1 Review of Probability

Definition 1: Probability Space.

The *probability space*, also called the *sample space*, is a set which contains *ALL* the possible outcomes. We usually label this with a Ω or an S. (Ω is the Greek letter "Omega"). The elements of the probability space are called *outcomes* also called *elementary events.*

Example 1: What is the probability space for rolling a 6 sided dice?

$$\Omega = \left\{\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{5}, \boxed{6}\right\}$$

This contains the six possible outcomes.

Example 2: What is the probability space for flipping a coin? What about for flipping a coin twice? Three times?

$$\begin{aligned}
\Omega_1 &= \left\{\boxed{\text{H}}, \boxed{\text{T}}\right\} \\
\Omega_2 &= \left\{\boxed{\text{H}|\text{H}}, \boxed{\text{H}|\text{T}}, \boxed{\text{T}|\text{H}}, \boxed{\text{T}|\text{T}}\right\} \\
\Omega_3 &= \left\{\boxed{\text{H}|\text{H}|\text{H}}, \boxed{\text{H}|\text{T}|\text{H}}, \boxed{\text{T}|\text{H}|\text{H}}, \boxed{\text{T}|\text{T}|\text{H}}, \boxed{\text{H}|\text{H}|\text{T}}, \boxed{\text{H}|\text{T}|\text{T}}, \boxed{\text{T}|\text{H}|\text{T}}, \boxed{\text{T}|\text{T}|\text{T}}\right\}
\end{aligned}$$

Definition 2. Events.

Events are just *subsets* of your probability space. That is they are collections of outcomes. We usually label these with capital letters.

Example 1: What is the event that the dice roll is even?

$$A = \left\{\boxed{2}, \boxed{4}, \boxed{6}\right\}$$

Example 2: What is the event that there are at least as many heads as their are tails? (Use the three probability spaces from above)

$$\begin{aligned}
A_1 &= \left\{\boxed{\text{H}}\right\} \\
A_2 &= \left\{\boxed{\text{H}|\text{H}}, \boxed{\text{H}|\text{T}}, \boxed{\text{T}|\text{H}}\right\} \\
A_3 &= \left\{\boxed{\text{H}|\text{H}|\text{H}}, \boxed{\text{H}|\text{T}|\text{H}}, \boxed{\text{T}|\text{H}|\text{H}}, \boxed{\text{H}|\text{H}|\text{T}}\right\}
\end{aligned}$$

Definition 3. Probability.

Probability is a number that is assigned to events. Every event has a probability. The probability is always a number between 0 and 1. We use $\mathbf{P}(A)$ to mean "the probability of the event A". The probability is defined as:

$$\mathbf{P}(A) = \frac{\#|A|}{\#|\Omega|}$$

Example 1: Calculate the probability of the event $A = \left\{\boxed{2}, \boxed{4}, \boxed{6}\right\}$ in the space $\Omega = \left\{\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{5}, \boxed{6}\right\}$.

$$\mathbf{P}(A) = \frac{\#|\left\{\boxed{2}, \boxed{4}, \boxed{6}\right\}|}{\#|\left\{\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{5}, \boxed{6}\right\}|} = \frac{3}{6} = \frac{1}{2}$$

Example 2: Calculate the probabilites from the previous Example 2:

$$\begin{aligned}
\mathbf{P}(A_1) &= \frac{1}{2} \\
\mathbf{P}(A_2) &= \frac{3}{4} \\
\mathbf{P}(A_3) &= \frac{4}{8} = \frac{1}{2}
\end{aligned}$$

Exercise 1: [BONUS CONTENT] In the coin flipping problem, suppose n is an odd number and you flip n coins. Let A_n be the event that there are at least as many $\boxed{\text{H}}$ as $\boxed{\text{T}}$. Can you prove that $\mathbf{P}(A_n) = \frac{1}{2}$ for every odd number n?

Remark: The definition here is the simplest way to define probability and it only works when all the outcomes are equally likely. There are more general ways to define probability that can handle unequal outcomes and other more complicated situations. There is enough interesting stuff going on in this simple case we wont need any of that!

2 Conditional Probability

Say you have a probability space Ω of all your possible outcomes. Then someone comes along and tells you: "Based on our new information, some of these outcomes are now impossible!" In this case, you must delete some outcomes from your space Ω to reflect this new information. This is called *conditioning*.

Definition 4: The conditional probability space and conditional probability.

Let Ω be a probability space and G an event. The *conditional probability space* is a new probability space, $\Omega_{|G}$, by by deleting everything from Ω which is not in G. By deleting everything not in G, you remain with the set G. We write $\mathbf{P}(A|G)$ for probabilities on this space. This is called the *probability of A given G.*

If you have an event A that you are interested in, you must also update A by deleting everything that is not in G. You will be left with the set $A \cap G$; the outcomes in A AND in G. To summarize:

$$\begin{aligned}
\Omega_{|G} &= G \\
A_{|G} &= A \cap G \\
&= \{\text{elements in both } A \text{ and in } G\} \\
\mathbf{P}(A|G) &= \frac{\#|A_{|G}|}{\#|\Omega_{|G}|} \\
&= \frac{\#|A \cap G|}{\#|G|}
\end{aligned}$$

Example 3: Suppose we roll a dice. Suppose we are given that the dice is even. (For example: maybe the dice rolled on the floor, and before we see the dice, someone yells out: "THAT DICE IS EVEN!!!!"). What is the probability that the dice roll is a $\boxed{5}$ or a $\boxed{6}$?

Solution: Original probability space:

$$\Omega = \left\{\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{5}, \boxed{6}\right\}$$

Event we are looking at

$$A = \left\{\boxed{5}, \boxed{6}\right\}$$

Event to condition on:

$$G = \left\{\boxed{2}, \boxed{4}, \boxed{6}\right\}$$

Conditional event:

$$A_{|G} = A \cap G = \left\{\boxed{6}\right\}$$

Conditional probability:

$$\mathbf{P}(A|G) = \frac{\#|A \cap G|}{\#|G|} = \frac{1}{3}$$

Theorem 1: Bayes' Theorem.

$$\begin{aligned}
\mathbf{P}(A|G) &= \frac{\mathbf{P}(A \cap G)}{\mathbf{P}(G)} \\
&= \mathbf{P}(G|A) \frac{\mathbf{P}(A)}{\mathbf{P}(G)}
\end{aligned}$$

Proof: To see that $\mathbf{P}(A|G) = \frac{\mathbf{P}(A \cap G)}{\mathbf{P}(G)}$ we do a sneaky multiply and divide by $\#|\Omega|$:

$$\begin{aligned}
\mathbf{P}(A|G) &= \frac{\#|A \cap G|}{\#|G|} \\
&= \frac{\#|A \cap G|}{\#|\Omega|} \cdot \frac{\#|\Omega|}{\#|G|} \\
&= \mathbf{P}(A \cap G) \cdot \frac{1}{\mathbf{P}(G)}
\end{aligned}$$

Once you have this, you can rearrange the two formulas $\mathbf{P}(A|G) = \frac{\mathbf{P}(A \cap G)}{\mathbf{P}(G)}$ and $\mathbf{P}(G|A) = \frac{\mathbf{P}(A \cap G)}{\mathbf{P}(A)}$ to get Bayes' theorem.

3 Problems

3.1 The Rare Disease Problem

Problem: Suppose there is a rare disease and you have 1 in a 10,001 chance of having it. You take a test to see if you have the disease. If you have the disease, the test will always be positive. If you don't have the disease, the test is 99.9% accurate; it will come back negative 999 out of 1000 times. 1 out of 1000 times it will return a false positive. Given that your test result was positive, what is the probability you have the disease?

Answer: $\frac{1}{11} \approx 9\%$

Solution: What are the possible outcomes? Before you take your test, you either have the disease or you dont. Since you are $10,000$ times more likely to not have the disease than to have it, we include $10,000$ outcomes where you don't have it and 1 outcome where you do have it:

$$\Omega = \{ \underbrace{\text{☹}}_{\text{1 disease}}, \underbrace{\text{☺}, \text{☺}, \text{☺}, \ldots, \text{☺}}_{\text{10,000 healthy}} \}$$

After you take the test, your probability space is pairs, your disease state and your test result. The disease state always leads to a positive test result. The healthy state leads to a negative test result 99.9% of the time and a positive test result 0.1% of the time. So of the $10,000$ healthy states, $10,000 \times 99.9\% = 9,990$ are negative and $10,000 \times 0.1\% = 10$ are positive. So we have:

$$\Omega = \{ \underbrace{\text{☹}+}_{\text{1 disease,+test}}, \underbrace{\text{☺}-, \text{☺}-, \text{☺}-, \ldots, \text{☺}-}_{\text{9,990 healthy,-test}}, \underbrace{\text{☺}+, \ldots, \text{☺}+}_{\text{10 healthy +test}} \}$$

What is the event we are conditioning on? The event is $G = \{\text{You tested positive}\}$. This is anything in the probability space that has a $+$ with it. This is:

$$G = \left\{ \underbrace{\text{☹}+}_{\text{1 disease,+test}}, \underbrace{\text{☺}+, , \ldots, \text{☺}+}_{\text{10 healthy +test}} \right\}$$

Now we are interested in the event $A = \{\text{You have the disease}\}$. There is only one state where you have the disease here:

$$A = \{\text{☹}+\}$$

Notice that this is in the set G, so $A = A \cap G$. So the conditional probability is:

$$\mathbf{P}(A|G) = \frac{\#|A \cap G|}{\#|G|} = \frac{1}{11} \approx 9\%$$

Remark: You can also solve this with Bayes' theorem, but if you do it this way, its a bit harder to visualize:

$$\mathbf{P}(A|G) = \mathbf{P}(G|A)\frac{\mathbf{P}(A)}{\mathbf{P}(G)} = (100\%) \times \frac{1/10,001}{(100\%) \times 1/10,001 + (0.1\%) \times 10,000/10,001} = \frac{1}{11}$$

3.2 Two Children Problem

Problem: I have two children.

a) What is the probability I have two girls?

b) Given that I have at least one girl, what is the probability I have two girls?

c) Given that I have at least one girl who was born in the AM, what is the probability I have two girls?

d) Given that I have at least one girl who was born on a Monday, what is the probability I have two girls?

e) Given that I have at least one girl who was born on April 20th, what is the probability I have two girls?

[Assumptions: Before conditioning, you should assume that the date/time of their birth is uniformly likely to be any date/time. Also assume that each child is equally likely to be a Boy or a Girl]

Answers:

a) $\frac{1}{4} \approx 25\%$ b) $\frac{1}{3} \approx 33\%$ c) $\frac{3}{7} \approx 43\%$ d) $\frac{13}{27} \approx 48\%$ e) $\frac{729}{1459} \approx 49.97\%$

Solutions:

a) The probability space is the same as that for two coin flips. (Use $\boxed{\text{B}}$ = Boy and $\boxed{\text{G}}$ = Girl)

$$\Omega = \left\{ \begin{matrix} \boxed{\text{B}}\boxed{\text{B}}, & \boxed{\text{B}}\boxed{\text{G}}, \\ \boxed{\text{G}}\boxed{\text{B}}, & \boxed{\text{G}}\boxed{\text{G}} \end{matrix} \right\}$$

The event that I have two girls is $A = \left\{\boxed{\text{G}\,|\,\text{G}}\right\}$. So we have:

$$\mathbf{P}(A) = \frac{\#|A|}{\#|\Omega|} = \frac{1}{4}$$

b) Now we condition on the event $G = \{\text{I have at least one girl}\}$. This is:

$$\Omega_{|G} = G = \left\{ \begin{array}{ccc} & , & \boxed{\text{B}\,|\,\text{G}}, \\ \boxed{\text{G}\,|\,\text{B}}, & & \boxed{\text{G}\,|\,\text{G}} \end{array} \right\}$$

So the conditional probability is:

$$\mathbf{P}(A|G) = \frac{\#|A \cap G|}{\#|\text{G}|} = \frac{1}{3}$$

b) This time, we have to distinguish what time of day the child was born on. This means we have a much bigger starting space:

$$\Omega = \left\{ \begin{array}{cccc} \boxed{\text{B-AM}\,|\,\text{B-AM}}, & \boxed{\text{B-PM}\,|\,\text{B-AM}}, & \boxed{\text{G-AM}\,|\,\text{B-AM}}, & \boxed{\text{G-PM}\,|\,\text{B-AM}}, \\ \boxed{\text{B-AM}\,|\,\text{B-PM}}, & \boxed{\text{B-PM}\,|\,\text{B-PM}}, & \boxed{\text{G-AM}\,|\,\text{B-PM}}, & \boxed{\text{G-PM}\,|\,\text{B-PM}}, \\ \boxed{\text{B-AM}\,|\,\text{G-AM}}, & \boxed{\text{B-PM}\,|\,\text{G-AM}}, & \boxed{\text{G-AM}\,|\,\text{G-AM}}, & \boxed{\text{G-PM}\,|\,\text{G-AM}}, \\ \boxed{\text{B-AM}\,|\,\text{G-PM}}, & \boxed{\text{B-PM}\,|\,\text{G-PM}}, & \boxed{\text{G-AM}\,|\,\text{G-PM}}, & \boxed{\text{G-PM}\,|\,\text{G-PM}} \end{array} \right\}$$

The event $G = \{\text{I have at least one girl born in the AM}\}$ is:

$$G = \left\{ \begin{array}{cccc} & & \boxed{\text{G-AM}\,|\,\text{B-AM}}, & \\ & & \boxed{\text{G-AM}\,|\,\text{B-PM}}, & \\ \boxed{\text{B-AM}\,|\,\text{G-AM}}, & \boxed{\text{B-PM}\,|\,\text{G-AM}}, & \boxed{\text{G-AM}\,|\,\text{G-AM}}, & \boxed{\text{G-PM}\,|\,\text{G-AM}} \\ & & \boxed{\text{G-AM}\,|\,\text{G-PM}}, & \end{array} \right\}$$

The event $A = \{\text{I have at least two girls }\}$is:

$$A = \left\{ \begin{array}{cc} \boxed{\text{G-AM}\,|\,\text{G-AM}}, & \boxed{\text{G-PM}\,|\,\text{G-AM}} \\ \boxed{\text{G-AM}\,|\,\text{G-PM}}, & \boxed{\text{G-PM}\,|\,\text{G-PM}} \end{array} \right\}$$

So $A \cap G$ is:

$$A \cap G = \left\{ \begin{array}{cc} \boxed{\text{G-AM}\,|\,\text{G-AM}}, & \boxed{\text{G-PM}\,|\,\text{G-AM}} \\ \boxed{\text{G-AM}\,|\,\text{G-PM}}, & \end{array} \right\}$$

Finally then:

$$\mathbf{P}(A|G) = \frac{\#|A \cap G|}{\#G|} = \frac{3}{7}$$

c) In the same way as b) if we draw G we see that it will consist of a row and a column that cross:

$$G = \left\{ \begin{array}{cccccc} & & & \boxed{\text{G-Mon}\,|\,\text{B-Mon}}, & & \\ & & & \boxed{\text{G-Mon}\,|\,\text{B-Tue}}, & & \\ & & & \vdots & & \\ \boxed{\text{B-Mon}\,|\,\text{G-Mon}}, & \boxed{\text{B-Tue}\,|\,\text{G-Mon}} & \dots & \boxed{\text{G-Mon}\,|\,\text{G-Mon}}, & \dots & \boxed{\text{G-Fri}\,|\,\text{G-Mon}} \\ & & & \vdots & & \\ & & & \boxed{\text{G-Mon}\,|\,\text{G-Fri}}, & & \end{array} \right\}$$

So $\#|G| = 14 + 14 - 1 = 27$. The set $A \cap G$ is the bottom piece of this:

$$A \cap G = \left\{ \begin{array}{ccc} \boxed{\text{G-Mon}\,|\,\text{G-Mon}}, & \dots & \boxed{\text{G-Fri}\,|\,\text{G-Mon}} \\ \vdots & & \\ \boxed{\text{G-Mon}\,|\,\text{G-Fri}}, & & \end{array} \right\}$$

So $\#|A \cap G| = 7 + 7 - 1 = 13$. Finally then:

$$\mathbf{P}(A|G) = \frac{\#|A \cap G|}{\#G|} = \frac{13}{27}$$

d) Following the patterns of the last two parts, $\#|G| = 2 \times 365 + 2 \times 365 - 1$ and $\#|A \cap G| = 365 + 365 - 1$ so we have:

$$\mathbf{P}(A|G) = \frac{\#|A \cap G|}{\#G|} = \frac{365 + 365 - 1}{2 \times 365 + 2 \times 365 - 1} = \frac{729}{1459}$$

4 More Info

4.1 Wikipedia

- Conditioning (probability)
- Conditional probability
- Bayes' theorem
- Monty Hall problem [see section on conditional probability]

4.2 Kahn Academy

- Probability and combinatorics