Random Graphs and Complex Networks

1 Branching Processes

A branching process is a simple model for something like a population evolving in time.

Example 1. Suppose we have some amoeba who, at each generation, each individual gives birth to a number of children given by some distribution: $p_i = \mathbf{P}$ (individual has exactly i children). Let Z_n be the number of individuals born in the n^{th} generation. A nice way to realize the Z_n is to set up an infinite array of i.i.d random variables $X_{j,n}$ each distributed by the vector $\vec{p}$ i.e. $\mathbf{P}(X = i) = p_i$. This will represent the number of children that the individual j will have in the n-th generation. We must be careful not to count individuals who haven't been born yet. Some thought leads us to the recursive expression for Z_n:

$$Z_n = \sum_{j=1}^{Z_{n-1}} X_{n,j}$$

Theorem 2. *Let $\eta = \mathbf{P}(\exists n : Z_n = 0)$ be the probability that the population becomes extinct at some time. For a branching process with i.i.d offspring given by X as above, we will have $\eta = 1$ when $\mathbf{E}(X) < 1$ while $\eta < 1$ for $\mathbf{E}(X) > 1$. If $\mathbf{E}(X) = 1$ and $\mathbf{P}(X = 1) < 1$ then $\eta = 1$. Moreover, if we let G_X be the probability generting function of X:*

$$G_X(s) = \mathbf{E}\left(s^X\right)$$

Then η is the smallest solution in $[0,1]$ of:

$$\eta = G_X(\eta)$$

Proof. The proof uses a slick trick with probability generating functions. Let G_n be the generating function for Z_n, so that $G_n(s) = \mathbf{E}\left(s^{Z_n}\right)$. let $\eta_n = \mathbf{P}(Z_n = 0) = G_n(0)$ be the probability that we are extict at time n. Since once we are extinct, we stay extinct forever, we know that η_n is monotone non-decreasing. Moreover, $\eta_n \to \eta$ as $n \to \infty$ since $\{Z_n = 0\} \uparrow \{\exists n : Z_n = 0\}$. Now, using the fact that $Z_n \sim \sum_{i=1}^{X} \tilde{Z_{n-1}}$ i.e. Z_n is distributed like X independent copies of Z_{n-1} (the X here is the number of guys in the first generation $X = Z_1$), and $Z_1 \sim X$ we get that:

$$\begin{aligned}
G_n(s) &= \mathbf{E}\left(s^{Z_n}\right) \\
&= \sum_{i=0}^{\infty} p_i \mathbf{E}\left(s^{Z_n} | Z_1 = i\right) \\
&= \sum_{i=0}^{\infty} p_i \mathbf{E}\left(s^{\sum_{j=1}^{Z_1} Z_{n-1}} | Z_1 = i\right) \\
&= \sum_{i=0}^{\infty} p_i \mathbf{E}\left(s^{Z_{n-1}}\right)^i \\
&= \sum_{i=0}^{\infty} p_i \left(G_{n-1}(s)\right)^i \\
&= G_X\left(G_{n-1}(s)\right)
\end{aligned}$$

Plugging in $s = 0$ gives the recurrence relation for η_n through $\eta_n = G_n(0)$:

$$\eta_n = G_X(\eta_{n-1})$$

Finally, since $\eta_n \to \eta$ and since G_X is continuous in $[0,1]$ (its analytic in at least the unit disk, since the coefficients of its power series are all less than 1), by taking $n \to \infty$ we have that η must satisfy:

$$\eta = G_X(\eta)$$

This is the main breakthough, the rest of the result follows by a bit of careful analysis.

If $\mathbf{P}(X = 1) = 1$,then $Z_n = 1$ a.s. and so there is nothing to prove. If $\mathbf{P}(X = 0) + \mathbf{P}(X = 1) = 1$, but $\mathbf{P}(X = 0) = p > 0$, then $G_X(s) = p + (1-p)s$, so the reucrrence above gives $\eta_n = 1 - (1-p)^n$ and $\eta = 1$.

Hence, with these cases covered, we may assume WOLOG that $\mathbf{P}(X \le 1) < 1$. Let $\psi \in [0,1]$ be any root of $\psi = G_X(\psi)$. Claim: $\eta_n \le \psi$ for all n. Proof: (by induction) $\eta_0 = 0 \le \psi$, and if $\eta_{n-1} \le \psi$ then $\eta_n = G_X(\eta_{n-1}) \le G_X(\psi) = \psi$ since G_X is an increasing function. Hence $\eta = \lim \eta_n \le \psi$ too. So η must be the *smallest* root of $\eta = G_X(\eta)$ in $[0,1]$.

Now, notice $G_X'' = \mathbf{E}(X(X-1)s^X) \ge 0$ so X is concave up, and recall that $G_X'(1) = \mathbf{E}(X)$. If $G_X' = \mathbf{E}(X) < 1$, then by convexity, there can be no intersection between $G_X(s)$ and s for $s \in [0,1]$, while for $G_X' = \mathbf{E}(X) > 1$ there must be at least one point of interesction less than 1 when we take into account the fact that $G_X(0) = \mathbf{P}(X=0) \ge 0$ (Draw a picture! It's like an easy ODE's type slope and region argument). If $G_X'(1) = \mathbf{E}(X) = 1$ then either $G_X(0) = 0$ and $G_X(s) \equiv s$ (which is the case we already covered $X = 1$ a.s.) or $G_X(0) > 0$ and there can be no intersection except at $s = 1$ again by convexity. (Draw the picture again!) □

Theorem 3. *Let $T = \sum_{n=0}^{\infty} Z_n$ be the total size of the population. Then the generating function for T, $G_T(s) = \mathbf{E}\left(s^T\right)$ satisfies:*

$$G_T(s) = sG_X\left(G_T(s)\right)$$

Notice that this is not normalized if the extenction probability$\eta < 1$ (i.e. its possible that $T = \infty$), instead of the usual $G_Y(1) = 1$, we have here $G_T(1) = \eta$.

Proof. This is the same type of proof as the reccurance in the above theorem, just condition on $Z_1 \sim X$ and use independence. □

Example 4. If we consider binary branching, $X = 0$ with probability p, and $X = 2$ with probability $q = 1-p$. Then:

$$G_T(s) = \frac{1 - \sqrt{1 - 4s^2pq}}{2sp}$$

Notice that if we plug in $s = 1$ here we get:

$$\begin{aligned} \eta &= \frac{1 - |2p-1|}{2p} \\ &= \begin{cases} 1 & p < \frac{1}{2} \\ \frac{1-p}{p} & p > \frac{1}{2} \end{cases} \end{aligned}$$

Remark 5. Notice this is exactly the form of the Lagrange implicit function theorem (see notes from Combinatorics, the formula in question is $R(x) = xG(R(x)) \Rightarrow [x^0]R(x) = 0$ and $[x^n]R(x) = \frac{1}{n}[u^{n-1}]G(u)^n$ whenever $[u^0]G(u) \ne 0$) Intepreted probabalistically, this says that, under the assumption $\mathbf{P}\{X = 0\} > 0$, we have:

$$\begin{aligned} \mathbf{P}\{T = 0\} &= 0 \\ \mathbf{P}\{T = n\} &= \frac{1}{n}\mathbf{P}\{X_1 + X_2 + X_3 + \ldots + X_n = n-1\} \end{aligned}$$

I am not sure what the combinatorial interpratation of this formula is. (Later on we will see why using the "hitting-time" theorem)

Theorem 6. *If $\mathbf{E}(X) = \mu$ then:*

$$\mathbf{E}(Z_n) = \mu^n$$

and moreover, $\mu^{-n}Z_n$ is a martingale.

Proof. (By induction). The base case is clear, as $Z_1 \sim X$. Now suppose it holds for n. To see the induction step, suppose that $\mathbf{E}(Z_{n-1}) = \mu^{n-1}$ and use the recurence:

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_{n,i}$$

We compute:

$$\begin{aligned} \mathbf{E}(Z_n) &= \mathbf{E}\left(\sum_{i=1}^{Z_{n-1}} X_{n,i}\right) \\ &= \sum_{k=0}^{\infty} \mathbf{E}\left(\left(\sum_{i=1}^{Z_{n-1}} X_{n,i}\right) \middle| Z_{n-1} = k\right) \mathbf{P}(Z_{n-1} = k) \\ &= \sum_{k=0}^{\infty} k\mu \mathbf{P}(Z_{n-1} = k) \\ &= \mu\left(\sum_{k=0}^{\infty} k\mathbf{P}(Z_{n-1} = k)\right) \\ &= \mu \mathbf{E}(Z_{n-1}) \\ &= \mu^n \end{aligned}$$

Finally, to see that $\mu^{-n}Z_n$ is a martingale, first notice that the recurence $Z_n = \sum_{i=1}^{Z_{n-1}} X_{n,i}$ shows that $\mathbf{E}\left(Z_n|Z_{n-1},\ldots Z_1\right) = \mathbf{E}\left(Z_n|Z_{n-1}\right)$ and also:

$$\begin{aligned}\mathbf{E}\left(Z_n|Z_{n-1}\right) &= \mathbf{E}\left(\sum_{i=1}^{Z_{n-1}} X_{n,i}|Z_{n-1}\right)\\ &= \mu Z_{n-1}\end{aligned}$$

So it follows that$\mathbf{E}\left(\mu^{-n}Z_n|\mu^{-(n-1)}Z_{n-1}\right) = \mu^{-n}\mu Z_{n-1} = \mu^{-(n-1)}Z_{n-1}$ so it is a martingale. □

Remark 7. By the martingale convergence theorem, we know that $\mu^{-n}Z_n \to_{a.s} W$ for some random variable W.

Remark 8. Notice that in the non-trivial case that has $\mathbf{E}(X) = 1$ we have that $Z_n \to 0$ in probability, but $\mathbf{E}(Z_n) = 1$ for every n. This is a bit weird!

Remark 9. By summing, this shows $\mathbf{E}(T) = \frac{1}{1-\mu}$.

1.1 Random-Walk perspective to the Branching Process

Here is an entirely different way to think about the branching process. After the branching process happens, we will count the number of people by the following exploring algorithm. We start by labeling the first dude as active. Now choose an active person. Then, in the next step, we label all his children as active and him as inactive. If there are no active people left we stop, if there are some active people left tepeat. Let S_n be the number of active people after n iterations. We have:

$$\begin{aligned}S_0 &= 1\\ S_k &= S_{k-1} + X_k - 1\\ &= X_1 + \ldots + X_k - (k-1)\\ T &= \min\{t : S_t = 0\}\\ &= \min\{t : X_1 + \ldots + X_k = k-1\}\end{aligned}$$

Let's find the probability distribution for this sequence. Let us create a probability distribution on $A \cup B$ where A, B are the following sets:

$$\begin{aligned}A &= \{(x_1, x_2, \ldots, x_n) : \inf\{k : x_1 + \ldots x_k = k-1\} = n\}\\ B &= \{(x_1, x_2, \ldots) : x_1 + \ldots x_k > k - 1 \forall k \in \mathbb{N}\}\end{aligned}$$

These are possible histories we may discover in our exploration algorithm, A corresponds to those histories which lead to extinction and B are those that never go extinct. We know that the probability of events in A given by:

$$\mathbf{P}\left(H = (x_1, x_2, \ldots x_n)\right) = \prod_{i=1}^{t} p_{x_i}$$

(Recall here that p_x is the distribution function for X, i.e. $\mathbf{P}(X = x) = p_x$). This is a probability distribution on A when we condition on the fact that the population eventually goes extinct. We will now condition on the fact that we want $x_1 + \ldots + x_n = n - 1$, and $x_1 + \ldots + x_k > k - 1$ for $k < n$ to get the probability distribution we want.

Define a new birth distribution called the *conjugate* birth distribution that we denote with a $\prime$:

$$p'_x = \eta^{x-1}p_x$$

. This is a new probability distribution on the integers, for normilization is guarnteed by the fact that $\eta = G_X(\eta)$:

$$\sum_k p'_k = \sum_k p_k \eta^{k-1} = \eta^{-1}G_X(\eta) = \eta^{-1}\eta = 1$$

Notice that the extinction probability for this process is 1, since:

$$G_{X'}(s) = \sum_{k=0}^{\infty} p_k \eta^{k-1}s^k = \eta\sum_{k=0}^{-1\infty} p_k(\eta s)^k = \eta^{-1}G_X(\eta s)$$

So since $\eta = G_X(\eta)$ is the smallest solution to that equation, the smallest solution to $\psi = G_{X'}(\psi)$ is:

$$\begin{aligned}\psi &= G_{X'}(\psi) = \eta^{-1}G_X(\eta\psi)\\ \eta\psi &= G_X(\eta\psi)\end{aligned}$$

Hence $\eta\psi = \eta$, so $\psi = 1$. Hence η' which is the smallest root must be 1 too.

Even more importantly, the branching process defined by X' is exactly the branching process define by X conditioned on the fact that extinction occurs:

$$\begin{aligned}
\mathbf{P}(H = (x_1, \ldots, x_n)| \text{ extinction}) &= \frac{\mathbf{P}(H = (x_1, \ldots, x_n) \cap \text{extinction})}{\mathbf{P}(\text{extinction})} \\
&= \frac{\mathbf{P}(H = (x_1, \ldots, x_n), n < \infty)}{\eta} \\
&= \eta^{-1} \prod_{i=1}^{n} p_{x_i}
\end{aligned}$$

Now, since $(x_1, \ldots, x_n) \in A$ (since $n < \infty$) we know that $\sum_{k=1}^{n} x_k = n - 1$ so rewrite this as:

$$\begin{aligned}
\mathbf{P}(H = (x_1, \ldots, x_n)| \text{ extinction}) &= \eta^{-1} \prod_{i=1}^{n} p_{x_i} \\
&= \eta^{-1} \prod_{i=1}^{n} p'_{x_i} \eta^{1-x_i} \\
&= \eta^{\sum x_i + n - 1} \prod_{i=1}^{n} p'_{x_i} \\
&= \prod_{i=1}^{n} p'_{x_i} \\
&= \mathbf{P}'(H = (x_1, \ldots, x_n))
\end{aligned}$$

Now we will return to the formula:

$$\begin{aligned}
\mathbf{P}\{T = n\} &= \frac{1}{n} \mathbf{P}\{X_1 + X_2 + X_3 + \ldots + X_n = n - 1\} \\
T &= \inf\{k : X_1 + X_2 + X_3 + \ldots + X_k = k - 1\}
\end{aligned}$$

This can be proven using the hitting time theorem below are rewriting $Y_i = X_i - 1$ and starting at $x_0 = 1$.

Theorem 10. *Let $\{Y_i\}$ be i.i.d steps for a random walk starting at $x_0 \geq 0$, $S_t = x_0 + \sum_{i=0}^{t} Y_i$. Suppose that:*

$$\mathbf{P}(Y \geq -1) = 1$$

Then the hitting time T_0 =inf $\{n : S_n = 0\}$ satisfies:

$$\mathbf{P}_{x_0}(T_0 = n) = \frac{x_0}{n} \mathbf{P}_{x_0}(S_n = 0)$$

($\mathbf{P}_{x_0}$ here makes the dependence of the measure on the initial state $S_0 = x_0$ explicit, which will be needed in the proof). This is a crazy theorem!

Proof. (We prove it for every possible $x_0 \geq 0$ by induction on $n \geq 1$)

Base case ($n = 1$): If $x_0 = 0$ we have $T_0 = 0$, so both sides are 0 above. If $x_0 = 1$ then both sides are equal to $\mathbf{P}(Y_1 = 0)$. If $x_0 \geq 2$, since $Y_{-1} \geq -1$, both sides are zero again.

Induction step ($n \geq 2$ and assume holds for $n - 1$): As in the base case, if $x_0 = 0$ both sides are zero, so assume $x_0 \geq 1$. Write:

$$\begin{aligned}
\mathbf{P}_{x_0}(T_0 = n) &= \sum_{s=-1}^{\infty} \mathbf{P}_{x_0}(T_0 = n | Y_1 = s) \mathbf{P}(Y_1 = s) \\
&= \sum_{s=-1}^{\infty} \mathbf{P}_{x_0+s}(T_0 = n - 1) \mathbf{P}(Y_1 = s) \\
&= \sum_{s=-1}^{\infty} \frac{x_0 + s}{n - 1} \mathbf{P}_{x_0+s}(S_{n-1} = 0) \mathbf{P}(Y_1 = s) \text{ by induction hyp.}
\end{aligned}$$

Now use $\mathbf{P}_{x_0+s}(S_{n-1}=0) = \mathbf{P}_{x_0}(S_n = 0|Y_1 = s)$ to get:

$$\begin{aligned}
(n-1)\mathbf{P}_{x_0}(T_0 = n) &= \sum_{s=-1}^{\infty}(x_0+s)\mathbf{P}_{x_0}(S_n = 0|Y_1 = s)\,\mathbf{P}(Y_1 = s) \\
&= \sum_{s=-1}^{\infty}(x_0+s)\mathbf{P}_{x_0}(S_n = 0 \cap Y_1 = s) \\
&= x_0\mathbf{P}(S_n = 0) + \sum_{s=-1}^{\infty} s\mathbf{E}_{x_0}\left(\mathbb{1}_{\{Y_1=s\}}\mathbb{1}_{\{S_n=0\}}\right) \\
&= x_0\mathbf{P}(S_n = 0) + \mathbf{E}_{x_0}\left(Y_1\mathbb{1}_{\{S_n=0\}}\right)
\end{aligned}$$

Now since $Y_i's$ are i.i.d we can write:

$$\begin{aligned}
\mathbf{E}_{x_0}\left(Y_1\mathbb{1}_{\{S_n=0\}}\right) &= \frac{1}{n}\sum_{k=1}^{n}\mathbf{E}_{x_0}\left(Y_k\mathbb{1}_{\{S_n=0\}}\right) \\
&= \frac{1}{n}\mathbf{E}_{x_0}\left(\left(\sum_{k=1}^{n}Y_k\right)\mathbb{1}_{\{S_n=0\}}\right) \\
&= \frac{1}{n}\mathbf{E}_{x_0}\left((S_n - x_0)\,\mathbb{1}_{\{S_n=0\}}\right) \\
&= \frac{-x_0}{n}\mathbf{P}_{x_0}(S_n = 0)
\end{aligned}$$

So finally then:

$$\begin{aligned}
\mathbf{P}_{x_0}(T_0 = n) &= \frac{1}{n-1}\left(x_0\mathbf{P}_{x_0}(S_n = 0) + \frac{-x_0}{n}\mathbf{P}_{x_0}(S_n = 0)\right) \\
&= \frac{x_0}{n-1}\left(1 - \frac{1}{n}\right)\mathbf{P}_{x_0}(S_n = 0) \\
&= \frac{x_0}{n}\mathbf{P}_{x_0}(S_n = 0)
\end{aligned}$$

Which completes the induction step.
I skipped the next couple sections. □

2 Configuration Model

The configuration model is a way to create a random graph with a prescribed degree sequence. Suppose we have a complex network of size n whose degrees are labeled with the set $[n] = \{1, 2, \ldots n\}$ and the vertex i has degree v_i. How can we create a random graph with this property?

We will create a random *multigraph* that has the above property. When we condition on the fact that the graph is simple, the resulting random graph will be uniform over all the possible graphs with the prescribed the degree sequence.

2.1 Introduction

Fix n and let $\mathbf{d} = (d_i)_{i\in[n]}$. This will be the vector that stores the distribution of the degrees. We will suppose $d_j \geq 0$, because otherwise we could just as well delete node j from our graph. Let $l_n = \sum_{i\in[n]} d_i$ be the sum of the degrees. Notice that $l_n = 2|E|$ must be even because of the handshake lemma. Hence we can only hope to create a simple random graph when l_n is even.

Definition 11. Fix n, $\mathbf{d} = (d_i)_{i\in[n]}$, and $l_n = \sum_{i\in[n]} d_i$. We will create a random multigraph with degree sequence $\mathbf{d}$ as follows. At every vertex j, draw d_j half-edges leaving vertex j but not yet connected to anything. Number the half-edges $1..l_n$. Choose a pair of half-edges uniformly at random and connect them to make an edge between the vertices they are from. Repeat this until all the $l_n$half edges are connected to something. (Notice they could leave some vertices connected to themselves, making a loop, or two vertices could be connected by multiple edges.) We call the resulting random graph $CM_n(\mathbf{d})$.

An equivalent way to think about this is to label the half-edges as $i \in l_n$ and then their partner is given by the bijection $\sigma(i) : [l_n] \to [l_n]$ which has the property that $\sigma(\sigma(i)) = i$ so that the configuration is stored in this bijection. Such a bijection is called a *configuration*.

Example 12. The number of possible configurations is $(2l_n - 1)!! = (2l_n - 1)(2l_n - 3)\ldots 3 \cdot 1$ since there are $2l_n - 1$ choices available for $\sigma(1)$, and then $2l_n - 3$ choices for the next index which hasnt been chosen yet etc.

To go from the configuration to the configuration model graph, identify the set $\{1, \ldots d_1\}$ with vertex 1, $\{d_1 + 1, \ldots d_1 + d_2\}$ to vertext 2 etc.

Proposition 13. *Let G be a multi graph on $[n]$ and let $x_{ij} = \#$of vertices from i to j. Let $d_i = x_{ii} + \sum_{j\in[n]} x_{ij}$ be the degree of vertex i. Then:*

$$\mathbf{P}\left(CM_n(\mathbf{d}) = G\right) = \frac{1}{(l_n-1)!!}\frac{\prod_{i\in[n]} d_i!}{\prod_{i\in[n]} 2^{x_i}\prod_{1\le i\le j\le n} x_{ij}!}$$

Proof. Since there are $(l_n-1)!!$ configurations from which to choose from, (and they are uniformly selected), we have:

$$\mathbf{P}\left(CM_n(\mathbf{d}) = G\right) = \frac{1}{(l_n-1)!!}N(G)$$

Where $N(G)$ is the number of configurations that give the graph. One can see this precisly $\frac{\prod_{i\in[n]} d_i!}{\prod_{i\in[n]} 2^{x_{ii}}\prod_{1\le i\le j\le n} x_{ij}!}$. There are $d_1!$ ways to permute the half-edges in $\{1\ldots d_1\}$ which are identified with the vertex 1, all of which will lead to the same outcome for the graph. Similarly for $d_i!$, so we see that $\prod_{i\in[n]} d_i!$ counts the relabeling possible of the half-edges without chaging which vertex they correspond too. However, this counts some configurations more than once. Every loop leads to a double counting. (say the loop is from half edges a, b which both correspond to vertex 1. We count it once in $\{...a, ...b, ...\}$ and again in the rearrangment where a and b have switched spots $\{..., b, ..., a...\}$) Whenever there is more than one edge from i to j we will double count by a factor of $x_{ij}!$, for if we permute the x_{12} edges connecting 1 to 2 by the same permutation in $\{1, \ldots d_1\}$ and in $\{d_1+1, \ldots d_1+d_2\}$ then we are back to the same configuration. So we divide by $\prod_{i\in[n]} 2^{x_{ii}}\prod_{1\le i\le j\le n} x_{ij}!$ to cancel these out. □

Corollary 14. If G is simple then $\mathbf{P}\left(CM_n(\mathbf{d}) = G\right) = const_{\mathbf{d}}$ so in particular, if $CM_n(\mathbf{d})$ is simple, then it is a uniform selection from the set of all simple graphs with degree sequence $d_1, \ldots d_n$.

Definition 15. Let V be a vertex chosen uniformly at random from $[n]$ and let $D_n =$ degree of vertex V a random variable. Notice that the cumulative distribution function for $D_n$is:

$$\begin{aligned} F_n(x) &= \mathbf{P}\left(D_n \le x\right) \\ &= \frac{1}{n}\sum_{j\in[n]} 1_{\{d_j\le x\}} \end{aligned}$$

Notice that the only jumps for this CDF happen when x an integer.

Definition 16. The following three regularity conditions will be useful:

I) Weak convergence of the distribution of vertex weights:

$$D_n \Rightarrow D$$

For some random variable D distributed on the integers. (Equivalently, $F_n(x)\to F(x)$ for some F and every $x\in\mathbb{R}$. This is equivalent here because D_n is distributed on the integers, so for bounded continuous functions f we have that $\mathbf{E}(D_n f) = \sum_{x\in N} f(x)\mathbf{P}(D_n = x)$, so choosing bump functions with $f(x) = \begin{cases} 0 & x\ne k \\ 1 & x = k\end{cases}$ and the definition of weak convergence lets us see $\mathbf{P}(D_n = x)\to\mathbf{P}(D = x)$ for every x, and similarly for F_n.

II) We will want $\mathbf{E}\left(D_n\right)\to\mathbf{E}(D)$ and $\mathbf{E}(D_n^2)\to\mathbf{E}(D^2)$. This is some sort of uniform integrability condition, see the notes on Billingsly.

Remark 17. Let $n_k = \sum 1_{\{d_i=k\}}$ be the number of vertices that have degree k. Up to relabeling of vertices, this uniquely determines $\mathbf{d}$ (just write n_1 copies of "1", n_2 copies of "2" etc.) Notice that $\mathbf{P}(D_n = k) = \frac{n_k}{n}$, so the weak convergence condition above is that:

$$\frac{n_k}{n}\to\mathbf{P}(D = k)$$

One canocical way to get this model is to choose D up front and then generate the $n_k's$ by:

$$n_k = \lceil n\mathbf{P}(D\le k\rceil - \lceil n\mathbf{P}(D\le(k-1)\rceil$$

Notice here that $F_n(k) = \frac{1}{n}\lceil nF(k)\rceil\ge F(k)$ which is enough to give us uniform integrablity for $D_n's$ so we can see $\mathbf{E}(D_n)\to\mathbf{E}(D)$.

Remark 18. Another way to get this is to choose the sequence $d_1, \ldots d_n$ to be an iid sequence of random variables $d_i\sim D$. Then $n_k = \sum 1_{\{d_i=k\}}$ is a sum of many iid Bernouli random variables, and $\frac{n_k}{n}\to p_k$ is the Law of large numbers, so we have the weak convergence. Moreover, the random variable D provides a bound that gives uniform integrability, so $\mathbf{E}(D_n)\to\mathbf{E}(D)$ too. Note that there is a problem here that l_n might not be even, but we can always just add an extra half edge in this case that will not change things too much.