Special Properties of Holomorphic Functions

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For all these theorems, we assume the function f is holomorphic. The first most basic special property of holomorphic functions is that they have power series $f(z) = \sum c_n z^n$. This, along with the Cauchy integral formula, can be used to get a few more properties that holomorphic functions *must* satisfy. The main ones for us are:

- Non-zero functions have discrete zeros
- Bounded functions are 0 (Louiville's Theorem)
- Mean Value Property/Maximum Modulous Principle
- Shwarz Lemma

Theorem 1. *(Discrete Zeros) If there is a sequence z_n that has an acumulation point and $f(z_n) = 0$ for every n, then $f \equiv 0$*

Proof. Suppose WOLOG that the accumulation point is at $z = 0$. Write $f(z) = \sum c_n z^n$. If this is non-zero, then there is a first non-zero term, say $f(z) = z^n (c_n + c_{n+1}z + \ldots)$. But then $c_n = \lim_{z\to 0}(c_n + c_{n+1}z + \ldots) = \lim_{k\to\infty}(c_n + c_{n+1}z_k + \ldots) = \lim_{k\to\infty}\frac{f(z_k)}{z_k^n} = \lim_{k\to\infty} 0 = 0$, so $c_n = 0$, which contradicts that this was the first non-zero coefficient. □

Remark 2. By subtracting a constant from f, the proof gives us a "discrete α's" theorem: the set $\{z : f(z) = \alpha\}$ cannot have accumulation points unless $f \equiv \alpha$

Theorem 3. *(Louiville's Theorem) If $f(z)$ is bounded, then $f(z) = cons't$.*

Proof. Write $f(z) = \sum c_n z^n$ and use the Cauchy Integral formula to estimate the c_n's (proof taken from wikipedia)

$$\begin{aligned} |a_k| &\le \frac{1}{2\pi}\oint_{C_r}\frac{|f(\zeta)|}{|\zeta|^{k+1}}\,|d\zeta| \\ &\le \frac{1}{2\pi}\oint_{C_r}\frac{M}{r^{k+1}}\,|d\zeta| \\ &= \frac{M}{r^k} \to 0 \text{ as } r\to\infty \end{aligned}$$

□

Corollary 4. *(Extended Louivilles Theorem) If $f(z)$ has polynomial growth, $|f(z)| \le A|z|^N + B$ and then $f(z)$ is a polynomial of order N.*

Proof. (Method 1) Same estimate as above that goes to zero only for $k > N$ (Method 2) Do a proof by induction by writing $g(z) = \frac{f(z)-f(0)}{z}$ to reduce N by 1. □

Theorem 5. *If $f(z) \to \infty$ as $z \to \infty$, then $f(z)$ is a polynomial. (More precisly, the condition is that $\forall M > 0, \exists R > 0$ so that $|f(z)| > M$ for all $|z| > R$)*

Proof. The idea is to see that $\frac{1}{f(z)}$ must be bounded-ish, and then apply the Louivlle theorem. The obstuction that one has to work around is the zeros of f which ruin the analyticity of $\frac{1}{f}$.

Claim: f has finitely many zeros

Pf: Choose R so that $|f(z)| > 1$ for $|z| < R$, so that f cannot have zeros outside of $\{z : |z| \le R\}$. Since this is a compact set, f cannot have infinetly many zeros here, because if it did they would have an accumulation point and then $f \equiv 0$. The only remaining possibility is that f has finitely many zeros.

Let $\alpha_1, \alpha_2, \dots$ are the zeros of f, then consider $g(z) = \frac{f(z)}{\prod(z-\alpha_i)}$. This is holomorphic! Notice that $\left|\frac{1}{g(z)}\right| \le \frac{1}{1}\prod|z-\alpha_i|$ has polynomial growth outside of $|z| = R$. By the extended Louiville theorem, $\frac{1}{g}$ is a polynomial. But this polynomial has no zeros, so it must be that $g \equiv cons't$. Then $f(z) = C\prod(z-\alpha_i)$ is a polynomial. ☐

Corollary 6. *If $f(z): \mathbb{C} \to \mathbb{C}$ is entire and has a holomorphic inverse, then $f(z) = az + b$ for $a, b \in \mathbb{C}$*

Proof. This f will have $f \to \infty$ as $z \to \infty$. Otherwise, we whould have an M and a sequence $z_n \to \infty$ with $|f(z_n)| \le M$. Let $w_n = f(z_n)$. Have then that $f^{-1}(w_n) \to \infty$ but $|w_i| < M$, which means that f^{-1} is unbounded on the ball of radius M. This is impossible since f is continuous. Hence f is a polynomial. Since it is 1-1 it must be a degree 1 polynomial, and the result follows.

(There is another proof of this theorem using the fac that f cannot have an essential singularity at ∞, because it would not be $1:1$ if that happened (This is because essential singularities give dense images). Taking the Laurent series expansion around ∞ will then reveal that f must be a polynomial.) ☐

Theorem 7. *(Mean Value Property) $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$.*

Proof. This is just a restatement of the Cauchy Integral formula. ☐

Theorem 8. *(Maximum Modulous Principle) The modulous, $|f(z)|$ cannot have a local maximum in the interior of the domain of definition unless $f \equiv cons't$*

Proof. By the mean value property, $|f(z)| \le \frac{1}{2\pi}\int_0^{2\pi}|f(z+re^{i\theta})|d\theta \le \max_\theta |f(z+re^{i\theta})|$. If for some value of r, this equality is strict (i.e. its $<$) then f is not a local maximum. Otherwise, if this is equality for every sufficiently small r, notice the second equality hold if $|f(z+re^{i\theta})| = \max_\theta |f(z+re^{i\theta})|$ for each θ, so it must be that $|f|$ is constant on every circle of radius r. By the "discrete zeros" property, f is then constant everywhere. ☐

Remark 9. If f has no zeros, you can apply the maximum modulous principle to $\frac{1}{f}$ to get a minimum modulous principle.

Theorem 10. *(Schwarz Lemma) Let D be the unit disk and let $f: D \to D$ be a holomorphic function with $f(0) = 0$. Then $|f(z)| \le |z|$ for all z AND $|f'(0)| \le 1$. Moreover, if either of these are equality at any point, then it must be that $f(z) = cons't$*

Proof. Apply the maximum modulous principle to $g(z) = \frac{f(z)}{z}$. On the boundary it has modulous 1, so $|g(z)| \le 1$ everywhere which gives the result. (There is a slight technicality : you have to do it on a disk of radius $r < 1$ and then take the limit $r \to 1$) ☐

Theorem 11. *Fractional Linear Tranformations (FLT) are the ONLY conformal maps from the unit disk to the unit disk.*

Proof. Suppose $f: D \to D$ is holomorphic. Say ϕ is any FLT. It suffices to show that $g = \phi \circ f$ is a fractional linear transformation, since FLTs are closed under composition. Choose ϕ to map $f(0) \xrightarrow{\phi} 0, D \xrightarrow{\phi} D$ so that $g(0) = 0$ and $g: D \to D$. By Shwarz lemma $|g(z)| \le |z|$. Since this is a conformal map, it has an inverse $g^{-1}: D \to D$. Notice $g^{-1}(0) = 0$ too, so applying Schwarz lemma we have $|g^{-1}(z)| \le |z|$ relableing $z = g(w)$ this is saying $|w| \le |g(w)|$. Since we have inequalities both ways, it must be that $|g(z)| = |z|$ everywhere. Hence g must be a consant of norm 1. ☐

Problem 12. (Jan 2008 Problem 1)

Part 1: Prove that the most general 1:1 conformal map of the upper half-plane onto itself is of the form:

$$z \to \frac{az+b}{cz+d}$$

with real a, b, c, d and $ad - bc = 1$

Part 2: Let f be a 1:1 analytic function from the plane to itself. What can it be? A full explanation from first principles is wanted.

Problem 13. (Sept 2006 Problem 4)

Part 1: Conformally map the region $A = z : Re(z) < 0, 0 < Im(z) < \pi$ onto the first quadrant.

Part 2 Describe all conformal maps from A to the first quadrant.