CONTOUR INTEGRALS

1. Theory

Definition 1. Let $\gamma : [a, b] \to \mathbb{C}$ be a C^1 curve and let $f : \mathbb{C} \to \mathbb{C}$. Define:

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t$$

Remark 2. This is just like a line integral in $\mathbb{R}$, so we can use tools from there e.g. the fundemental theorem of calculus. One can check using the change of variables formula that if we re-parametrize the curve, then the value of the integral does not change. If you reverse the direction of the curve, you get a multiplicative factor of -1.

Theorem 3. *[Cauchy-Goursat Theorem] If $f(z)$ is holomorphic, and γ is the closed curve over a triangle, then $\int_\gamma f(z)dz$=0.*

Proof. (super rough sketch) Cut triangle into four pieces (midpoints), and integrate along each piece. There is one piece so that $|\int \text{piece}| > 1/4|\int \text{whole triangle}|$. Doing this recursivly, gives a sequence of triangles Δ^k with$|\int \Delta^k| > \frac{1}{4^k}|\int \text{original triangle}|$ But since triangle pieces are closed sets, there is a single point in the intersection $\Delta^k \to \{z_0\}$. Use the fact that f is holomorphic to write a Taylor series approx near z_0, with error to 0 as Δ^k as $k \to \infty$. Then plug approx into $\int \Delta^k$ to get an upper bound estimate and conclude$\int$ original = 0. ☐

Corollary 4. *Holomorphic function have antiderivatives in convex (or starlike) domains.*

Proof. Define $F(z) = \int_{[z_0,z]} f(\zeta)\mathrm{d}\zeta$. Show that F is continuous by the Cauchy Goursat theorem. The check F is differentiable by the Cauchy Riemann Equantions (again, by C-G) and the derivative is f. ☐

Corollary 5. *If f holomorphic in a convex (or starlike) domain then$\int_\gamma f(z)dz = 0$ for any closed curve γ.*

Proof. By last corolarry it has an antiderivative, so integral is zero by Fundemental Theorem of Calculus. ☐

Corollary 6. *If γ_1 and γ_2 are* homotopic *(i.e. they can be continuously deforemed into each other) in a region Ω where f is holomorphic, then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$*

Proof. Requires a little bit of work and set up....but not too hard from where we are. ☐

Theorem 7. *[Cauchy Integral Formula] Suppose that f is holomorphic in a disk of radius R centered at some point z_0. Then for any a in the disk and $r < R$:*

$$f(a) = \frac{1}{2\pi i} \int\limits_{|z-z_0|=r} \frac{f(z)}{z-a} dz$$

Proof. By the previous theorem, the integral on the RHS does not depend on r. By taking $r \to 0$ we can see that this value has to be $f(a)$. ☐

Corollary 8. *Suppose f is holomorphic in $A_{r,R} = \{z : r < z < R\}$ .The f has Laurent Series expansion, $f(a) = \sum_{n=-\infty}^{\infty} a^n c_n$ for $a \in A_{r,R}$. Moreover, the coefficients c_n are given explicitly in terms of an integrals of $f(z)$*

Proof. Suppose WOLOG we are working in Draw a picture then use the homotopic curves theorem and the Cauchy Integral Formula to see that for any $a \in A_{r,R}$:

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int\limits_{|z-a|=\epsilon} \frac{f(z)}{z-a}\mathrm{d}z \\ &= \frac{1}{2\pi i} \int\limits_{|z|=R} \frac{f(z)}{z-a}\mathrm{d}z - \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{f(z)}{z-a}\mathrm{d}z \end{aligned}$$

Then do some manipulations to force a geometric sum out:

$$\begin{aligned}\int_{|z|=R} \frac{f(z)}{z-a}\mathrm{d}z &= \int_{|z|=R} \frac{1}{1-\frac{a}{z}}\frac{f(z)}{z}\mathrm{d}z \\ &= \int_{|z|=R} \sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n \frac{f(z)}{z}\mathrm{d}z \\ &= \sum_{n=0}^{\infty} a^n \left(\int_{|z|=R} \frac{f(z)}{z^{n+1}}\mathrm{d}z\right)\end{aligned}$$

Exchanging the sum with the integral here is ok because the sum is *uniformly* convergent as $a \in A_{r,R}$ has $\left|\frac{a}{z}\right| < 1$ for $|z| = R$. We do the same thing on the inner circle of radius r, where this time $\left|\frac{z}{a}\right| < 1$ for $|z| = r$:

$$\begin{aligned}-\int_{|z|=r} \frac{f(z)}{z-a}\mathrm{d}z &= \int_{|z|=r} \frac{-1}{\frac{z}{a}-1}\frac{f(z)}{a}\mathrm{d}z \\ &= \int_{|z|=r} \sum_{n=0}^{\infty}\left(\frac{z}{a}\right)^n \frac{f(z)}{a}\mathrm{d}z \\ &= \sum_{n=0}^{\infty} a^{-(n+1)} \left(\int_{|z|=r} f(z)z^n\mathrm{d}z\right)\end{aligned}$$

So finally then:

$$\begin{aligned}f(a) &= \sum_{n=-\infty}^{\infty} a^n c_n \\ c_n &= \frac{1}{2\pi i}\int_{|z|=R} \frac{f(z)}{z^{n+1}}\mathrm{d}z \text{ for } n \geq 0 \\ c_{-n} &= \frac{1}{2\pi i}\int_{|z|=r} f(z)z^{n-1}\mathrm{d}z \text{ for } n \geq 1\end{aligned}$$

□

Corollary 9. *In the above setting,* $\int_\gamma f(z)dz = 2\pi i c_{-1}$ *is the coefficient of the exponent* -1 *in the Laurent series expansion of* f*.*

Definition 10. The coefficient c_{-1} above is called the *Residue* of f at $z = 0$ This is because the annulus $A_{r,R}$ was centered at $z = 0$. In general $Res_{z=z_0}f := c_{-1}$ where $c_{-1}$is the Laurent series expansion for f in an annulus centered at z_0. There are lots of different notations for the residue of f.

Theorem 11. *[Residue Theorem]* $\int_\gamma f(z)dz = 2\pi i \sum_{z_k} Wind_\gamma(z_k) \cdot Res_{z=z_k} f$

Proof. This is not very far from the corollary we just saw! □

2. Computing Residues

Method_1 Do some algebraic manipulations to get the Laurent series expansion of f and look at c_{-1}. Usually, you do this by combining some power series you already know e.g. $(1+z)^\alpha = 1 + \alpha z + \ldots$ or $\sin(z) = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \ldots$

Example Show$Res_{z=0}\frac{1}{z-\sin(z)}=\frac{3!}{5\cdot4}$

$$\begin{aligned}\frac{1}{z-\sin(z)} &= \frac{1}{z}\left(1-\frac{\sin z}{z}\right)^{-1}\\ &= \frac{1}{z}\left(1-1+\frac{1}{3!}z^2-\frac{1}{5!}z^4\dots\right)^{-1}\\ &= \frac{1}{z}\frac{3!}{z^2}\left(1-\frac{1}{5\cdot4}z^2+\dots\right)^{-1}\\ &= \frac{3!}{z^3}\left(1+\frac{1}{5\cdot4}z^2+\dots\right)\\ &= \frac{3!}{z^3}+\frac{3!}{5\cdot4}\frac{1}{z}+\dots\end{aligned}$$

Method_2 If you know the *order* of the singularity and it is finite, then you know that $f(z)=c_{-k}\left(z-z_0\right)^{-k}+\dots$, so $\left(z-z_0\right)^k f(z)=c_{-k}+\dots$ is an ordinary power series. The coefficient c_{-1} is the coefficient of $(z-z_0)^{k-1}$in $f(z)$, so its not hard to see that:

$$Res_{z=z_0}f=\lim_{z\to z_0}\frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\left[\left(z-z_0\right)^k f(z)\right]$$

For a *simple* pole, this is:

$$Res_{z=0}f=\lim_{z\to0}zf(z)$$

If $f(z)=\frac{g(z)}{h(z)}$ and $h(0)=0,h'(0)\neq0$ then:

$$\begin{aligned}Res_{z=0}f &= \lim_{z\to0}z\frac{g(z)}{h(z)}\\ &= \lim_{z\to0}\frac{g(z)}{h(z)/z}\\ &= \frac{g(0)}{h'(0)}\end{aligned}$$

3. Making Estimates

It often comes up that you want estimates on $\left|\int_\gamma f(z)\mathrm{d}z\right|$ to show that certain contour integrals are small. Here are two that are good.

Theorem 12. *[Estimation Lemma] Let $M=\sup_{z\in\gamma}|f(z)|$ and let $L=\int|\gamma'(t)|dt$ be the* length *of the curve γ. Then $\left|\int_\gamma f(z)dz\right|\le ML$*

Proof. From the definition, $\left|\int_\gamma f(z)\mathrm{d}z\right|=\left|\int f(\gamma(t))\gamma'(t)\mathrm{d}t\right|\le\int|f(\gamma(t))|\,|\gamma'(t)|\,\mathrm{d}t\le\int M\,|\gamma'(t)|\,\mathrm{d}t=ML$ □

Theorem 13. *[Jordan's Lemma] Suppose $f(z)$ can be written $f(z)=e^{iaz}g(z)$Let $C_R=\{z:z=Re^{i\theta},\theta\in[0,\pi]\}$ be a curve in the upper half plane. Then:*

$$\left|\int_{C_R}f(z)\,dz\right|\le\frac{\pi}{a}\max_{\theta\in[0,\pi]}\left|g\left(Re^{i\theta}\right)\right|$$

Proof. (From Wikipedia) Let $I_R=\left|\int_{C_R}f(z)\,dz\right|$ and $M_R=\max_{\theta\in[0,\pi]}\left|g\left(Re^{i\theta}\right)\right|$ for convenience. Have:

$$\begin{aligned}\int_{C_R}f(z)\,dz &= \int_0^\pi g(Re^{i\theta})\,e^{iaR(\cos\theta+i\sin\theta)}\,iRe^{i\theta}\,d\theta\\ &= R\int_0^\pi g(Re^{i\theta})\,e^{aR(i\cos\theta-\sin\theta)}\,ie^{i\theta}\,d\theta\,.\end{aligned}$$

So then:

$$\begin{aligned} I_R = \left| \int_{C_R} f(z)\, dz \right| &\leq R \int_0^{\pi} \left|g(Re^{i\theta})\right| \left| e^{aR(i\cos\theta - \sin\theta)}\, ie^{i\theta} \right| d\theta \\ &= R \int_0^{\pi} \left|g(Re^{i\theta})\right| e^{-aR\sin\theta} \mathrm{d}\theta \\ &\leq M_R R \int_0^{\pi} e^{-aR\sin\theta} \mathrm{d}\theta \\ &= 2M_R R \int_0^{\pi/2} e^{-aR\sin\theta} \mathrm{d}\theta \end{aligned}$$

Now use the inequality $\sin\theta \geq \frac{2\theta}{\pi}$ (This holds since sin is concave down for $\theta \in [0, \pi/2]$; draw a picture to see what the inequality is saying), to get:

$$\begin{aligned} I_R &\leq 2M_R R \int_0^{\pi/2} e^{-aR2\theta/\pi} \mathrm{d}\theta \\ &= 2M_R R \frac{\pi}{2aR} \left(1 - e^{-aR}\right) \\ &\leq \frac{\pi}{a} M_R \end{aligned}$$

□

Remark 14. Jordan's lemma will work nicely for many applications. If you need a bit more power, it is possible to improve it. One way is to split the integral above into $\int_0^{\theta_R}$ and then from $\int_{\theta_R}^{\pi/2}$. If you choose θ_R to be small (i.e. $\theta_R \to 0$ as $R \to \infty$) then the integral $\int_0^{\theta_R}$is controlled since θ_R is small, while $\int_{\theta_R}^{\pi/2}$ is controlled since $y > \theta_R$ here. This works well for functions for which $|f(x+iy)|$ is small for large y.

Remark 15. Wikipedia has a really really great section on methods of Contour Integration with a few nice worked examples. http://en.wikipedia.org/wiki/Methods_of_contour_integration

Example 16. $\int_0^{\infty} \frac{\log x}{(1+x^2)^2}\, \mathrm{d}x = ?$

Proof. (From Wikipedia) To do this integral, it turns out it is useful to look at the function $f(z) = \frac{\log(z)^2}{(1+z^2)^2}$. (We use the branch of log where $\mathrm{Arg}(z) \in (-\pi, \pi)$) Draw a keyhole contour around this branch of logarithm. The outside and inside circular contours $\to 0$ by the estimation lemma. The keyhole bits hit the axis from above and below, and so converge to:

$$\int_0^{\infty} \frac{(\log x + i\pi)^2}{(1+x^2)^2}\, \mathrm{d}x - \int_0^{\infty} \frac{(\log x - i\pi)^2}{(1+x^2)^2}\, \mathrm{d}x.$$

This is equal to:

$$4\pi i \int_0^{\infty} \frac{\log x}{(1+x^2)^2} \mathrm{d}x$$

It remains only to calculate the resdiues at $z = \pm i$. These are poles of order 2, so using method 2 above, we calculate:

$$\begin{aligned} \lim_{z \to \pm i} \frac{\mathrm{d}}{\mathrm{d}z} \left[(z \mp i)^2 \frac{\log(z)^2}{(1+z^2)^2} \right] &= \lim_{z \to \pm i} \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{\log(z)^2}{(z \pm i)^2} \right) \\ &= \lim_{z \to \pm i} \frac{2\log(z)(z \pm i)^2 - 2(z \pm i)\log(z)^2}{(z \pm i)^4} \\ &= -\frac{\pi}{4} \pm \frac{1}{16} i\pi^2 \end{aligned}$$

So we have by the residue theorem:

$$\begin{aligned} 4\pi i \int_0^{\infty} \frac{\log x}{(1+x^2)^2} \mathrm{d}x &= 2\pi i \left(-\frac{\pi}{4} + \frac{1}{16} i - \frac{\pi}{4} - \frac{1}{16} i \right) \\ \int_0^{\infty} \frac{\log x}{(1+x^2)^2} \mathrm{d}x &= -\frac{\pi}{4} \end{aligned}$$

□