Problems

March 25, 2014

Exercise. Find the 2×2 matrix that gives the orthogonal projection onto the line $y = 2x$ in the plane.

Exercise. Prove that for a matrix A with eigenvalues λ_i, that $\mathrm{Tr}(A^k) = \sum_i \lambda_i^k$. Does this still hold if A is not diagonalizable?

Exercise. Let $f(t) = \det(A + tB)$. What is $f'(0)$? (Hint: first do the case $B = Id$, then the case B is invertable)

Problem. (#4 Jan 04) Prove that the non-zero eigenvalues of AB and BA are the same, where A and B are rectangular matrices of size $n \times m$ and $m \times n$ respectively. Express the corresponding eigenvalues of AB in terms of those of BA.

Problem. (#5 Sept 05) (Note the wording on this is hard to read!!) Let V be a real finite dimensional vector space with inner product. A set of k vectors $(y_1, \ldots, y_k \in V$ defines the linear operator $A : V \to V$ by $Ax = \sum_{j=1}^{k} \langle y_j, x\rangle y_j$. Define the $k \times k$ matrix, M, with entries $M_{ij} = \langle y_i, y_j\rangle$. The norm of A is the operator norm coming from the vector norm $\|x\|^2 = \langle x, x\rangle$. The norm of M is the matrix norm coming from the vector norm of $\mathbb{R}^k$ given by $\|z\|^2 = \sum_{j=1}^{k} z_j^2$. Show that $\|A\| = \|M\|$.

Problem. (#2 Jan 05) Let C be an $n \times n$ matrix,. Prove that $\mathrm{Tr}(C) = 0$ if and only if C is similar to a matrix D with all diagonal elements equal to 0. (Hint: Make one zero at a time on the diagonal). Show that this condition is both necessary and suffient for C to be equal to $AB - BA$ for some $n \times n$ matrices A, B .

Problem. (#1 Sept 02) Let A be the $n \times n$ matrix:

$$A = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 1 & 1 & \ldots & 1 & 1 \end{bmatrix}$$

a) Is A diagonalizable? Orthogonal? Gives its rank and a basis of its kernel.
b) Give a relation between its nonzero eigenvalues.
c) Compute A^2 and deduce another relation and finally the values of its eigenvalues.
d) Give the minimal polynomial of A.