1. Theory

Exercise 1. Prove the *limit comparison test:* Suppose that a_n, b_n are positive sequences so that $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

2. Problems

Problem 2. (Sept '03 #3) Determine which of these converge: a) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ b) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ c) $\sum_{n=2}^{\infty} \frac{\cos(\log(n))}{n\log(n)}$

(Note: we did part a and part b in class already. We had some ideas for part c, but I don't think we had a solid proof. The difficulty here is that $\cos(\log(n))$ is not behaving in a monotone way, so we can't apply the Cauchy Condensation Test or the Integral test directly)

Problem 3. (Jan '05 #1) Let (x_n) be a sequence converging to a. Show that $\left(x_1 x_2^2 \cdots x_n^n\right)^{1/n^2} \to \sqrt{a}$.

Problem 4. (Sept '09 #1) Suppose $\sum_{n=1}^{\infty} x_n$ is absolutely convergent. Show that, for any increasing sequence (a_n) of positive numbers with $a_n \to \infty$, we have:

$$\lim_{N\to\infty} \frac{1}{a_N} \sum_{n=1}^{N} a_n x_n = 0$$

Problem 5. (Berkeley Problem Su85) Let k be fixed and $A_1 \geq A_2 \geq \ldots \geq A_k \geq 0$. Compute:

$$\lim_{n\to\infty} (A_1^n + A_2^n \ldots + A_k^n)^{\frac{1}{n}}$$

Problem 6. (Berkeley Problem Sp96) Compute:

$$\lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{\frac{1}{n}}$$

Problem 7. (Berkeley Problem Sp89) Let $a_1, a_2, \ldots$ be a positive sequence so that $\sum_{n=1}^{\infty} a_n < \infty$. Prove that there exists a sequence of positive numbers c_n so that:

$$\lim_{n\to\infty} c_n = \infty \text{ and } \sum_{n=1}^{\infty} c_n a_n < \infty$$