1 Contour Integrals

Problem 1. (Jan 2001 Problem 1) Explain why the function $\sqrt{1-z^2}$ can be thought of as single valued in a plane cut along $-1 \leq z \leq 1$. Then the integral:

$$I = \int \frac{\mathrm{d}z}{\sqrt{1-z^2}}$$

Taken around the circle $|z| = R > 1$ makes sense. How is I related to $\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{\pi}{2}$? How does I change as $R \to \infty$. Compute I by "pure thought" in light of these remarks.

Problem 2. (September 2004 Problem 2)

1. Evaluate $\int_0^\infty \frac{\sqrt{x}\log(x)}{x^2+1}\mathrm{d}x$
2. Evaluate $\int_0^\infty \frac{\log(x)}{x^2+1}\mathrm{d}x$ and $\int_0^\infty \frac{\mathrm{d}x}{x^2+1}$ by integrating $\int_C \frac{(\log z)^2}{x^2+1}\mathrm{d}z$ over an appropriate contour.
3. Evaluate $\int_0^\infty \frac{(\log(x))^2}{x^2+1}\mathrm{d}x$

2 Counting Zeros

Theorem 3. *(Argument Principle and generalized argument principle)*

$$\begin{aligned} \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz &= \#zeros - \#poles \\ \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}g(z)dz &= \sum_k g(z_k) - \sum_k g(p_k) \end{aligned}$$

Proof. Follows from the Residue theorem. □

Theorem 4. *(Rouche's Theorem-A practical version)*
Say $f(z)$,$\epsilon(z)$ are never zero on some contour γ and that $|f(z)| > |\epsilon(z)|$ on the contour. Then $f(z)+\epsilon(z)$ has the same number of zeros as $f(z)$

Proof. Let $h_t(z) = \frac{f(z)+t\epsilon(z)}{f(z)}$, apply the argument principle integral to $h_t(z)$. The integral is a continuous function of t , but must always be an integer so it must be constant. At $t = 0$ this is1, so the integral is 0. At $t = 1$ it gives the number of zeros minus the number of poles of $\frac{f(z)+t\epsilon(z)}{f(z)}$, so this must be equal to 0 too. But the only poles come from roots of f, so number roots of f = number of roots of $f + \epsilon$. The condition $|\epsilon(z)| < |f(z)|$ ensures that $(1+t) > \left|\frac{f(z)+t\epsilon(z)}{f(z)}\right| > (1-t)$ has no roots or poles so that the integral is continuous. □

Example 5. (January 2009 Problem 3) Determine the number of zeros of the polynomial $z^5 + z^2 - 6z + 3$ in the annulus $\frac{1}{3} \leq |z| \leq 1$.

On $|z| = \frac{1}{3}$, the term $f(z) = 3$ dominates $\epsilon(z) = z^5 + z^2 - 6z$. So the number of roots is the same as the number of roots of f, namely 0.

On $|z| = 1$, the term $f(z) = -6z$ dominates $\epsilon(z) = z^5 + z^2 + 3$. So the number of roots is the same as the number of roots of f, namely 1.

Hence there is exactly 1 root in the given annulus.

Problem 6. (September 2009 Problem 2) [Easy] Show that the function $p(z) = z^5 + 7z + 3$ has exactly 4 roots in the annulus $1 < |z| < 2$ and that the only root in $|z| = 1$ is $z_1 = \frac{1}{2\pi i}\int_{|z|=1} \frac{z(5z^4+7)}{z^5+7z+3}\mathrm{d}z$.

Problem 7. (Jannuary 2011 Problem 2) [Hard] How many roots does $z^{10} - 6z^6 + 3z^4 - 1 = 0$ have inside the circle of: 1. radius 1, radius $\frac{1}{2}$, radius $\frac{3}{2}$.

3 Conformal Maps

3.1 Powers

Say $w = z^n$, and $z = x + iy$, $w = u + iv$. Best seen in polar coordinates, $re^{i\theta} \leftrightarrow r^n e^{in\theta}$. The map z^2 is particularly nice $(x+iy)^2 = x^2 - y^2 + 2ixy$, so the level curves $u = u_0$ or $v = v_0$ get mapped to *hyperbolas* $x^2 - y^2 = u_0$ or $xy = -\frac{v_0}{2}$. The lines $x = x_0$ and $y = y_0$ get mapped to *parabolas* $v^2 = 4x_0^2\left(x_0^2 - u\right)$ or $u^2 = 4y_0^2\left(y_0^2 - v\right)$

3.2 Exp and Log

They map "cartesian coordinates" to "polar coordinates" in a way$\exp(x + iy) = \exp(x)\exp(iy)$ or $\log(re^{i\theta}) = \log(r) + i\theta$

3.3 Fractional Linear Tranformations

Also called Mobius Transformations

- Map "circles" to "circles"
- Map conjugate points to each other
- Act like "matrices" (e.g. inverse formula) to see this use homogenous coordinates.
- Have exactly two fixed points (since it is essentially a quadratic equation) Corollary: It is uniquely specified by the action of 3 points
- The Fractional Linear Transformations have some uniqueness properties (i.e. they are the only maps from the unit disk to itself) that can be proved with Shwarz lemma (will do next time)

Problem 8. How do you find the FLT that maps (z_1, z_2, z_3) to (w_1, w_2, w_3)?

Two good ones to memorize (can be derived on the spot using the fact that FLT's preserve symetric points):

1. $e^{i\theta}\frac{z - z_0}{1 - z\bar{z}_0}$ maps the unit disk to the unit disk, and $z_0 \to 0$.
2. $e^{i\theta}\frac{z - z_0}{z - \bar{z}_0}$ maps the upper half plane to the unit disk

Problem 9. (Sept 2009 Problem 4) [Easy] Find the linear fractional transformation that carries the circle $|z| = 2$ into the circle $|z + 1| = 1$, the point -2 into the origin, and the origin into i.

3.4 Joukowsky transform

This is the map $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ whose inverse is $z = w - \sqrt{w^2 - 1}$ (as in previous problem this is set up so that it is defined on $\mathbb{C}\backslash[-1, 1]$. Has (here $H^{\pm}$is the upper or lower half plane, D is the unit disk and $D^+$is the top half of the unit disk):

$$\begin{aligned} H^+\backslash D^+ &\to H^+ \\ D^+ &\to H^- \\ H^+ &\to \mathbb{C}\backslash[-1, 1] \end{aligned}$$

Can see all this from writing $z = re^{i\theta} = r\cos(\theta) + ir\sin(\theta)$, and $\frac{1}{z} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}\cos(\theta) - i\frac{1}{r}\sin(\theta)$ then

$$w = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos(\theta) + i\frac{1}{2}\left(r - \frac{1}{r}\right)\sin(\theta)$$

So we that the cirlce $r = r_0$ get mapped to *ellipses* $\frac{u^2}{\frac{1}{2}(r+\frac{1}{r})^2} + \frac{v^2}{\frac{1}{2}(r-\frac{1}{r})^2} = 1$ with semimajor axis $r + \frac{1}{r}$ and semiminor axis $r - \frac{1}{r}$. The rays $\theta = \theta_0$ get mapped to the *hyperbolas* $\frac{u^2}{\cos(\theta_0)^2} - \frac{v^2}{\sin(\theta_0)^2} = 1$.

Problem 10. (January 2011 Problem #5) The picture shows what the function $f : \mathbb{C} \to \mathbb{C} + \infty$ does on the complex plane. The values 0 at 0, 1 at ± 1, and ∞ at $\pm i$ are specified. The signatures $+/-$ indicate that the regions so marked are mapped 1 to 1 onto the upper/lower half-plane. What is f? Explain why it cannot be otherwise.