Intermediate Disorder Limits for Multi-layer Random Polymers

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Abstract

The **intermediate disorder** regime is a scaling limit for random polymers where the strength of the random environment is scaled to zero as the system size grows to infinity. We prove intermediate disorder limits for models with **multiple non-intersecting** polymer paths, both for purely discrete polymers [2] and semi-discrete polymers [3]. In both cases, the limiting object is the continuum polymer related to the multi-layer extension of the stochastic heat equation introduced by O'Connell and Warren in [4]. The semi-discrete convergence resolves an outstanding conjecture by Corwin and Hammond [1] and gives a characterization of the KPZ line ensemble.

References

[1] I. Corwin and A. Hammond. KPZ line ensemble. *Probab. Theo. Rel. Fields*, 2015.

[2] I. Corwin and M. Nica. Intermediate disorder directed polymers and the multi-layer extension of the stochastic heat equation. *arXiv math/1603.08168*, 2016.

[3] M. Nica. Intermediate disorder limits for multi-layer semi-discrete directed polymers. *arXiv math/1609.00298*, 2016.

[4] N. O'Connell and J. Warren. A multi-layer extension of the stochastic heat equation. *Commun. Math. Phys.*, 2015.

Continuum Polymer: $t \in (0,\infty)$, $z \in \mathbb{R}$

Let $\vec{D}^{(t,z)}(\cdot)$ denote d *non-intersecting Brownian bridges* with endpoints $\vec{D}^{(t,z)}(0) = (0,0,\dots,0)$, $\vec{D}^{(t,z)}(t) = (z,z,\dots,z)$. This is also known as *Brownian watermelon*; a sample path is shown in Figure 1.

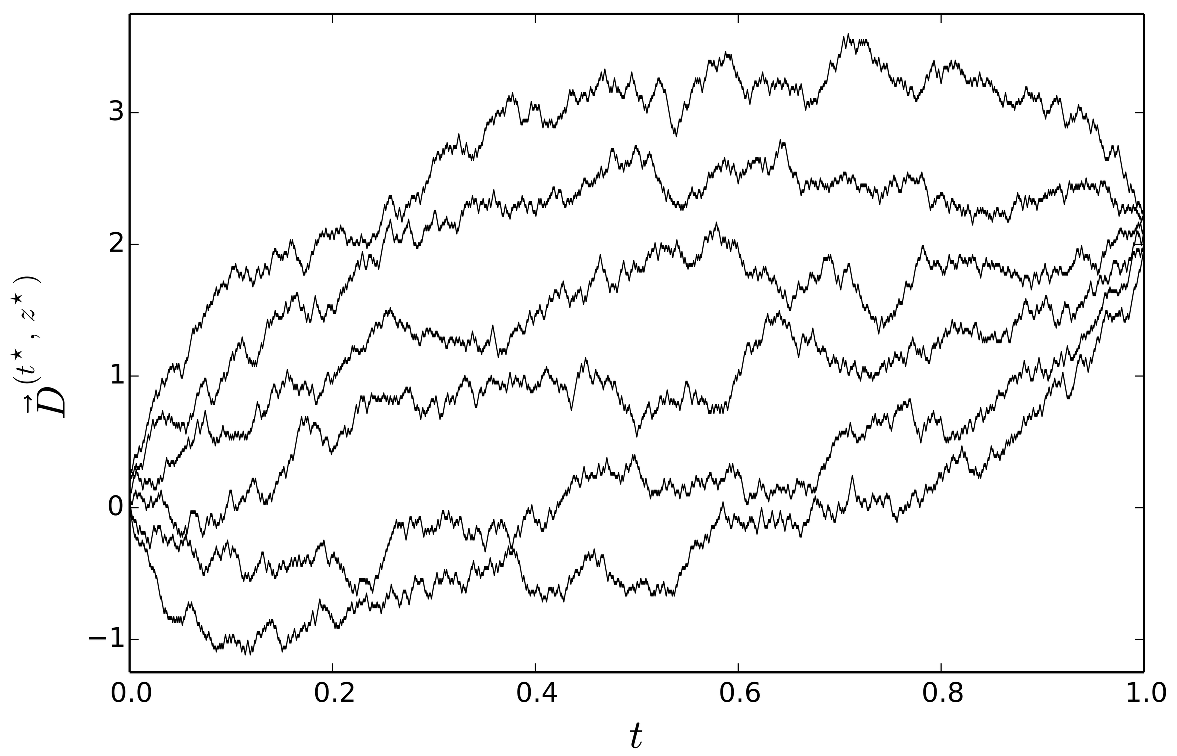

Figure 1: Sample path for non-intersecting Brownian bridges $\vec{D}^{(t,z)}$ when $d = 6$, $t = 1.0$, $z = 2.0$.

Let $\xi(t,z)$ denote a *Gaussian white noise environment*. The polymer partition function at inverse temperature $\beta > 0$, first defined in [4], is given by a *Wick exponential*:

$$\mathcal{Z}^{\beta}_d(t,z) = \rho(t,z)^d \mathbb{E}\left[: \exp : \left\{ \beta \sum_{j=1}^{d} \int_0^t \xi\big(s, D^{(t,z)}_j(s)\big) \mathrm{d}s \right\} \right],$$

where $\mathbb{E}$ is expectation over $\vec{D}^{(t,z)}$ and $\rho(t,z) := (2\pi t)^{-\frac{1}{2}} \exp\left(-z^2/2t\right)$. $\mathcal{Z}^{\beta}_d$ is the limiting object in the convergence Theorems from [2] and [3]. Formally $\mathcal{Z}^{\beta}_d$ is a *Wiener chaos series*:

$$\mathcal{Z}^{\beta}_d(t,z) := \rho(t,z)^d \sum_{k=0}^{\infty} \beta^k \iint_{\substack{\vec{t} \in \Delta_k(0,t) \\ \vec{z} \in \mathbb{R}^k}} \psi^{(t,z)}_k\big((t_1,z_1),\dots,(t_k,z_k)\big) \xi^{\otimes k}(\mathrm{d}\vec{t}, \mathrm{d}\vec{z}),$$

where $\Delta_k(s,s') = \left\{\vec{t} : s < t_1 < \cdots < t_k < s'\right\} \subset (0,\infty)^k$, and the functions $\psi^{(t,z)}_k$ are the *k-point correlation functions* for $\vec{D}^{(t,z)}$.

Discrete Polymer: $n \in \mathbb{N}$, $x \in \mathbb{Z}$

Let $\vec{X}^{(n,x)}(\cdot)$ denote d *non-intersecting random walks* with endpoints $\vec{X}^{(n,x)}(0) = (0,2,\dots,2d-2)$, $\vec{X}^{(n,x)}(n) = (x, x+2, \dots, x+2d-2)$. These are in bijection with *non-intersecting up/right lattice paths*.

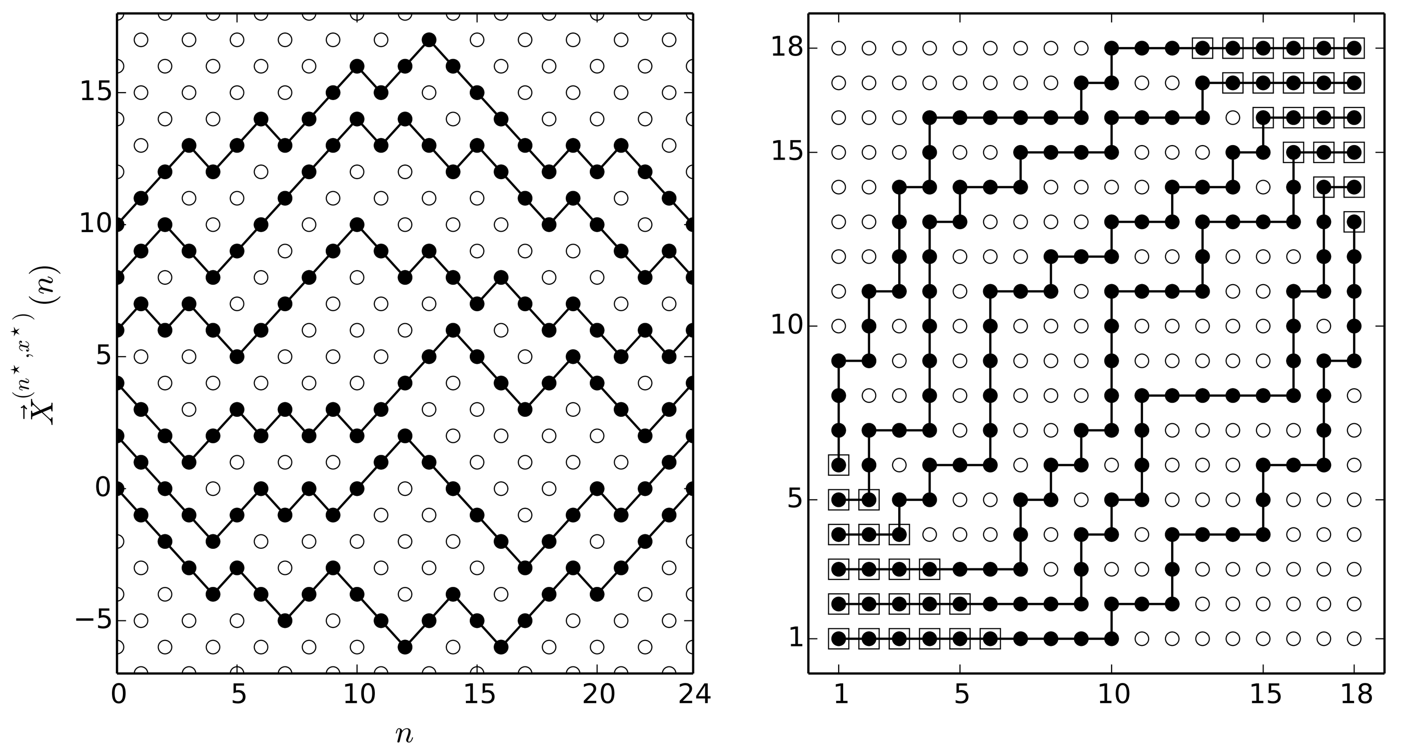

Figure 2: (Left) A sample path of $\vec{X}^{(n,x)}$ when $d = 6$, $n = 24$, $x = 0$. (Right) Rotation by 45° transforms these into up-right lattice paths.

Let $\omega = \{\omega(n,x)\}_{n\in\mathbb{N}, x\in\mathbb{Z}}$ be an *environment of i.i.d. random variables*. The polymer partition function at inverse temperature $\beta > 0$ is:

$$Z^{\beta}_d(n,x) := \mathbb{E}\left[\exp\left(\beta \sum_{j=1}^{d} \sum_{i=1}^{n-1} \omega\left(i, X^{(n,x)}_j(i)\right)\right)\right],$$

where $\mathbb{E}$ denotes expectation over $\vec{X}^{(n,x)}$.

Semi-discrete Polymer: $\tau \in (0,\infty)$, $x \in \mathbb{N}$

Let $\vec{S}^{(\tau,x)}(\cdot)$ denote d *non-intersecting semi-discrete paths* with endpoints $\vec{S}^{(\tau,x)}(0) = (1,2,\dots,d)$, $\vec{S}^{(\tau,x)}(\tau) = (x+1, x+2, \dots, x+d)$. A sample path is shown in Figure 3.

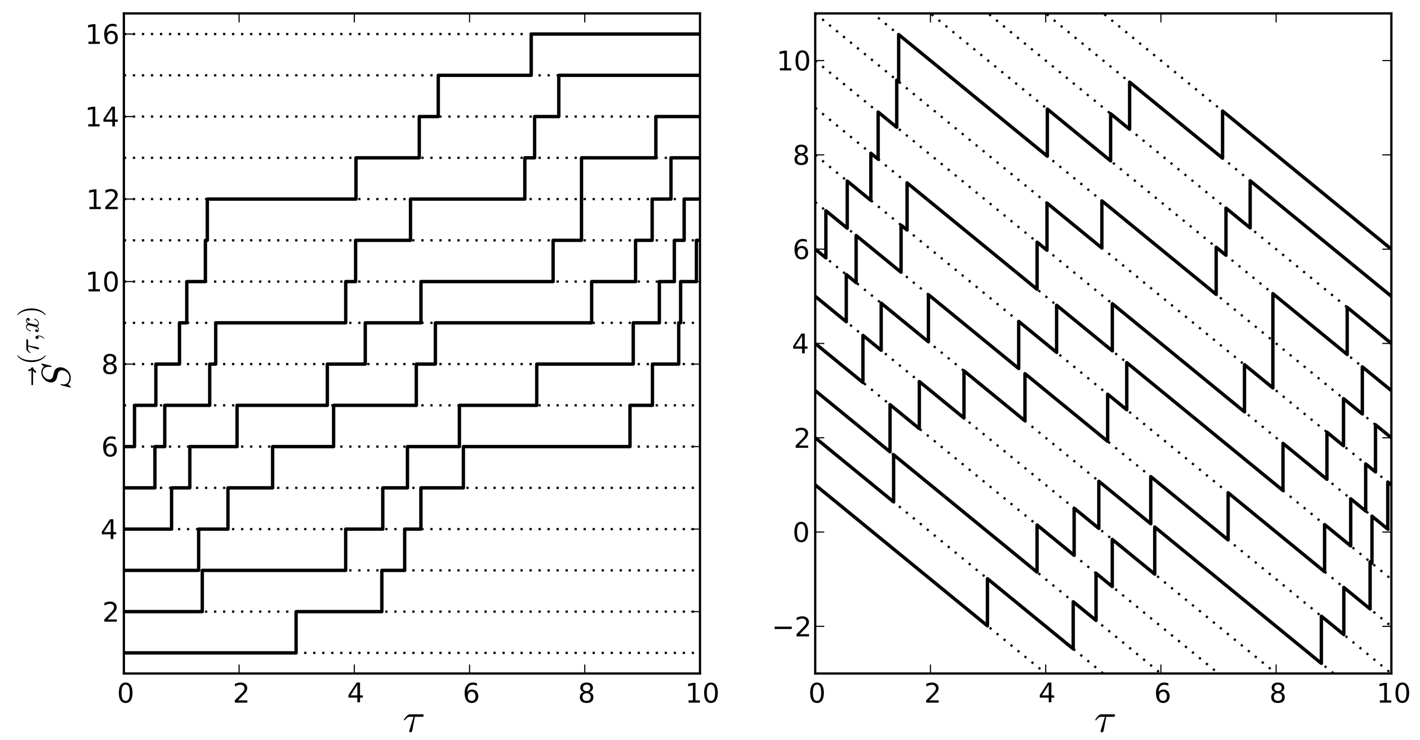

Figure 3: (Left) A sample path of $\vec{S}^{(\tau,x)}$ when $d = 6$, $\tau = 10$, $x = 10$. (Right) Compensating these processes turns these into "sawtooth walks"

Let $B = \{B_x(\tau)\}_{\tau\in(0,\infty), x\in\mathbb{N}}$ be an *environment of i.i.d. Brownian motions*. The polymer partition function at inverse temperature $\beta > 0$ is:

$$Z^{\beta,\mathrm{sd}}_d(\tau,x) := \mathbb{E}\left[\exp\left(\beta \sum_{j=1}^{d} \int_0^{\tau} \mathrm{d}B_{S^{(\tau,x)}_j(s)}(s)\right)\right],$$

where $\mathbb{E}$ denotes expectation over $\vec{S}^{(\tau,x)}$.

Convergence of Discrete Polymer

Theorem [2] (joint with Ivan Corwin). Suppose $\omega(0,0)$ is *any* random variable with mean zero, unit variance, and finite exponential moments $\Lambda(\beta) := \log\left(\mathcal{E}\left(e^{\beta\omega(0,0)}\right)\right)$. For any $\beta > 0$, set $\beta_N := N^{-\frac{1}{4}}\beta$. As $N \to \infty$, one has:

$$Z^{\beta_N}_d\left(\lfloor Nt \rfloor, \lfloor \sqrt{N} z \rfloor\right) \exp\Big(-dNt\Lambda(\beta_N)\Big) \Rightarrow \frac{\mathcal{Z}^{\sqrt{2}\beta}_d(t,z)}{\rho(t,z)^d}.$$

Convergence of Semi-discrete Polymer

Theorem [3]. For any $\beta > 0$, set $\beta_N := N^{-\frac{1}{4}}\beta$. As $N \to \infty$, one has:

$$Z^{\beta_N,\mathrm{sd}}_d\left(Nt, \lfloor Nt + \sqrt{N} z \rfloor\right) \exp\left(-\frac{1}{2} dNt\beta_N^2\right) \Rightarrow \frac{\mathcal{Z}^{\beta}_d(t,z)}{\rho(t,z)^d}.$$

Moreover, it is possible to couple the partition functions so that the convergence is in L^p for any $p \geq 1$.