1. Generators/Backward Eqn

	$t\in\mathbb{N},\mathcal{S}=\mathbb{N}$	$t\in\mathbb{R},\mathcal{S}=\mathbb{N}$	$t\in\mathbb{R},\mathcal{S}=\mathbb{R}$
$Lf(x):=$	$\mathbf{E}\left[f(X_1)\vert X_0=x\right]$	$\lim_{\Delta t\to 0}\frac{\mathbf{E}[f(X_{\Delta t})\vert X_0=x]-f(x)}{\Delta t}$	
$L=?$	$L_{i,j}=\mathbf{P}\left(i\to j\right)$	$L_{i,j}\mathrm{d}t+\delta_{i,j}=\mathbf{P}\left(i\overset{\mathrm{d}t}{\to}j\right)$	$\mathrm{d}\vec{X}_t=\vec{a}\mathrm{d}t+\overline{h}\mathrm{d}\vec{W}_t$
$\langle u_n,f_n\rangle$	$u_n^Tf_n$	$L=\sum a_i\partial_{x_i}+\frac{1}{2}\sum\left(\overline{hh}^T\right)_{ij}\partial_{x_ix_j}$	
$\langle u,f\rangle$	$\int u(x,t)f(x,t)\mathrm{d}x$	$\sum\partial_{x_i}\left(a_i\,\cdot\right)+\frac{1}{2}\sum\partial_{x_ix_j}\left(\left(\overline{hh}^T\right)_{ij}\,\cdot\right)$	

Fact 1. *Let $f(x,t)=\mathbf{E}\left[V(X_T,T)\vert X_t=x\right]$. For $t=\mathbb{N}$: $f_{n-1}=Lf_n$. For $t=\mathbb{R}$: $\partial_tf+Lf=0$*

Proof. Write (by defn of L) $\mathbf{E}\left(f(X_{t+\Delta t},t+\Delta t)\vert X_t=x\right)=f(x,t)+Lf(x,t)\Delta t+\partial_tf(x,t)\Delta t+o(\Delta t)$, and also notice that this is equal to $f(x,t)$ by the tower property. Gives $0=Lf+\partial_tf$. You can also Taylor expand this to get the formula for L. ☐

Same idea works for $f(x,t)=\mathbf{E}\left[\int_t^TV(X_s)\mathrm{d}s\right]$ or $\mathbf{E}\left[\exp\left(\int_t^TV(X_s)\mathrm{d}s\right)\right]$. When you use the tower property this time, instead of getting $f(x,t)$ you get $f(x,t)-\Delta tV(x)+O(\Delta t^2)$ or $f(x,t)\left(1-\Delta tV(x)+O(\Delta t^2)\right)$. Which lead to the backwards equations $-V=Lf+\partial_tf$ or $-Vf=Lf+\partial_tf$.

Fact 2. *Let $u(x,t)$ the p.d.f (column vector). For $t=\mathbb{N}$: $u_{n-1}=L^\star u_{n-1}$. For $t=\mathbb{R}$: $\partial_tu+L^\star u=0$*

Proof. Hit the bck eqn with the correct inner product and use $\langle u,f\rangle=\mathbf{E}\left(V(T)\right)=const$. Change L to $L^\star$and move over in inner product. You can use integration by parts to calculate $L^\star$. ☐

2. Special Examples

Definition 3. (Strong vs Weak Sol'n) A weak sol'n to the SDE $\mathrm{d}X_t=a(X_t)\mathrm{d}t+b(X_t)\mathrm{d}W_t$ is a process X_t that has $\mathbf{E}\left(\Delta X_t\vert\mathcal{F}_t\right)=a(X_t)\Delta t+O(\Delta t^2)$ and $\mathbf{E}\left((\Delta X_t)^2\vert\mathcal{F}_t\right)=b(X_t)^2\Delta t+O(\Delta t^2)$. A strong solution is a function g so that $X_t=g(W_t,t)$

Example 4. Brownian Martingales: W_t,W_t^2-t, any derivative of $\exp\left(\theta W_t-\frac{1}{2}\theta^2t\right)$ at $\theta=0$.

Fact 5. $\mathbf{E}\left(\exp\left(\theta Z\right)\right)=\exp\left(\theta\mu+\frac{1}{2}\theta\sigma^2\right)$ *for $Z\sim N(\mu,\sigma^2)$ (Use the above martingales)*

Example 6. 1. Let $\xi(s)$ be a deterministic function and $\theta(t)=\int_0^t\xi(s)\mathrm{d}s$. Then $W_{\theta(t)}\sim\int_0^t\xi(s)\mathrm{d}W_s$. 2. Brownian scaling: $B_t=\frac{1}{\sqrt{c}}W_{ct}$, and $B_t=tW_{\frac{1}{t}}$ still B.M.

Definition 7. OU process: $X_t=-\gamma X_t\mathrm{d}t+\sigma\mathrm{d}W_t$.

Fact 8. *Has $X_T=X_0e^{-\gamma T}+\mu(1-e^{-\gamma T})+\int_0^T\sigma e^{\gamma(s-T)}\,dW_s$. by Ito isometry*$\mathbf{Cov}(X_s,X_t)=\frac{\sigma^2}{2\gamma}\left(e^{-\gamma(t-s)}-e^{-\gamma(t+s)}\right)$. *(To prove this apply Ito Lemma to $X_te^{\gamma t}$*

Proposition 9. *OU process has $X_t=Ke^{-\gamma t}W_{\frac{\sigma}{2\gamma K^2}\left(e^{2\gamma t}-1\right)}=\frac{\sigma}{\sqrt{2\gamma}}e^{-\gamma t}W_{\left(e^{2\gamma t}-1\right)}$.*

Proof. Write $Y_t=a(t)W_{s(t)}$ where a,s are to be determined. Then Taylor expand $Y_{t+\Delta t}-Y_t$ to find that: $\mathbf{E}\left[Y_{t+\Delta t}-Y_t\vert\mathcal{F}_t\right]=a'(t)W_{s(t)}\Delta t=\frac{a'(t)}{a(t)}Y_t\Delta t$ and $\mathbf{E}\left[\left(Y_{t+\Delta t}-Y_t\right)^2\vert\mathcal{F}_t\right]=a^2(t)s'(t)\Delta t$. Equating this to $-\gamma Y_t$ and σ give ODES for a,s which we can solve. (This method can be improved to $x_t=x_0e^{-\gamma t}+\mu(1-e^{-\gamma t})+\frac{\sigma}{\sqrt{2\gamma}}e^{-\gamma t}W_{e^{2\gamma t}-1}$.) ☐

Definition 10. Geometric Brownian Motion: $S_t=\mu S_t\mathrm{d}t+\sigma S_t\mathrm{d}W_t$

Proposition 11. *Has $S_t=s_0\exp\left[\sigma W_t+(\mu-\frac{1}{2}\sigma^2)t\right]$*

Proof. Do Ito's lemma on $\log S_t$:$\mathrm{d}\log S_t=0\mathrm{d}t+\frac{1}{S_t}\mathrm{d}S_t-\frac{1}{2}\frac{1}{S_t^2}\left(\mathrm{d}S_t\right)^2=\mu\mathrm{d}t+\sigma\mathrm{d}W_t\left(-\frac{1}{2}\sigma^2\mathrm{d}t\right)$. So then $\log S_t=\left(\mu-\frac{1}{2}\sigma^2\right)\mathrm{d}t+\sigma W_t+\log S_0$ is a sol'n. Can also be done with the exponential Ansatz $S_t=A_te^{\sigma W_t}$. Can also find a change of variables in the backward equation which makes this work. ☐

3. Convergence Proofs

Lemma 12. *(Borel Cantelli Lemma) $\sum\mathbf{E}\left(A_n\right)<\infty\Rightarrow\sum A_n<\infty$ almost surely $\Rightarrow A_n\to 0$ almost surely. Often useful to apply to A_n^2.*

Lemma 13. *(Cauchy Shwarz Ineq) $\mathbf{E}\left(uv\right)\leq\mathbf{E}\left(u^2\right)^{\frac{1}{2}}\mathbf{E}\left(v^2\right)^{\frac{1}{2}}$. Examples: $\mathbf{E}\left(\vert\Delta W\vert\right)\leq\mathbf{E}\left(\Delta W^2\right)^{\frac{1}{2}}\mathbf{E}\left(sgn(W)^2\right)^{\frac{1}{2}}=(\Delta t)^{\frac{1}{2}}\left(1\right)^{\frac{1}{2}}=O(\Delta t^{\frac{1}{2}})$ and $\mathbf{E}\left(\vert\Delta W\vert^3\right)\leq\mathbf{E}\left(\vert\Delta W\vert\right)O(\Delta t^{\frac{3}{2}})$*

Proposition 14. *Ito Integral: $\int_0^tf_sdW_s=\lim_{\Delta t\to 0}\sum_{t<t_k}f_{t_k}\left(W_{t_{k+1}}-W_{t_k}\right)$ converges almost surely. (Technical condition: need f_t to be adapted, and $\mathbf{E}\left(\left(f_{t+\Delta t}-f_t\right)^2\vert\mathcal{F}_t\right)=O(\Delta t)$*

Proof. Use $\Delta t=2^{-m}$and write $_k$ instead of t_k everywhere. Let $S_t^m=\sum_{t_k<t}f_k\left(W_{k+1}-W_k\right)$. Then, cutting Δt in half, we have: $S_t^{m+1}=\sum_{t_k<t}f_{k+\frac{1}{2}}\left(W_{k+1}-W_{k+\frac{1}{2}}\right)+f_k\left(W_{k+\frac{1}{2}}-W_k\right)$ and factoring gives:

$$S_t^{m+1}-S_t^m=\sum_{t<t_k}\left(f_{k+\frac{1}{2}}-f_k\right)\left(W_{k+1}-W_{k+\frac{1}{2}}\right):=\sum Y_k$$

Each Y_k has $\mathbf{E}(Y_k\vert\mathcal{F}_{k+\frac{1}{2}})=(f_{k+\frac{1}{2}}-f_k)\mathbf{E}\left(W_{k+1}-W_{k+\frac{1}{2}}\vert\mathcal{F}_{k+\frac{1}{2}}\right)=0$, and so for $j<k$ we have $\mathbf{E}(Y_jY_k\vert\mathcal{F}_{k+\frac{1}{2}})=Y_j\mathbf{E}\left(Y_k\vert\mathcal{F}_{k+\frac{1}{2}}\right)=0$. Moreover, $\mathbf{E}\left(Y_k^2\vert\mathcal{F}_{k+\frac{1}{2}}\right)=(f_{k+\frac{1}{2}}-f_k)^2\mathbf{E}\left(\left(W_{k+1}-W_{k+\frac{1}{2}}\right)^2\vert\mathcal{F}_{k+\frac{1}{2}}\right)=(f_{k+\frac{1}{2}}-f_k)^2\left(\frac{\Delta t}{2}\right)$. Hence, $\mathbf{E}(Y_k^2\vert\mathcal{F}_k)=\mathbf{E}((f_{k+\frac{1}{2}}-f_k)^2\left(\frac{\Delta t}{2}\right)\vert\mathcal{F}_k)\leq C\Delta t^2$ by the assumption on f.

So then let $A_m=S_t^{m+1}-S_t^m$ and get $\mathbf{E}\left(A_m^2\right)=\sum_{t_k<t}\mathbf{E}\left(Y_k^2\right)+2\sum_{t_k<t_j<t}\mathbf{E}\left(Y_kY_j\right)\leq\sum_{t_k<t}C\Delta t^2+0=Ct\Delta t$. Hence, $\sum\mathbf{E}(A_m)\leq\sum\mathbf{E}(A_m^2)^{\frac{1}{2}}\leq\sum C\sqrt{t}\left(2^{-m/2}\right)<\infty$ so by Borel Canteli, $\sum A_m<\infty$ almost surely. But this shows that S_t^m is a Cauchy Sequence, so it has an almost sure limit! ☐

Proposition 15. *Quadratic Variation : $\sum_{t_k<T}f_t\left(W_{t_{k+1}}-W_{t_k}\right)^2\to\int_0^Tf_tdt$ a.s. (Technically need: $\int\mathbf{E}\left(f_t^2\right)dt<\infty$)*

Corollary 16. *The same proof shows that for a diffusion process $dX_t=a(X_t,t)dt+b(X_t,t)dW_t$ has $\sum_{t_k<T}f_{t_k}\left(X_{t_{k+1}}-X_{t_k}\right)^2\to\int_0^Tf_tb^2(X_t,t)dt$ a.s. The assumption on diffusion proceses that $\mathbf{E}\left(\Delta X^4\vert\mathcal{F}_t\right)=O(\Delta t^2)$ is needed! This shows that $[X]_t=\int_0^Tb^2(X_t,t)dt$*

Proof. Write: $A_m:=\sum_{t_k<T}f_k\left[\left(W_{k+1}-W_k\right)^2-\Delta t\right]:=\sum Y_k$ Notice that $\mathbf{E}\left(Y_k\vert\mathcal{F}_k\right)=f_k\mathbf{E}\left(\left(W_{k+1}-W_k\right)^2-\Delta t\vert\mathcal{F}_k\right)=0$ and so for $j<k$ we have $\mathbf{E}\left(Y_jY_k\vert\mathcal{F}_k\right)=Y_j\mathbf{E}\left(Y_k\vert\mathcal{F}_k\right)=0$. Moreover, $\mathbf{E}\left(Y_k^2\vert\mathcal{F}_k\right)=f_k\mathbf{E}\left(\left(\left(W_{k+1}-W_k\right)^2-\Delta t\right)^2\vert\mathcal{F}_k\right)=f_k\mathbf{Var}\left(\left(W_{k+1}-W_k\right)^2\right)=f_k\left(3\Delta t^2-\Delta t^2\right)=2f_k\Delta t^2$. Then:$\mathbf{E}\left(A_m^2\right)=\sum_k\mathbf{E}\left(f_k^2\right)2\Delta t^2+0=2\Delta t\int\mathbf{E}\left(f_t^2\right)\mathrm{d}t+o(\Delta t^2)$.

So since $\Delta t = 2^{-m}$ we see that $\sum \mathbf{E}(A_m^2)$ is a geometric series and by Borel Cantelli conclude that $A_m \to 0$ almost surely. □

Proposition 17. *Ito Isometry: Let* $X_t = \int_0^t f(W_s, s) dW_s$ *and* $Y_t = \int_0^t g(W_s, s) dW_s$ *then* $\mathbf{E}(X_t Y_t) = \int_0^t \mathbf{E}(f(W_s, s) g(W_s, s))\, ds$.

Proof. We approximate all the integrals by Riemann sums to see the result. Notice that :$\mathbf{E}\left(\sum_{t_j<t}\sum_{t_k<t} f_j g_k \Delta W_j \Delta W_k\right) = \sum_{t_j<t} \mathbf{E}\left(f_j g_j \Delta W_j^2\right) + 2\sum_{j<k} \mathbf{E}\left(f_j g_k \Delta W_j \Delta W_k\right) = \ldots = \sum_{t_j<t} \mathbf{E}\left(f_j g_j \Delta t\right) + 0$ □

Proposition 18. *Ito Lemma: If* $f(x,t)$ *is a smooth function, then* $df(W_t, t) = \left(\partial_t f(W_t, t) + \frac{1}{2}\partial_{xx} f(W_t, t)\right) dt + \left(\partial_x f(W_t, t)\right) dW_t$ *in then sense that if you integrate this expression, the random variable on the LHS and RHS are a.s. equal. (Technical conditions: need f is 3 times differentiable with bounded third derivatives)*

Proof. After integrating the LHS is $f(W_b, b) - f(W_a, a)$. Write this as a telescoping sum using time steps of size Δt, so that we need to consider $\sum \Delta f$. As in the proof of the existence of Ito integrals, we consider each Δf individually first, then we show that the series of sums is Cauchy with Borel Cantelli. Taylor expand each Δf we have $f(W_{t+\Delta t}, t+\Delta t) - f(W_t, t) = \Delta t \partial_t f(W_t, t) + \Delta W_t f(W_t, t) + \frac{1}{2}(\Delta W_t)^2 \partial_{xx} f(W_t, t) + O(\Delta t^2) + O(\Delta t |\Delta W|) + O(\Delta W^3)$

(The bounded-ness of the third derivitives is needed to controll the last three error terms here) By the tricks with C-S inequality, all the error terms are at least $O(\Delta t^{\frac{3}{2}})$ and so after summing, they are still neglible. The terms $\Delta t \partial_t f(W_t, t)$ and $\Delta W_t \partial_x f(W_t, t)$ when are $\int \partial_t f(W_t, t) \mathrm{d}t$ and $\int \partial_x f(W_t, t) \mathrm{d}W_t$. The term $\frac{1}{2}(\Delta W_t)^2 \partial_{xx} f(W_t, t)$ when summed converges to $\int \frac{1}{2}\partial_{xx} f(W_t, t) \mathrm{d}t$ by the QUADRATIC VARIATION for Brownian motion proposition. □

4. Multidimensional Formula

If $\vec{X}$ is a vector diffusion process with $\mathrm{d}X_{i,t} = a_{i,t}\mathrm{d}t + \sum_{j=1}^n h_{ij,t} \mathrm{d}W_{j,t}$. Then the Ito formula is:

$$\begin{aligned} \mathrm{d}f(X_t, t) &= \frac{\partial f}{\partial t}\mathrm{d}t + \sum_j \frac{\partial f}{\partial x_j}\mathrm{d}X_{j,t} + \frac{1}{2}\sum_{j,k}\frac{\partial^2 f}{\partial x_j x_k}\mathrm{d}X_{j,t}\mathrm{d}X_{k,t} \\ &= \left(\frac{\partial f}{\partial t} + \sum_j a_{j,t}\frac{\partial f}{\partial x_j} + \frac{1}{2}\sum_{j,k,p} h_{jp,t} h_{kp,t}\frac{\partial^2 f}{\partial x_j x_k}\right)\mathrm{d}t + \sum_{j,p} h_{jp,t}\frac{\partial f}{\partial x_j}\mathrm{d}W_{p,t} \end{aligned}$$

5. Maximum of BM / Stopping Times

Proposition 19. *Let* $B(t)$ *be a Brownian motion starting at* $B(0) = 0$ *and let* τ_a *be the first time* $B(t)$ *hits* a. *Then* $\mathbf{P}(\tau_a < t) = 2\mathbf{P}(B(t) > a)$

Proof. $\mathbf{P}(B(t) > a) = \mathbf{P}(B(t) > a | \tau_a > t)\mathbf{P}(\tau_a > t) + \mathbf{P}(B(t) > a | \tau_a < t)\mathbf{P}(\tau_a < t) = 0 + \frac{1}{2}\mathbf{P}(\tau_1 < t)$ □

Fact 20. $\mathbf{E}(X_\tau) = \mathbf{E}(X_0) \iff \mathbf{E}\left(\lim_{n\to\infty} X_{\min(\tau,n)}\right) = \lim_{n\to\infty}\mathbf{E}\left(X_{\min(\tau,n)}\right)$; $\mathbf{P}(\tau < \infty) = 1$ *and* $\mathbf{E}(|X_\tau|) < \infty$, *and* $\lim_{n\to\infty}\mathbf{E}(X_n 1_{\tau>n}) = 0$, *then* $\mathbf{E}(X_\tau) = \mathbf{E}(X_0)$.

Proposition 21. *To get* $\mathbf{E}(\tau)$ *for reasonable stopping times* τ *use the martingales like* W_t *or*$W_t^2 - t$ *and the Doob optional stopping time theorem. Example: Say* $W(0) = 0$ *and let* $\tau = \tau_a \wedge \tau_{-b}$ *be the first time it exists the interval* $[-b, a]$ *(here* $a, b > 0$*). Then* $\mathbf{P}(\tau = \tau_a) =$, $\mathbf{P}(\tau = \tau_{-b}) =$ *and* $\mathbf{E}(\tau) = ab$

Proof. Have a linear equation with two unknowns for $p_a = \mathbf{P}(\tau = \tau_a)$ and $p_b = \mathbf{P}(\tau = \tau_b)$:$p_a + p_b = 1$ $0 = \mathbf{E}(W_\tau) = \mathbf{E}(W_\tau | \tau = \tau_a) p_a + \mathbf{E}(W_\tau | \tau = \tau_{-b}) p_b = a p_a - b p_b$. Solving gives $p_a = \frac{b}{a+b}$ $p_b = \frac{a}{a+b}$. Now using the martingale $W_t^2 - t$, $\mathbf{E}(\tau) = b^2 p_b + a^2 p_a = ab$. □

Fact 22. *Let* W_t *be a Brownian motion,* $M_t = \sup_{s<t} W_s$ *The joint density is:* $\rho(M_T = b, B_T = a)$ $\frac{2(2b-a)}{\sqrt{2\pi}T^{\frac{3}{2}}}\exp\left(-\frac{(2b-a)^2}{2T}\right)$ *for* $b > a$ *and* $b > 0$. *(density is 0 otherwise)*

6. Green's Function For the Heat Eqn

The sol'n for the heat equation can be seen by remembering the formula for adding random variables (this is a convolution $\rho(X_t = \cdot) = G(\cdot, t-s) * \rho(X_s = \cdot)$)

$$\begin{aligned} \rho(X_t = x) &= \int \rho(X_t - X_s = x - y)\rho(X_s = y)\mathrm{d}y \\ &= \int \frac{1}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{(x-y)^2}{2(t-s)}\right)\rho(X_s = y)\mathrm{d}y \end{aligned}$$

7. Backward Equation for stopping times

The function $f(x,t) = \mathbf{E}\left(V(X_T) 1_{\{\tau > T\}} | \tau > t, X_t = x\right)$ satisfies the backward equation.

Example 23. Find $\mathbf{P}(\tau_0 > t)$ for a Brownian motion starting at $X_0 = 1$.

Proof. Choose $V \equiv 1$ in the backward equation to get the PDE $\partial_t f + \frac{1}{2}\partial_{xx} f = 0$ with boundary condition $f(x, T) = 1$ for $x > 0$. and $f(0, t) = 0$ for all $t \in [0, T]$. On this domain we can solve it with the method of images by doing an odd extension of f. Do a change of variable $s = T - t$ to get $\partial_s f = \frac{1}{2}\partial_{xx} f$ has a sol'n $f(s, x) = \int_0^\infty \frac{1}{\sqrt{2\pi s}}\exp\left(-\frac{(x-y)^2}{2s}\right)\mathrm{d}y - \int_{-\infty}^0 \frac{1}{\sqrt{2\pi s}}\exp\left(-\frac{(x-y)^2}{2s}\right)\mathrm{d}y$ so $f(t, x) = N\left(\frac{x}{\sqrt{T-t}}\right) - N\left(\frac{-x}{\sqrt{T-t}}\right)$ where $N(z) = \mathbf{P}(Z \le z)$ for Z a standard normal. □

Example 24. The p.d.f. for X_t starting at X_0 conditioned on $\tau_0 > t$ is $\frac{1}{\mathbf{P}(\tau > t)}\frac{1}{\sqrt{2\pi t}}\left(\exp(-(x - x_0)^2/2t)\right.$

Proof. Use boundary condition $u(0, t) = 0$ and method of images to get the p.d.f. for Brownian motion killed at $x = 0$. Then divide by $\mathbf{P}(\tau > t)$ to normalize it. □

8. Girsanov Formula

Proposition 25. *Let* P, Q *be measures on path space so that:*$P : dX_t = 0 + dW_t Q : dX_t = a_t dt + dW_t$
Then $\frac{dQ}{dP} = L$ *exists and* $L = \exp\left(\int_0^T a_t dX_t - \frac{1}{2}\int_0^T a_t^2 dt\right)$

Proof. Write the (approximate) density function for a path:$u_P(x_1, \ldots, x_n) = C\prod\exp\left(-\frac{(x_i - x_{i+1})^2}{2b_i^2\Delta t}\right)$; $C\prod\exp\left(-\frac{(x_i - x_{i+1} - a_i\Delta t)^2}{2b_i^2\Delta t}\right)$ So the ratio is:

$$\begin{aligned} L(x_1, \ldots, x_n) &= u_Q(x_1, \ldots, x_n)/u_P(x_1, \ldots, x_n) \\ &= \exp\left(-\tfrac{1}{2}a_i^2\Delta t + a_i(x_i - x_{i+1})\right) \to \exp\left(\int_0^T \frac{a_t}{b_t^2}\mathrm{d}X_t - \frac{1}{2}\int_0^T \frac{a_t^2}{b_t^2}\mathrm{d}t\right) \end{aligned}$$

□

Proposition 26. *For* $P : dX_t = 0 + b_t dW_t Q : dX_t = a_t dt + b_t dW_t$ *then Then* $\frac{dQ}{dP} = L$ *exists and* $L = \exp\left(\int_0^T \frac{a_t}{b_t} dW_t - \frac{1}{2}\int_0^T \frac{a_t^2}{b_t^2} dt\right)$

Proof. First factor out a b_t to get $\mathrm{d}X = b_t(\mu_t \mathrm{d}t + \mathrm{d}W_t)$. Then set $\mu_t = \frac{a_t}{b_t}$ to get the result. □