-\alpha}\left(1+\frac{a-\alpha}{z-\alpha}+\left(\frac{a-\alpha}{z-\alpha}\right)^2+\dots\right) \end{aligned}$$

The sum converges uniformly inside the circle because $|a-\alpha| < |z-\alpha| = R$ on the conotur in question. Have then:

$$\begin{aligned} \int_{C_\rho} \frac{1}{z-a}\mathrm{d}z &= \int \frac{1}{z-\alpha}\mathrm{d}z + (a-\alpha)\int_{C_\rho} \frac{1}{(z-\alpha)^2}\mathrm{d}z + \dots \\ &= 2\pi i + 0 + 0 + \dots \end{aligned}$$

The interchange of integral with the infinite sum is ok because the sum converges uniformly. □

THEOREM. *(5.5.) (Taylor Expansion of an Entire Function)*
If f is entire, it has a power series representation. In fact $f^{(k)}(0)$ exists for all k and:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

PROOF. By the Cauchy integral formula, we can choose a circle C containing the point z and then have:

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\mathrm{d}w$$

Now we can do the following expansion for $\frac{1}{w-z}$:

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{w}\frac{1}{1-\frac{z}{w}} \\ &= \frac{1}{w}\left(1+\frac{z}{w}+\left(\frac{z}{w}\right)^2+\dots\right)\end{aligned}$$

We know that this converges uniformly inside the circle since $|z| < |w| = R$ on the contour in question. Have then:

$$f(z) = \left(\frac{1}{2\pi i}\int_C \frac{f(w)}{w}\mathrm{d}w\right) + \left(\frac{1}{2\pi i}\int_C \frac{f(w)}{w^2}\mathrm{d}w\right)z + \left(\frac{1}{2\pi i}\int_C \frac{f(w)}{w^3}\mathrm{d}w\right)z^2 + \dots$$

The switch of the integral and the infinte sum is ok becasue the sum converges uniformly here.

This shows that $f(z)$ is equal to (in the circle we chose at least) an infinte power series. By our study of power series, we know then that $\left(\frac{1}{2\pi i}\int_C \frac{f(w)}{w^2}\mathrm{d}w\right) = \frac{f^{(k)}(0)}{k!}$ and this completes the proof. $\square$

Corollary. *(5.6.) An entire function is infinitely differentiable.*

Proof. We saw that power series are infitetly differentiable, the above shows that every entire function is a power series. $\square$

Proposition. *(5.8) If f is entire and we define $g(z) = \frac{f(z)-f(z)}{z-a}$ for $z \neq a$ and $f'(a)$ for $z = a$ then g is entire.*

Proof. Let $h(z) = f^{'}(a) + \frac{f''(a)}{2!}(z-a) + \dots$. By the Taylor series expansion for f when $z \neq a$ we see that $g = h$. We also notice that $h(a) = g(a) = f'(a)$. Hence $g \equiv h$. Since g is a power series, it is entire. $\square$

Corollary. *(5.9.) If f is entire with zeros at $a_1, a_2, \dots$ then we may define:*

$$g(z) = \frac{f(z)}{(z-a_1)\dots(z-a_n)} \text{ for } z \neq a_j$$

and defined by $g(a_k) = \lim_{z\to a_k} g(z)$ when these limits exists. Then g is entire.

Proof. Induction using the last proposition. $\square$

6.14. Liouville Theorems and the Fundemental Theorem of Algebra

Theorem. *(5.10) (Liouville's Theorem)*
A bounded entire function is constant

Proof. (The idea is to get the value of the function from the Cauchy integral formula. If the function is bounded we have a bound on the integral from the ML theorem)

Take any two points a and b and then take a circle of radius R centerd at the origin and large enough to contain both points. Then have the by the CIF that:

$$\begin{aligned} f(b)-f(a) &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z-b}\mathrm{d}z - \frac{1}{2\pi i}\int \frac{f(z)}{z-a}\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_C \frac{f(z)(b-a)}{(z-a)(z-b)}\mathrm{d}z \\ &\ll \frac{M\left|b-a\right|R}{\left(R-\left|a\right|\right)\left(R-\left|b\right|\right)} \\ &\to 0 \end{aligned}$$

as $R \to \infty$. Hence $f(b) = f(a)$ $\square$

THEOREM. *(5.11) (Extended Liouville Theorem)*
If f is entire and $\left|f(z)\right| \leq A + B\left|z\right|^k$ then f is a polynomial of degree at most k.

PROOF. There are two proofs
(1) By induction using $g(z) = \left(f(z) - f(0)\right)/z$ which effectivly reduces the value of k by one.
(2) Use the "extended Cauchy Integral formula" that we found earlier that:

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_C \frac{f(w)}{w^k}\mathrm{d}w$$

and then do the same type of argument as from the ordinarty Louiville theorem. $\square$

THEOREM. *(5.12) (Fundemental Theorem of Algebra)*
Every non-constant (with complex valued coefficients) polynomial has a zero in $\mathbb{C}$

PROOF. Otherwise $1/P(z)$ is an entire function, and it is bounded since $\left|P(z)\right| \to \infty$ as $\left|z\right| \to \infty$ (highest order term dominates eventaully). By Louivlle's thm, $1/P(z)$ must be constant, a contradiciton. $\square$

Properties of Analytic Functions

These are notes from Chapter 6 of [**1**].

We will now look at functions which are not entire, but instead are only analytic in some region.

7.15. The Power Series Representation for Functions Analytic in a Disc

THEOREM. *(6.1.) Supposef is analytic in $D = D(\alpha; r)$. If the closed rectanlge R and the point a are both contined in D and Γ is the boundary of R then:*

$$\int_\Gamma f(z)dz = \int_\Gamma \frac{f(z) - f(a)}{z - a} dz = 0$$

PROOF. Same idea as for entire functions. □

DEFINITION. From now on we write $g(z) = \frac{f(z)-f(a)}{z-a}$ to mean this when $z \neq a$ and its limit $\lim_{z\to a}$ when $z = a$.

THEOREM. *(6.2.) If f is analytic in $D(\alpha; r)$ and $a \in D(\alpha; r)$ then there exist functions F and G anayltic in D such that:*

$$F'(z) = f(z),\ G'(z) = \frac{f(z) - f(a)}{z - a}$$

PROOF. Same idea as for entire functions. □

THEOREM. *(6.3.) If f and a are as above, and C is any smooth closed curve contained in $D(\alpha; r)$ then:*

$$\int_C f(z)dz = \int_C \frac{f(z) - f(a)}{z - a} dz = 0$$

PROOF. Same idea as for entire functions. □

THEOREM. *(6.4.) (Cauchy Integral Formula) If f is analytic in $D(\alpha; r)$ and $0 < \rho < r$ and $|a - \alpha| < \rho$ then:*

$$f(a) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z - a} dz$$

PROOF. Same idea as for entire functions. □

THEOREM. *(6.5.) Power Series Representation for Functions Analytic in a Disc*

If f is analytic in $D(\alpha; r)$ there exist constants C_k such that:

$$f(z) = \sum_{k=0}^{\infty} C_k (z - \alpha)^k$$

for all $z \in D(\alpha; r)$

PROOF. Same idea as for entire functions. □

7.16. Analytic in an Arbitarty Open Set

The above methods cannot work in an arbitary open set because we really on the fact that $\left|\frac{a-\alpha}{w-\alpha}\right| < 1$ to hold in our open set. The best we can do is:

THEOREM. *(6.6.) If f is analytic in an arbitary open domain D, then for each $\alpha \in D$ there exist constants $C_k$such that:*

$$f(z) = \sum_{k=0}^{\infty} C_k (z-\alpha)^k$$

for all points z inside the largest disc centered at α and contained in D.

7.17. The Uniqueness, Mean-Value and Maximum-Modulus Theorems

We now explore consequences of the power series representation.
Here is a local version of Prop 5.8.

PROPOSITION. *(6.7.) If f is analytic at α, so is:*

$$g(z) = \begin{cases} \frac{f(z)-f(\alpha)}{z-\alpha} & z \neq \alpha \\ f'(\alpha) & z = \alpha \end{cases}$$

PROOF. By Thm 6.6., in some n'h'd of α we have a power series rep for f. Thus we have a power series rep for g in the same n'h'd, so g is analytic. □

THEOREM. *(6.8.) If f is analytic at z, then f is infintely diff at z*

PROOF. If its analytic, then it has a power series rep, so its infitenly diff. □

THEOREM. *(6.9.) Uniqueness Thm*
Suppose that f is analytic in a region D and that $f(z_n) = 0$ where z_n is a sequence of distinct points $z_n \to z_0 \in D$. Then $f \equiv 0$ in D.

PROOF. By the power series rep for f in some disk centered at z_0, and the uniqueness theorem for power series, we know that $f \equiv 0$ in some disk containing z_0. To see that f is identically 0 now, do a connectedness argument with the set $A = \{z \in D : z \text{ is a limit of zeros of } f\}$ and A^c. □

REMARK. The point $z_0 \in D$ is required. For example $\sin(\frac{1}{z})$ has zeros with a limit point at 0, but it is not identlically zero as $0 \notin D$

THEOREM. *(6.11.) If f is entire and $f(z) \to \infty$ as $z \to \infty$, then f is a polynomial.*

PROOF. f is $\gg 1$ outside some disk R, so it can have no zeros outside R. It can have only finetly many zeros inside R or else it would be identically zero by Bolzanno Weirestrass+Uniqueness thm. Divide out those zeros to get a function with no zeros. Then look at $1/g = (\text{zero monomials})/f \leq A + |z|^N$ shows that $1/g$ is a polynomial by extended Louiville thm. □

REMARK. The uniqueness thm can be used to show that things like $e^{z_1+z_2} = e^{z_1}e^{z_2}$ which are true on the real line, are true everywhere.

We now examine the local behaviour of analytic functions

THEOREM. *(6.12.) Mean Value Theorem*

If f is analytic in D then $f(\alpha)$ is equal to the mean value of f taken around the boundary of any disc centered at α and contained in D that is:

$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})d\theta$$

PROOF. This is just the Cauchy Integral Formula with the paramaterization $z = \alpha + re^{i\theta}$ □

DEFINITION. We call a point z a **relative maximum** for a function f if $|f(z)| \geq |f(w)|$ for all w in a n'h'd of z. A relative maximum is defined similarly,

THEOREM. *(6.13) Maximum-Modulus Theorem*

A non-constant analytic function in a region D does not have any interior maximum points.

PROOF. If z was an interior maximum, then it would violate the mean value theorem unless it is constant on a disk. □

DEFINITION. We say that f is "$C-$**analytic in a region** D" to mean that f is analytic in D and is continous on $\bar{D}$.

THEOREM. *Maximum-Modulus Theorem 2*

If f is $C-$analytic in a compact domain D, then f achieves its maximum on ∂D.

THEOREM. *(6.14) Minimum Modulous Principle*

If f is a non-constant analytic fucntion in a region D, then no point $z \in D$ can be a relative minimum of f unless $f(z) \equiv 0$

PROOF. Assume by contradiction that $f(z) \neq 0$. Restrict attention to a disk around z where $f \neq 0$ by continuity, then apply the maximum modulous thm to $g = 1/f$ in this n'h'd. □

REMARK. The Maximum-Modulous Principle can also be seen by examing local power series. Around any point α we have a power series:

$$f(z) = C_0 + C_k(z-\alpha)^k + \text{higher order terms}\ldots$$

where k is the first non-zero term. Choose θ so that C_0 and $C_k e^{i\theta}$ have the same direction, and we will have:

$$f(\alpha + \delta e^{i\theta/k}) = C_0 + C_k\delta^k e^{i\theta} + \text{higher order terms}$$

But then:

$$\begin{aligned} \left|f(\alpha + \delta e^{i\theta/k})\right| &\geq \left|C_0 + C_k\delta^k e^{i\theta}\right| - |\text{higher order terms}| \\ &= |C_0| + |C_k|\,\delta^k - O(\delta^{k+1}) \end{aligned}$$

Where we have used the face that $C_k e^{i\theta}$ is the same direction as C_0. Then taking $\delta \to 0$ the δ^k term will dominate the higher order terms and we will have a point which has higher modulus than $|f(\alpha)| = C_0$.

THEOREM. *(6.15) Anti-Calculus Proposition*

Suppose f is analytic throughout a closed disc and assumes its maximumum modulus at the boundary point α. Then $f^{'}(\alpha) \neq 0$ unless f is constant.

PROOF. Goes by examinging the power series, as we did above: if $C_1 = 0$ then there are at leat two directions which we can find a maximum in (Arg$= \theta/k + 2\pi n/k$ will work for any $0 \leq n < k$) and one of those directions must be on the interior of the disc. $\square$

Further Properties

These are notes from Chapter 7 of [**1**].

8.18. The Open Mapping Theorem; Schwarz' Lemma

The uniqueness theorem states that a non-constant function in aregion cannot be constant on any open-set. Similarly, $|f|$ cannot be constant on an open set.

Geometrically, this is saying that a non-constant analytic function cannot map an open set to a point or to a circular arc. By applyting the maximum modulous principle, we can derive the following sharper result.

THEOREM. *(7.1.) Open Mapping Theorem*
The image of an open set under an analytic function f is open.

PROOF. We will show that the image under f of some small disc centered at α will contain a disc about $f(\alpha)$. WOLOG center the picture so that $f(\alpha) = 0$

By the uniqueness theorem, find a circle C centered at α so that $f(z) \neq 0$ for $z \in C$ (if every circle had a zero, αwould be a limit points of zeros and the uniqueness theorem would tell us f is always zero). Let $2\epsilon = \min_{z \in C} |f(z)|$, we will shows that the image of the disk bounded by C contains the disk $D(0; \epsilon)$.

Suppose $w \in D(0; \epsilon)$. We wish to show that $w = f(z)$ for some z in the disk bounded by C. Notice that for all $z \in C$ we have that $|f(z) - w| \geq |f(z)| - |w| \geq 2\epsilon - \epsilon = \epsilon$. Also notice that $|f(\alpha) - w| = |0 - w| < \epsilon$. Since the disc bounded by C is a compact set, we know that $f(\text{disk}) - w$ must have a minimum somewhere. It cannot be on the boundary, (since we just showed that$|f(z) - w| \geq \epsilon$ while $|f(\alpha) - w| < \epsilon$). Hence it is an interior maximum! By the minimum modulous principle, the value there must be 0. But that is exactly saying that there exists a point with $f(z) = w$ in this region. $\square$

THEOREM. *(7.2.) Schwarz' Lemma*
Suppose that f is analytic in the unit disk, that $f \ll 1$ on the unit disk and that $f(0) = 0$. Then:
i) $|f(z)| \leq |z|$
ii) $|f'(0)| \leq 1$
with equality in either of the above if and only if $f(z) = e^{i\theta} z$

PROOF. The idea is to apply the maximum modulous to the analytic secant function $g(z) = \frac{f(z)}{z}$ in a smart way. (One hiccup is that you can only apply max mod for open sets) Take the open disc or radius r and notice that since $f \ll 1$ we know that $g \ll 1/r$ on the circle of radius r. Taking $r \to 1$ gives that $g \ll 1$ thoughout the unit disk, and this proves i) and ii). If we have equality, then the maximum mod principle says that g must be constant, so therefore $f = e^{i\theta} z$ $\square$

REMARK. The class of conformal maps that map the unit disk to the unit disk are (recall there are 3 real paramters...(since if a mobius tranformation fixes 3 points it is the identity...boils down to quadratic))

$$e^{i\theta}B_{\alpha}(z)=e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$$

These functions can be composed/dividided in order to force the condition that $f(0)=0$ needed for the Schwarz lemma.

EXAMPLE. (1)

Suppose that f is analytic and bounded by 1 in the unit disk and that $f(\frac{1}{2})=0$ we wish to estimate $\left|f(\frac{3}{4})\right|$. Since $f(\frac{1}{2})=0$ let $h=f\circ B_{\frac{1}{2}}^{-1}$. Notice that $h(\frac{1}{2})=f(\frac{1}{2})=0$ and $h\ll 1$ since $f\ll 1$ and $B_{1/2}$ preserves the unit disc. By Schwarz lemma, $|h(z)|\leq|z|$, hence $|f(3/4)|=\left|h(B_{1/2}(3/4))\right|\leq\left|B_{1/2}(3/4)\right|=2/5$ and there is equaltiy only when $f=B_{1/2}$ so that $h\equiv$id.

(In the book example he unpacked this and used the max-modulous principle instead of Schwarz lemma)

EXAMPLE. (2)

<u>Claim:</u> among all functions f which are analytic and bounded by 1 in the unit disc, $\max|f'(1/3)|$ is maximized by the function f which has $f(1/3)=0$

<u>Pf:</u> Suppose $f(1/3)\neq 0$. Then let:

$$g(z)=\frac{f(z)-f(1/3)}{1-\overline{f(1/3)}f(z)}=B_{f(1/3)}\circ f$$

(Notice that $g(1/3)=0$)$g\ll 1$ in the unit disc since $f\ll 1$ in the unit disc and $B_{f(1/3)}$ preserves the unit disc. A direct calculation shows that:

$$g'(1/3)=B'_{f(1/3)}(f(1/3))f'(1/3)$$

And $B'_{\alpha}(\alpha)=\frac{1}{1-|\alpha|^2}>1$ when $\alpha\neq 0$.

A similar phenonmenon will come into play for the Riemann mapping theorem.

PROPOSITION. *(7.3.)*

If f is entire and satisfies $|f(z)|\leq 1/|Imz|$ then $f\equiv 0$

PROOF. f is bounded outside of $-1\leq \mathrm{Im}z\leq 1$, so we just have to check its bounded there too. To do this we will use the maximum modulus principle in a clever way. I'm going to skip the details for now. □

8.19. The Converse of Cauchy's Theorem: Morera's Theorem; The Schwarz Reflection Principle

THEOREM. *(7.4.) Morera's Theorem*

Let f be continuous on an open set D. If:

$$\int_{\Gamma}f(z)dz=0$$

whenever Γ is the boundary of a closed rectangle in D, then f is analytic on D.

PROOF. Since we have essentially a "rectangle theorem" for f, we get an integral theorem, so that there is an $F'=f$. But then since F is infitenly differentiable, we have $f'=F''$ and so f is analytic. □

REMARK. Morera's theorem is often used to establish the analyticicy of functions given in integral form. For example in the left half plane (where $\mathrm{Re} z < 0$) conisder:

$$f(z) = \int_0^\infty \frac{e^{zt}}{t+1} \mathrm{d}t$$

For any rectangle Γ examing $\int_\Gamma f(z)\mathrm{d}z = \int_\Gamma \int_0^\infty \frac{e^{zt}}{t+1}\mathrm{d}t\mathrm{d}z$ which we can interchange by Fubini/Tonelli (check its absolutly convergegent!) , and when we do this $\int_\Gamma \frac{e^{zt}}{t+1}\mathrm{d}z$ vanishes for each t by the closed curve theorem.

DEFINITION. (7.5.) Suppose f_n and f are defined in D. We will say that f_n converges to f **uniformly on comapcta** if $f_n \to f$ uniformly on every compact subset $K \subset D$.

THEOREM. *(7.6.) If f_n converges to f uniformly on compacta in a domain D and the f_n are analytic in D, then f is analytic in D.*

PROOF. Every rectangle is contained in a comapct set and so $\int_\Gamma f = \int_\Gamma \lim f_n = \lim_n \int_\Gamma f_n = 0$ (interchange ok because $f_n \to f$ uniformly here), so by Morera f is analytic. □

THEOREM. *(7.7.) If f is contious in a set D and analytic there except possibly a line L, then f is analytic there.*

PROOF. Any rectangle can be approximated by rectanlges not touching L (via continuity of f) all of these vanish, so by Morera f is analytic. □

THEOREM. *(7.8.) Schwarz Reflection Principle*

You can extend a function which is analytic in the upper half plane and real on the real axis, to the whole complex plane by:

$$g(z) = \begin{cases} f(z) & z \in H^+ \\ \overline{f(\bar{z})} & z \in H^- \end{cases}$$

PROOF. Clearly analytic everywhere except possibly on the real line. It is continous on the real line, so by the last theorem it is actually analytic everywhre. □

Simply Connected Domains

These are notes from Chapter 8 of [**1**].

9.20. The General Cauchy Closed Curve Theorem

The begining of this chapter deals with generalizing the results above from entire functions on the whole complex plane to functions on a simply connected domain. This book defines simply connected as:

DEFINITION. (8.1) A region is **simply connected** if its complement is connected with ϵ to ∞.

He then does some stuff using polygonal paths and things.....which I am not a huge fan of. Instead I like doing it the way that I learned in Prof Hang's class on complex variables:

DEFINITION. A region is **simply connected** if every point is homotopic to a point.

And then everything we will want will follow from the theorem that:

THEOREM. *If γ_1 is homotopic to γ_2 in Ω and f is analytic in Ω then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$*

The proof of this relies on a few lemmas, which use the method of "slicing and dicing" to prove things:

LEMMA. *If Ω is ANY open set, $f \in H(\Omega)$ then if $A \subset \Omega$ is a compact set with a curve ∂A as its boundary, then $\int_{\partial A} f(z)dz = 0$*

PROOF. Chop up A so that each piece is a convex region, and the integral is zero on each piece. $\square$

LEMMA. *If $\phi : A \to \Omega$ is piecewise smooth, A is compact set with a curve ∂A as its boundary, f is analytic in $\phi(A)$ then $\int_{\phi(\partial A)} f(z)dz = 0$*

PROOF. Same idea as the first lemma, just chop things up. $\square$

Finally then then homopy theorem is proven by using $A = [0,1] \times [0,1]$ and the map ϕ is the homotopy.

This gives us a general Cauchy closed curve theorem on simply connected domains, and thus antiderivatives, a Cuachy integral formula etc.

9.21. The Analytic Function $\log z$

DEFINITION. (8.7) We will say f is **an analytic branch of** $\log z$ in a domain D if:

i) f is analytic in D

ii) $\exp(f(z)) = z$ for all $z \in D$

REMARK. Notice that it must be the case if $f(z) = u(z)+iv(z)$ then $\exp(f(z)) = \exp(u(z))\exp(iv(z))$ so we must have that $u(z) = \log|z|$ and $\exp(iv(z)) = \exp(i\arg(z))$. In other words, it must be that $f(z) = \log|z| + i\arg(z)$ where arg is some argument function.

THEOREM. *(8.8) If D is simply connected and $0 \notin D$ then for any choice of $z_0 \in D$, the following function is always an analytic branch of $\log z$:*

$$\begin{aligned} f(z) &= \int_{z_0}^{z} \frac{dw}{w} + \log(z_0) \\ &= \int_{z_0}^{z} \frac{dw}{w} + \log|z_0| + i\arg(z_0) \end{aligned}$$

PROOF. This is well defined since $1/z$ is analytic in the simply connected region D. To see that its a branch of log, define $g(z) = ze^{-f(z)}$ and check that $g'(z) = 0$ and $g(z_0) = 1$. $\square$

THEOREM. *If $f \neq 0$ in a region D then we can define an analytic branch of $\log f(z)$ by:*

$$\log f(z) = \int_{z_0}^{z} \frac{f'(w)}{f(w)} dw + \log(f(z_0))$$

PROOF. Same as above.
Note that we can define things like $\sqrt{z}$ anywhere we can define log. $\square$

PROPOSITION. *If f, g are branches of $\log$ is some set D, then $f - g = 2\pi ik$ for some k*

PROOF. Have $e^{f(z)} = z = e^{g(z)}$ so $e^{f(z)-g(z)} = 1 \implies f(z) - g(z) = 2\pi ik(z)$ but $k(z)$ must be continuous, so it is constant on connected components. $\square$

Isolated Singularities of an Analytic Function

These are notes from Chapter 9 of [**1**].

10.22. Classification of Isolated Singularties; Riemann's Principle and the Casorati-Weirestrass Theorem

Definition. (9.1.) We say f has an **isolated singularity at** z_0 if f is analytic in a punctured disk of z_0 but is not analytic at z_0.

These are classified as follows:

Name	Def'n	Characterization
Removable	$\exists g(z)$ analytic everywhere in Ω and $f(z) = g(z)$ in Ω	$\lim_{z\to z_0}(z-z_0)f(z) = 0$
Pole	$\exists A, B$ analytic in a n'h'd of z_0 so that $f(z) = A(z)/B(z)$ (called a pole of order k if B has a zero of order k)	$\exists k$ so that $\lim_{z\to z_0}(z-z_0)^k f(z) \neq 0$ and $\lim_{z\to z_0}(z-z_0)^{k+1} f(z) = 0$
Essential	Neither of the above	$R = \{f(z) : z \in D - z_0\}$ is dense in $\mathbb{C}$.

Proofs of these characterizations:

Theorem. *(9.3.) (Riemanns Principle of Removable Singularities)*
f has a removable singularity at z_0 if and only if $\lim_{z\to z_0}(z-z_0)f(z) = 0$

Proof. ($\Longrightarrow$) is clear since $\lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0}(z-z_0)g(z) = 0 \cdot g(z_0) = 0$

($\Longleftarrow$) follows by letting $h(z) = (z-z_0)f(z)$ for $z \neq z_0$ and 0 at z_0. h is analytic except at z_0, and continuous at z_0 so it is analytic everywhere by Morera. Then let $g(z) = h(z)/(z-z_0)$ and $g(z_0) = h'(z_0)$ be the usual "secant" function which is analytic and equal to f everywhere. $\square$

Corollary. *(9.4.) If f is bounded in a n'h'd of an isolated singularity, then it is a removable singularity.*

Theorem. *(9.5.) f has a pole of order k if and only if $\exists k$ so that* $\lim_{z\to z_0}(z-z_0)^k f(z) \neq 0$ *and* $\lim_{z\to z_0}(z-z_0)^{k+1} f(z) = 0$

Proof. ($\Longrightarrow$) $f(z) = A(z)/B(z) = A(z)(z-z_0)^k/B(z)(z-z_0)^k$ works.

($\Longleftarrow$) Same idea as the last proof with $h(z) = (z-z_0)^{k+1}f(z)$ and 0 at $z = z_0$ $\square$

Remark. According to the last two theorems, there is no analytic function that is a "fractional power" at a singularity, i.e. if $|f(z)| \leq \frac{1}{|z|^{5/2}}$ then actually $|f(z)| \leq \frac{1}{|z|^{\lceil 5/2 \rceil}} = \frac{1}{z^3}$.

THEOREM. *(9.6.) Casorati-Weierstrass Theorem*
If f has an essential singularity at z_0 then $\{f(z) : z \in D\}$ is dense in the complex plane

PROOF. Suppose by contradiciton there is a disk $B(w_0, \delta)$ which is is overlooked. Then $|f(z) - w_0| > \delta$ for all z and consequently $\frac{1}{|f(z)-w_0|} < \frac{1}{\delta}$ is bounded, and hence the function $\frac{1}{f(z)-w_0}$ has at worst a removable singularity. Hence there is an analytic version , $g(z) = \frac{1}{f(z)-w_0}$ i.e. $f(z) = w_0 + \frac{`1}{g(z)}$. Since g is analytic, this shows that either f has a pole at at z_0 if $(g(z_0) = 0)$ or has a removable singularity (if $g(z_0) \neq 0$) ☐

REMARK. The Picard theorem is an extension of this result which says actually that only a single point could possibly be ommited by R, all other points are there.

10.23. Laurent Expansions

DEFINITION. (9.7) We say that $\sum_{k=-\infty}^{\infty} \mu_k = L$ if both $\sum_{k=0}^{\infty} \mu_k$ and $\sum_{k=1}^{\infty} \mu_{-k}$ converge and their sum is L.

THEOREM. *(9.8) $f(z) = \sum_{-\infty}^{\infty} a_k z^k$ is convergent in the domain:*

$$D = \{z : R_1 < |z| < R_2\}$$

With:

$$\begin{aligned} R_1 &= \limsup |a_{-k}|^{1/k} \\ R_2^{-1} &= \limsup |a_k|^{1/k} \end{aligned}$$

PROOF. This is essentially the same proof as that for power series. ☐

THEOREM. *(9.9) [Cauchy Integral formula in an annulus]*
If f is analytic in an an annulus $A : R_1 < |z| < R_2$ then for any $z \in A$, f has:

$$f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w-z} dw$$

Where C_1, C_2 are circles with radii $R_1^+ < |z| < R_2^-$

PROOF. Start with:

$$\int_{C_1} \frac{f(w) - f(z)}{w-z} \mathrm{d}w = \int_{C_2} \frac{f(w) - f(z)}{w-z} \mathrm{d}w$$

(they are equal because the curves are homotopic and the "secant" function is analytic in A. Notice If they were homotopic to zero, then we could put zero for one side of the equation and recover the Cauhcy integral formula)
Hence:

$$\begin{aligned} \int_{C_2 - C_1} \frac{f(w)}{w-z} \mathrm{d}w &= f(z) \int_{C_2 - C_1} \frac{1}{w-z} \mathrm{d}w \\ &= 2\pi i f(z) \end{aligned}$$

Have then:

$$f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w-z} \mathrm{d}w - \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w-z} \mathrm{d}w$$

And now we can expand these in the same we did for the Cauchy integral formul ☐

THEOREM. *If f is analytic in an an annulus $A : R_1 < |z| < R_2$ then f has a Laurent expansion there.*

PROOF. Start with $f(z) = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\mathrm{d}w - \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\mathrm{d}w$ and then use $\frac{1}{1-x} = 1+x+\ldots$ to expand it out with $x = \frac{w}{z}$ on C_1 where $|w| < |z|$ use $x = \frac{z}{w}$ on C_2 whree and $|z| < |w|$. The part on C_1 gives the "negative" side of the Laurent expansion, while the part on C_2 gives the positive part. Carrying out the details we get to the slightly more specific result: □

COROLLARY. *(9.10) If f is analytic in the annulus $R_1 < |z - z_0| < R_2$ then f has a unique representation:*

$$f(z) = \sum_{-\infty}^{\infty} a_k (z - z_0)^k$$

With:

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{k+1}}\,dz$$

DEFINITION. The side of the Laurent expansion with the negative power terms is called the **principle part** of the Laurent expansion. The half with the positive power terms is called the **analytic part**.

THEOREM. *f has a removable singularity if and only if the principle part of the Laurent expansion vanises.*

PROOF. The Laurent expansion for f has to agree with the Taylor expansion for an analytic function g, so it cannot have a negative part. □

THEOREM. *f has a pole of order k if and only if it has k terms in the principle part of the expansion*

PROOF. Write $f(z) = Q(z)/(z - z_0)^k$ with Q analytic and then observe how the Laurent expansion from f relates to the one of Q. □

THEOREM. *f has an essential singularity if and only if it has infinelty many terms in the principle part,*

PROOF. This follows by process of elimination with the last two theorems in hand. □

THEOREM. *(9.13) (Partial Fraction Decomposition of Rational Functions) Any proper rational function:*

$$R(z) = \frac{P(z)}{Q(z)} = \frac{P(z)}{(z - z_1)^{k_1}\ldots(z - z_n)^{k_n}}$$

For polynomials P and Q can be expanded as a sum of polynomials in the terms $\frac{1}{z-z_j}$

PROOF. The key observation is that R has only poles as singularites, and so the principle part of the Laurent expansion for R about any of the poles z_j will have finetly many terms in the principle part. Hence you can write:

$$R(z) = P_1\left(\frac{1}{z - z_1}\right) + A_1(z)$$

Where P_1 is the "flipped" version of the principle part of the Laurent expansion about z_1 and A_1 is the analytic expansion there. Repeating this for A_1 (one checks its still a rational polynomial) and so on gives the result.

In fact, we get a slightly stronger result that:

$$R(z) = P_1\left(\frac{1}{z-z_1}\right) + P_2\left(\frac{1}{z-z_2}\right) + \ldots + P_n\left(\frac{1}{z-z_n}\right)$$

□

Introduction to Conformal Mapping

These are notes from Chapter 13 of [**1**].

11.24. Conformal Equivalence

Definition. (13.1.) Suppsoe two smooth curves C_1 and C_2 intersect at z_0. The angle from C_1 to C_2 at z_0 is measured as the angle counterclockwise from the tangent of C_1 at z_0 to the tangent of C_2 at z_0. We write this as $\angle C_1, C_2$.

Definition. (13.2) Suppose f is defined in a neigbouthood of z_0. f is said to be **conformal at** z_0 if f preserves angles there. That is to say, for every pair of smooth curves C_1, C_2 intersecting at z_0, we have $\angle C_1, C_2 = \angle f(C_1), f(C_2)$. We say f is conformal in a region D if it is conformal at every point $z \in D$.

Remark. We will see that holomorphic functions are conformal except at points where $f'(z) = 0$. For example, $f(z) = z^2$ is *not* conformal at $z = 0$. One way to see this for example is to notice that the angle between the positive real and positive imaginary axis goes from $\pi/2$ to π under the map $z \to z^2$.

Definition. (13.3.)
a) f is **locally 1-1 at** z_0 if for some $\delta > 0$ and any distinct $z_1, z_2 \in D(z_0; \delta)$ we have $f(z_1) \neq f(z_2)$
b) f is **locally 1-1 throughout a region** D if f is locally 1-1 at every point $z \in D$.
c) f is **(globally) 1-1 function in a region** D if for every distinct $z_1, z_2 \in D$ $f(z_1) \neq f(z_2)$

Theorem. *(13.4) Suppose f is analytic at z_0 and $f'(z_0) \neq 0$. Then f is conformal and locally $1-1$ at z_0.*

Proof. (Conformal) Take any curve $z(t) = x(t) + iy(t)$, then the tangent to the curve at a point $z_0 = z(t_0)$ is given by the derivative $\dot{z}(t_0) = x'(t_0) + iy'(t_0)$. Examine now $\Gamma = f(C)$. This is parametrized by $w(t) = f(z(t))$ and consequently we calculate the tangent by the chain rule $\dot{w}(t_0) = f'(z_0)\dot{z}(t_0)$. Hence:

$$\arg(\dot{w}(t_0)) = \arg\left(f'(z_0)\dot{z}(t_0)\right) = \arg(f'(z_0)) + \arg(\dot{z}(t_0))$$

This shows that the difference $\arg(\dot{w}(t_0)) - \arg(\dot{z}(t_0)) = \arg(f'(z_0))$ does not depend on the curves at all...only on the function f and the point z_0. Since every curve has its tangent line rotated by the same amount, the map is conformal.

(Notice that the argument breaks down if $f'(z_0) = 0$ since $\arg(f'(z_0))$ is not defined. (No matter how one defines it: the equality $\arg\left(f'(z_0)\dot{z}(t_0)\right) = \arg(f'(z_0)) + \arg(\dot{z}(t_0))$ cannot possibly hold if $f'(z_0) = 0$)) $\square$

(Locally 1-1 at z_0) This essentially uses the argument principle applied around the two contours Take $f(z_0) = \alpha$ and take $\delta_0 > 0$ small enough so that $f(z) - \alpha$ has no zeros in $D(z_0; \delta_0)$. Let C be the curve that goes over the boundary of $D(z_0; \delta_0)$ and let $\Gamma = f(C)$. By the argument principle we have that:

$$\#zeros = 1 = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z) - \alpha} \mathrm{d}z = \frac{1}{2\pi i} \int_\Gamma \frac{\mathrm{d}\omega}{\omega - \alpha} = \frac{1}{2\pi i} \int \frac{\mathrm{d}\omega}{\omega - \beta}$$

Where β is any number so that $|\alpha - \beta| \leq \max_{\omega \in D(z_0, \delta_0)} |\omega|$. This shows the each value β in this disk is achieved *exactly* once by the function f. Hence f is locally 1-1. (If it was not $1 - 1$ it would take some value twice!)

Example. (1) $f(z) = e^z$ has a nonzero derivative at all points, and hence is everywhere conformal and locally $1 - 1$. We see that this function does a transformation from polar to rectangular coordinates in a way: $f(x + iy) = e^x e^{iy}$ so the x coordinate gets mapped to the radius $r = e^x$ and the y coordinate gets mapped to the argument of the polar coordinate $\theta = y \mod 2\pi$. Notice that the map is NOT globally 1-1...it maps each vertical strip of with 2π conformally to the whole plane.

Example. (2) $f(z) = z^2$ has $f'(z) = 2z$ which is non-zero everywhere except at $z = 0$. Hence f is a conformal map except at $z = 0$. Notice that $\mathrm{Re} f(x + iy) = x^2 - y^2$ while $\mathrm{Im}(f(x + iy)) = 2xy$. Fro this we can see that the x-axis and y-axis are mapped through f into hyperbolas that intesect at right angles. Notice that each hyperbola has two branches...this reflects the fact that f is "globally 2-1" in some sense....

Definition. (13.5.) Let k be a positive integer. We say f is a **k-to-1** mapping of D_1 onto D_2 if for every $\alpha \in D_2$ the equation $f(z) = \alpha$ has k roots (counting multiplicity) in D_2.

Lemma. *(13.6) Let $f(z) = z^k$ for k a positive integer. Then f magnifies angles at 0 by a factor of k and maps the disk $D(0; \delta)$ to the disc $D(0, \delta^k)$ in a $k - to - 1$ manner.*

Proof. The rotation bit is clear since $f(re^{i\theta}) = r^k e^{ik\theta}$so $arg(f(z)) = k \arg(z)$ here. The k roots of unity scaled up by size δ and rotated show the k-to-1 mapping criteria. □

Theorem. *(13.7.) Suppose f is analytic with $f'(z_0) = 0$. Assume that f is not constant. Let k be the smallest positive integer for which $f^{(k)}(z_0) \neq 0$. Then in some sufficiently small open set containing z_0 we have that f is a k-to-1 mapping and f magnifies angles at z_0 by a factor of k. (i.e. f "locally looks like" $z \to z^k$.)*

Proof. Assume WOLOG that $f(z_0) = 0$. We have from the power series expansion that $f(z) = (z - z_0)^k g(z)$ where $g(z)$ is given by a power series about z_0 and $g(z_0) \neq 0$. Since $g \neq 0$ in a n'h'd of z_0, we can define a k-th root of g in this n'h'd. (We can define a k-th root/a branch of log in any simply connected region not containing 0) Let $h(z) = (z - z_0) g^{1/k}(z)$ then so that $f(z) = [h(z)]^k$. notice that at h is an analytic function with $h(z_0) =$ and $h'(z_0) = g^{1/k}(z_0) \neq 0$. This means that h is locally 1-1. Since f is a composition of a locally $1 - 1$ map h and the k-to-1 map z^k we see that f must be k-to-1 too. □

REMARK. An alternative proof is to go though the same kind of argument principle we did for the 1-1 proof. In this case you'll have there are exactly k roots in the n'h'd in question. One advantage of going through the function h is that you now view f as the composition of $z \to z^k$ and the map $z \to h(z)$ both of which are easier to understand than f.

THEOREM. *(13.8) Suppose f is a 1-1 analytic function in a region D. Then:*
a) f^{-1} exists and is analytic in $f(D)$
b) f and f^{-1} are conformal in D and $f(D)$ respectivly.

PROOF. Since f is $1-1$, $f' \neq 0$. Hence f^{-1} is also analytic and $(f^{-1})' = 1/f'$ shows that f^{-1} also has non-zero derivatives in $f(D)$ and so they are both conformal. □

DEFINITION. (13.9) In the setting of the above theorem we call f a conformal map and we say that D and $f(D)$ are conformally equivalent. More generally:
i) a 1-1 analytic mapping is called a **conformal mapping**
ii) Two regions D_1 and D_2 are said to be **conformally equaivalent** if there exists a conformal mapping of D_1 onto D_2.

11.25. Special Mappings

This section goes through power laws, and fractional linear transformations. The Fractional Linear Transformations are proved to be the unique mappings from the unit disk to the unit disc by the Schwarz lemma. For two conformal regions D_1 and D_2, there will be a 3 parameter family of conformal mapping theorems by composing with the right things to bring it to the form of mappings from the unit disk to itself (this requires the Riemann mapping theorem).

The Riemann Mapping Theorem

These are notes from Chapter 14 of [**1**].

12.26. Conformal Mapping and Hydrodynamics

Irrotational, source free flows correspond to holomorphic functions. (We have $\int_C \overline{g(z)}\mathrm{d}z = \sigma + i\tau$ where σ is the circulation and τ is the flux. If this is always 0, then the fucntion $f(z) = \overline{g(z)}$ is analytic by Morera's theorem.)

12.27. The Riemann Mapping Theorem

The Riemann Mapping Theorem, in its most common form asserts that any two simply connected proper subdomains of the complex plane are conformally equivalent. Note that neither region can be all of $\mathbb{C}$ or we would get a violation of Loiuville's theorem. To prove the theorem it suffices to show that there is a mapping from any region R to the open disc U. By composing mappings of this sort we will get mappings between any two regions. By composing with the conformal maps $U \to U$ (the FLT's) we can get a map with the property that $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$ (by imposing this it makes the map uniquely determined!).

Theorem. *(The Riemann Mapping Theorem)*
For any simply connected domain $R (\neq \mathbb{C})$ and $z_0 \in R$, there exists a unique conformal mapping φ of R onto U such that $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$.

Proof. (Uniqueness) By composing two such maps we would get a mapping from the unit disc to the unit disc. But this map would have to be the identity because of the proeperites specified $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$ and because we know exactly the 3 parameter family of maps from the unit disc to itself.

(Existence) We will phrase the problem as a *extreme problem* of finding the function φ from a certain family of functions that is the most extreme in some way.

Recall that the 1-1 mappings from the unit disk to itself that maximize $|\varphi'(\alpha)|$ are precisly those of the form:

$$\varphi(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$$

These also happen to be the ones that map $\alpha \to 0$ and map U to U . This suggests that we should look for the map that maximizes $|\varphi'(z_0)|$. We divide the details of the proof into three steps:

Step 0: Define $\mathcal{F} = \{f : R \to U : f'(z_0) > 0\ f \text{ is 1-1 here}\}$ (Notice $f(z_0) = 0$ is not imposed)

Step 1: Show that $\mathcal{F}$ is non-empty

Step 2: Show that $\sup_{f\in\mathcal{F}} f'(z_0) = M < \infty$ and that there exists a function $\varphi \in \mathcal{F}$ such that $\varphi'(z_0) = M$

Step 3: Show that φ is the function we are looking for! i.e. φ is a conformal map from R onto U such that $\varphi(z_0)$ and $\varphi'(z_0) > 0$. $\square$

Proof of Step 1:

The idea is to start with a point ρ_0 outside of R. Such a point exists since $R \neq \mathbb{C}$.

If we could find a disk around ρ_0 say of radius δ then the map $f(z) = \frac{\delta}{z-\rho_0}$ would have $|f(z)| < 1$ for $z \in R$ and we would be done!

If we cannot find such a disk, we must be more clever. Since the map $z \to z - \rho_0$ is never 0 on R, we can define a square root/branch of log for this function. Define: $g(z) = \sqrt{\frac{z-\rho_0}{z_0-\rho_0}}$ with the branches chosen so that $g(z_0) = 1$. We claim now that g stays bounded away from the value -1. Otherwise, if $g(\xi_n) = \sqrt{\frac{\xi_n-\rho_0}{z_0-\rho_0}} \to -1$ we will have $\xi_n \to z_0$ but then $g(\xi_n) \to 1 \neq -1$. Now $f(z) = \frac{\delta}{g(z)-(-1)}$ does the trick for us. By multiplying by a constant $e^{i\theta}$ we can suppose WOLOG that $f'(z_0) > 0$.

Proof of Step 2:

First we notice that $f'(z_0)$ is bounded by using a Cauchy-integral-formula estimate since $|f| \leq 1$ is known. Take a disk $D(z_0; 2\delta) \subset R$ and have:

$$|f'(z_0)| = \left| \frac{1}{2\pi i} \int_{C(z_0;\delta)} \frac{f(z)}{(z-z_0)^2} \mathrm{d}z \right| \leq \frac{1}{2\pi} \frac{1}{\delta^2} |2\pi i \delta| = \frac{1}{\delta} < \infty$$

Hence $\sup_{f\in\mathcal{F}} f'(z_0) = M < \infty$ exists.

Idea of the rest of the proof: The family of functions $f \in \mathcal{F}$ is uniformly bounded ($|f| < 1$ again) and it is roughly an equicontinuous family since we have control on the derivatives in terms of Cauchy-integral-formula estimates. This will allow us to use an Arzel-Ascoli type argument to get us a limit function φ that we want.

First take a sequence $\{f_n\} \subset \mathcal{F}$ so that $f_n'(z_0) \to M$ as $n \to \infty$. Take $\xi_1, \ldots$ a countable dense collection of points of R. For each k, the sequence $f_n(\xi_k)$ is bounded (since $|f| \leq 1$) and so has a convergence subsequence. Taking subsequences of subsequences in this way, we get a "diagonal" sequence call it $\varphi_n \subset \{f_n\}$ so that $\varphi_n(\xi_k)$ converges for all $k \in \mathbb{N}$. Define φ by $\varphi(\xi_k) = \lim_{k\to\infty} \varphi_n(\xi_k)$ and extend it by continiuty to all of R.

We will now show that φ_n converges everywhere in R (not just on the dense set) and that the convergence is uniform on compact subsets $\mathcal{K} \subset R$. Since every compact subset is contained in a finite union of closed discs, it suffices to check that φ_n converges uniformly on closed discs in R. Let $2d = d(\mathcal{K}, R^c)$ be the distance from the closed disc $\mathcal{K}$ to the outside of the region R. Now we have a uniform estimate on the magnitude of the derivative $|\varphi_n'(z)|$ from the Cauchy integral formula (again since $|\varphi| \leq 1$:

$$|\varphi_n'(z)| = \left| \frac{1}{2\pi i} \int_{C(z,d)} \frac{\varphi_n(\xi)}{(\xi - z)^2} \right| \leq \frac{1}{2\pi} \frac{2\pi d}{d^2} = \frac{1}{d}$$

and hence we have:

$$|\varphi_n(z_1) - \varphi_n(z_2)| \leq \left| \int_{z_1}^{z_2} \varphi_n'(z) \mathrm{d}z \right| \leq \frac{|z_1 - z_2|}{d}$$

This shows the family φ_n is equicontinuous (in fact they are uniformly Lipshitz on $\mathcal{K}$ with Lipshitz constant $\frac{1}{d}$)! This shows that φ_n is convergent not just at the ξ_k's but for any z since:

$$\begin{aligned} |\varphi_n(z) - \varphi_m(z)| &\leq |\varphi_n(z) - \varphi_n(\xi)| + |\varphi_n(\xi) - \varphi_m(\xi)| + |\varphi_m(\xi) - \varphi_m(z)| \\ &\leq |z - \xi| \frac{1}{d} + |\varphi_n(\xi) - \varphi_m(\xi)| + |z - \xi| \frac{1}{d} \end{aligned}$$

And the three terms on the RHS can be made arbitarily small by choosing ξ very close to z and using the fact that $\varphi_n(\xi)$ converges so its a Cauchy sequence. Moreover, the limit function φ will also be Liphitz on $\mathcal{K}$ with Lipshitz constant $\frac{1}{d}$ (just write it out to see this)

Finally, to see that $\varphi_n \to \varphi$ uniformly on $\mathcal{K}$, we apply the follow standard argument. (We are using the fact that an equicontinuous family that converges pointwise on a compact set actually converges uniformly) Fix $\epsilon > 0$ and let $S_j = \{z \in \mathcal{K} : |\varphi_n(z) - \varphi(z)| < \epsilon \text{ for all } n > j\}$. Since the family φ_n is equicontinuous, each set S_j is open. Since $\varphi_n \to \varphi$ pointwise, we know that S_j covers all of $\mathcal{K}$. Now since $\mathcal{K}$ is compact, we can find a finite subcover and thus get an integer N so large so that $\mathcal{K} \subset \cup_{j=1}^{N} S_j$. But then $|\varphi_n(z) - \varphi(z)| < \epsilon$ for all $z \in \mathcal{K}$ whenever $n > N$ and the convergence is uniform.

Since $\varphi_n \to \varphi$ uniformly on compacta, φ is analytic (by e.g. Morera's theroem). Moreover, $\varphi_n'(z_0) \to \varphi'(z_0)$ (e.g. by Cauchy integral formula), and hence $\varphi'(z_0) = \lim_{n\to\infty} \varphi_n'(z_0) = M$. φ is also 1-1 since it is the unifrom limit of 1-1 function (e.g. by the Argument principle)

Proof of Step 3:

Recall the maps:

$$e^{i\theta} B_\alpha(z) = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha} z}$$

These are the maps that map $\alpha \to 0$ and the unit disc $U \to U$. To see that $\varphi(z_0) = 0$ notice that if $\varphi(z_0) = \alpha \neq 0$ then composing $B_\alpha \circ \varphi$ gives a map whose derivative at 0 is $\frac{\varphi'(z_0)}{1-|\alpha|^2}$, contradiciting the maximality of φ.

To see that the image of φ is exactly the unit disc, suppose by contradiction that some point $-t^2 e^{i\theta}$ is missed. By rotating the whole disk by θ we may assume WOLOG that the point $-t^2$ that is missed is on the negative real axis. If $\varphi \neq -t^2$ ever then the map $B_{-t^2} \circ \varphi$ is never 0. Hence we can define its square root: $\sqrt{B_{-t^2} \circ \varphi}$. Now compose this to get a new map from $R \to D$ namely: $B_t \circ \sqrt{B_{-t^2} \circ \varphi}$. One can check that this still maps into U and that its derivative at z_0 is $\frac{\varphi'(z_0)(1+t^2)}{2t} \gg \varphi'(z_0)$ which is a contradiction of the choice of φ!

Remark. Step 2 is a bit quicker if you just say "By Arzela-Ascoli"....we reproduced the proof of that theorem in our proof of step 2. (Recall Arzela Ascoli says that if you have a family of function f that are bounded and are equicontinuous, the there is a uniformly convergent subsequence) The two steps in the proof of Arzela-Ascoli are:

1. Choose a coutnable dense set of points. By the Bolzanno-Weirestrass theorem, there is a convergent subsequence at each point. Take sub-sub-sequences to get a diagonal sequence that covnerges at every point of the dense set.

2. By equicontinuity, we actually have convergence everywhere. ($\epsilon/3$ argument $|\varphi_n(z) - \varphi_m(z)| \leq |\varphi_n(z) - \varphi_n(\xi)| + |\varphi_n(\xi) - \varphi_m(\xi)| + |\varphi_m(\xi) - \varphi_m(z)|$)

2. Let $S_j = \{z \in \mathcal{K} : |\varphi_n(z) - \varphi(z)| < \epsilon \text{ for all } n > j\}$ and then thse are open by equicontinuity. They form an open cover for the compact set in question and so they have a finite subcover. The finite subcover shows that the converge is uniform.

12.28. Connection to the Dirichlet problem for the Laplace Eqn

PROPOSITION. *The real part of any holomorphic function is harmonic* $\Delta u = 0$

PROOF. This is just from the Cauchy Riemann equations $u_x = -v_y$ and $u_y = v_x$ because then we can rewrite: $u_{xx} + u_{yy} = \frac{\partial^2}{\partial x \partial y} v - \frac{\partial^2}{\partial x \partial y} v = 0$ □

PROPOSITION. *Conversely, given any harmonic function* $\Delta u = 0$*, we can find a conjugate function* v *so that* $u + iv$ *is a holomorphic function*

PROOF. The trick is to go through the derivative first, and then integrate at the end. If we define $\partial_z := \partial_x - i\partial_y$ and $\partial_{\bar{z}} := \partial_x + i\partial_y$ then one can check that $\Delta = 4\partial_{\bar{z}}\partial_z$. Moreover, the Cauchy Riemann equations tell us that a function is holomorphic if and only if $\partial_{\bar{z}} f = 0$. Now if $\Delta u = 0$ then we have $\partial_{\bar{z}} (\partial_z u) = 0$. Hence $\partial_z u$ is a holomorphic function! Since holomorphic function have anti-derivatives, there is a holomoprhic function F so that $F' = \partial_z u$. We claim now that $u = 2\mathrm{Re}(F) + const$. Indeed if we write $F = a + ib$ then $\partial_z u = F' \implies \partial_x u - i\partial_y u = 2\partial_x a - 2i\partial_y a$ (Use Cauchy Riemann equations to manipulate what complex differentiation means in terms of real and imaginary parts). Equating these we get $\partial_x u = 2\partial_x a$ and $\partial_y a = 2\partial_y a$ from which we conclud that it must be the case that $u = a + const = \mathrm{Re}(F) + const$ □

PROPOSITION. *Let* R *be a simply connected domain with a smooth boundary and suppose* $0 \in R$*. Then the Diriclet problem:*

$$\begin{cases} \Delta u = 0 & inR \\ u\,|_{\partial R} = -\log|z| & on\ \partial R \end{cases}$$

Then the analytic function corresponding to the harmonic function u *that has* $f(z) = u + iv$*. Define* φ *by*

$$\varphi(z) = ze^{f(z)}$$

Then φ *is the map* $R \to U$

PROOF. Check that on ∂R we have $|\varphi(z)| = |z| \left|e^{f(z)}\right| = |z| e^{u(z)} = |z| e^{-\log|z|} = 1$ so indeed the boundary of R is mapped to the unit disc. Moreover, we check that φ is 1-1 as follows. For any point $\xi \in U$ in the unit disk, since $|\varphi(z)| = 1$ for $z \in \partial R$, we can find an region $(\partial R)_\epsilon$ close to the boundary ∂R so that $|\varphi(z)| > 1 - |\xi|/2$ in this region. But then, taking a closed curve γ_ϵ that stays in this region we have by the argument prinicple (/Rouche's theorem) that:

$$\frac{1}{2\pi i} \int_{\gamma_\epsilon} \frac{\varphi'(z)}{\varphi(z) - \xi} = \frac{1}{2\pi i} \int_{\gamma_\epsilon} \frac{\varphi'(z)}{\varphi(z)} = 1$$

This is 1 since it counts the number of times φ is equal to zero in R (Argument principle) and since $\varphi(z) = ze^{f(z)}$ is only zero at $z = 0$. □

12.29. Complex Theorems to Memorize

THEOREM. *(Riemann Mapping Theorem)*
Any simply connected domain that is not all of $\mathbb{C}$ *is conformally equivalent to the unit disc.*

THEOREM. *(Little Picard Theorem)*
If $f : \mathbb{C} \to \mathbb{C}$ *is entire and non-constant, then* $\{w : \exists f(x) = w\}$ *is the whole complex plane or the whole complex plane minus a point.*
(e^z *hits every value except* 0*)*

THEOREM. *(Big Picard)*
If an analytic function has an essential singularity at a point w*, then on a punctured n'h'd of* w*,* $f(z)$ *takes on all possible complex values with at most a single exception.*
($e^{1/z}$ *has an essential singularity at* 0 *but still never attains* 0 *as a value)*

THEOREM. *(Product Representation for sin):*
$\sin(\pi z) = (\pi z) \prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right) e^{z/k} \prod (1 + \frac{z}{k}) e^{-z/k}$

Bibliography

[1] J. Bak and D.J. Newman. *Complex Analysis*. Encyclopaedia of Mathematical Sciences. Springer, 1997.