LIST OF ORAL EXAM TOPICS

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Abstract. Some topics to know for the general topics: PDE, Functional Analysis, Complex Analysis, Real Analysis and Probability Theory.

1. PDE

1.1. **General First Order Stuff and Burger's Equation.** Basic Facts:

- Characteristic surfaces - consitencey/strip conditions
- Classification of elliptic/hyperbolic/parabolic for quasilinear second order equations
- (Hint: parametrize any curve with respect to s, then differentiate everything you can w.r.t. s)
- Burger's equation $u_y + uu_x = 0$:
  - implicit solution $u = h(x - yu)$,
  - finite time blow up: (look at char curves or take ∂_x of implict solution
  - conservation form $\frac{\partial R(u(x,y))}{\partial y} + \frac{\partial S(u(x,y))}{\partial x} = 0$ and $S'(u) = uR'(u)$
- Monge cone, nonlinear equations!

1.2. **Laplace Equation.** Basic Facts:

- Green's Identity: (apply divergence theorem to $u\nabla v$)
- Energy Identity: (Green's idenity when $u = v$ $\int |\nabla u|^2 \,\mathrm{d}x + \int u\Delta u \mathrm{d}x = \int_{\partial\Omega} u \frac{\mathrm{d}u}{\mathrm{d}n} \mathrm{d}S$
- Fundemental solution (only spherically symmetric solution, $\psi(r)$ satisfies $\psi''(r) + \frac{n-1}{r}\psi'(r) = r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\psi\right) = 0$..has $\Delta\psi = \delta_0$ in the weak sense)
- Integral representation (involves both u and $\frac{\mathrm{d}u}{\mathrm{d}\vec{n}}$) (implies Cauchy problem in general has no solution)
- Gauss's law of the arithmetic mean $u(\xi) = \frac{1}{\omega_n \rho^{n-1}} \int_{|x-\xi|=\rho} u(x)\mathrm{d}S_x$, can also be done with $\phi(r)$ =average over r
- Solution of Poisson equation (integration against the kernal)
- Poisson Integral formula - other Green's functions - method of images!
- Connection with Analytic Functions from Complex - Conjugate harmonic stuff
- Maximum Principle / Subharmonic and superharmonic functions
- Fourier series - why is $\sin(n\theta)r^n$ harmonic in the disk?

Advanced Facts:

- Basics of Perrons method - finding a maximum
- Hilbert space methods - the space H^1, weak solutions, Lax-Milgram, Dirchlet principle for energy minimization
- Boundedness and completeness of families - Arzela Ascoli, Relich Compactness
- Eigenvalues of the Laplacian
- Harnack Inequality (for compact sets V exists C so that $\inf_V u \le C \sup_V u$. Can be seen on balls by the averaging property, then on compact sets stitch togher many balls)

Typical Questions:

- Smoothness? (Answer: *Real analytic in interior of the domain.* Comes from the integral formula $u(\xi) = -\int_{\partial\Omega}\Big(K(x,\xi)\frac{\mathrm{d}u}{\mathrm{d}\vec{n}} - u(x)\mathrm{d}K$ so can see its inf-diff right away. In fact, since $K(x,\xi)$ is not just C^∞ but can actually be thought of as being complex-analytic away from $r = 0$. Hence it has a convergent taylor expansion. So for $x \in \Omega$, $u(\xi)$ is the integral of some analytic stuff. Another way to see this is because u has to be the real part of a real analytic function.)
- Uniqueness? (Answer using Energy: From the energy, we see that we must have uniqueness for the Neumann and Dirichlet boundary values. Answer using Maximum Principle: Since it satisfies a maximum principle, the Dirichlet problem solution must be unique.)
- Existence? (Answer using complex variables: In $\mathbb{R}^2$ Dirichlet problem always has a solution and Neumann problem has a sol'n as long as $\int \frac{\mathrm{d}u}{\mathrm{d}n}\mathrm{d}x = 0$. By Riemann Mapping Theorem, it suffices to solve on the open disc, and there is a Poisson integral formula for Dirichlet problem there. Neumann condition can be reduced to Dirichlet conditions for the conjugate harmonic function by $\frac{\mathrm{d}u}{\mathrm{d}n} = \frac{\mathrm{d}v}{\mathrm{d}s}$ and $\frac{\mathrm{d}v}{\mathrm{d}n} = -\frac{\mathrm{d}u}{\mathrm{d}s}$. Answer in higher dimensions: Can use Perron's method (finding the maximum) assuming that the boundary is regular enough (must have "barrier functions" which are subharmonic that are zero at one point and negative at all others) For example, an interior cusp is not allowed) Answer: Weak solutions can be found to exist by using Sobolev space methods/calculus of variation methods)
- Solution by Fourier Series? ***
- Solution on a half place? Existence/Uniqueness? ***

1.3. **Heat Equation.** Basic Facts

- Fundamental Solution (gaussian ... motivation by scaling)
- Existence/Uniqueness in unbounded domains assuming $u(x,t) \leq Me^{a|x|^2}$
- Counterxample: non-unqiness if its allowed to blow up
- Duhamel's Principle - non-homegenous problem
- Mean value formula - heat ball
- Maximum principle - by heat ball or by checking maxima
- Regularity
- Uniqueness of solutions - energy methods or maximum principle.
- Backward heat equation: energy is log-convex can't go to zero
- Fourier series solutions

1.4. **Wave Equation.** Facts to know:

- D'Alemberts solution to 1d
- How to deal with Boundary data - reflections
- Method of spherical means - Euler-Poisson-Darboux equation
- When $n = 3$ look at rU to get back the 1D Wave equation (when $n = 2k + 1$ is odd, do $\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}U\right)$)
- When $n = 2$, use method of descent to get the solution.
- Energy methods: uniqueness and domain of dependnece

1.5. **Cauchy-Kowalevski Theorem.**

- On non-characterstic surfaces, PDEs with real analytic data and real analytic coeffcients have a solution.
- "counter"example for heat equation $u_t - u_{xx} = 0$ with $u(x,0) = \frac{1}{1+x^2}$

2. Functional Analysis

2.1. **Hilbert Spaces.** Facts to know:

- Cauchy Schwarz Ineq
- Polarization Idenity - $\|f \pm g\|^2 = \|f\|^2 \pm 2\text{Re}\langle f, g\rangle + \|g\|^2$
- Parallelogram law: $\|f+g\|^2 + \|f-g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right)$
- Pythagorean Theorem - used to prove convergence
- Minimizing the distance to closed convex sets - use parralelogram law and convexity to prove that a minimizing sequence is Cauchy
- Projections onto closed subspaces. The minimizer is orthogogonal!
- Riesz Representation theorem - The element you are looking for is the projection onto $\ker(L)^\perp$
- Bases - maximal orthonormal sets
- Bessel Inequality - $\sum_{n=1}^{\infty} |\langle h, e_n\rangle| \leq \|h\|^2$ for any orthonormal set e_n (follows by Pythagoras)
- $\{e_n\}$ a basis $\iff$ if $h \perp e_n$ for all n then $h = 0$ $\iff$ span $\{e_n\} = \mathcal{H} \iff h = \sum \langle h, e_n\rangle e_n$ $\iff \langle g, h\rangle = \sum \langle g, e_n\rangle \langle e_n, h\rangle$ $\iff \|h\|^2 = \sum |\langle h, e_n\rangle|^2$ (Pareseval's Identity)
- Any two basis have the same dimension (get an injection by $e_n \to \{f_n : \langle e_n, f_n\rangle \neq 0\}$ which is countable)
- $\mathcal{H}$ is separable if and only if it has a countable basis. ($\Longrightarrow$: any onb gives you a collection of disjoint balls. Since $\mathcal{H}$separable, get an injection $\{$*onb balls*$\}$ $\to$seperable set shows that the number of balls is countable $\Longleftarrow$: do something like the rational span of the basis)
- Two hilbert spaces are isomorphic iff they have the same dimension (show all are isomorphic to $\mathcal{H} \to \ell^2(\mathcal{E})$ by $h \to \hat{h}$ with $\hat{h}(e) = \langle h, e\rangle$. Itsw an isometry by Parseval.)

2.2. **The Hanh-Banach Theorem.** Facts to know:

- The Hanh-Banach theorem
- Applications:
  - Seperation theorems
  - Banach limits

2.3. **Baire Category Theorem and Consequences.**

- Open mapping theorem:
- If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $A : \mathcal{X} \to \mathcal{Y}$ is a continuous linear surjection, then $A(G)$ is open in $\mathcal{Y}$ whenever G is open in $\mathcal{X}$.(Hint: find r so that i) $0 \in \text{int cl} A(B_r(0))$ (uses Baire category $Y = \cup_n A(B_n(0))$) then prove $\text{cl}\left(A\left(B_{r/2}(0)\right)\right) \subset A(B_r(0))$. Hence $A(B_r(0))$ has non-empty interior, which means A is open.)

2.4. **Locally Convex Spaces.**

- Gauge functional
- Seperation by hyperplanes

2.5. **Weak and Weak* topologies.** Facts to know:

- Statement of Alaoglu's theorem
- Concrete: Classical spaces
  - $(c_0)^* = \ell_1$, $\ell_1^* = \ell_\infty$, ℓ_∞^* =crazy space
  - Riesz Representation Theorem: For X compact: $C(X)^*$ =signed measures on X
  - $(L^p)^* = L^q$, $(L^1)^* = L^\infty$ (Hint: define $\nu(A) = \phi(1_A)$ and use Radon-Nikodym)
  - If X is not compact,$C_0(X)$ is NOT the dual space of anything (Hint: show that its unit ball has no extreme points (you can always add and subtract a bit from any $C_0(X)$ function in the bit before it $\to 0$. Averaging these give you back the function). By Alaoglu, the closed unit ball is weak* compact, so by Krein-Millman it should have extreme points, so this is a contradiction)

2.6. **The Spectrum of Operators.** Facts to know:

- Compact operators; show that integration against a continuous kernal is compact (hint: show they are equicontinous by continuity of K and then use Arzela-Ascoli.) or integration against an L^2 kernal (hint: write everything out in a basis)
- The spectral theorem for compact self adjoint operators on a Hilbert space
  - Spectrum is compact non-empty set
  - Dividing up the spectrum: point, continuous, residual, approximate point
  - First resolvent formula
  - Spectral radius formula $\rho(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$
  - Norm formula for self adjoint operators $\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$
  - (See section VI.5 Compact Operators, pg 198 in Reed-Simon and the appendix D.5 in Evans)
  - If λin the residual spectrum for T, then λ in the point spectrum for T^*
  - If λ in the point spectrium for T, then λ is either in the point spectrium or residual spectrum for T^*
- The spectral theory for compact operators on a Hilbert space (i.e. the Fredholm alternative)
- Concrete: Calculation of the spectrum for the left and right shift operators

Advanced:

- Riesz functional calculus ***
- The spectral theorem for self-adjoint (non-compact) operators on a Hilbert space ***

Typical questions:

- What is the Fredholm alternative? (Answer: For a compact operator on a Hilbert space $K : \mathcal{H} \to \mathcal{H}$ the spectrum of K is only eigenvalues and $\{0\}$. This is called the Fredholm alternative because either:
  - $\lambda u - Ku = f$ has a solution for every f
  - $\lambda u - Ku = 0$ has non-trivial solutions. In the latter case, $\lambda u - Ku = f$ has solutions only if $f \in \ker(\lambda I - K^*)^\perp$.
- State the spectral theorem? (Answer: The spectral theorem for compact self-adjoint operators is that $\exists \lambda_n$ and finite dimensional projection operators P_n which are orthogonal $P_nP_m = P_mP_n = 0$ for $n \neq m$ that diagonalize T, namely $T = \sum_{n=1}^\infty \lambda_n P_n$)
- Define spectrum, point spectrum, residual spectrum and continuous spectrum? (Answer: The spectrum is the set $\{\lambda\}$ where the operator $\{\lambda I - T\}$ is not invertible. The point spectrum is the set of eigenvalues $\ker(\lambda I - T) \neq (0)$; $\lambda I - T$ is not infective. The continuous spectrum is where $\ker(\lambda I - T) = \{0\}$ and $\operatorname{ran}(\lambda I - T) \neq \mathcal{H}$ but we do have $\overline{\operatorname{ran}(\lambda I - T)} = \mathcal{H}$; $(\lambda I - T)^{-1}$ exists as a densely defined map on $\mathcal{H}$ but it is not bounded. The residual spectrum is where $\overline{\operatorname{ran}(\lambda I - T)} \subsetneq \mathcal{H}$; $\lambda I - T$ is not surjective.
- Calculate the spectrum of the unilateral left/right shift operators on $\ell^1(\mathbb{N})$? (Answer: The leftshift operator eats the first components; the right shift operator introduces a 0. So $LR = I$ but RL is the operator that replaces the first entry with 0. Any $|\lambda| < 1$ is in the point spectrum for L is easy. Any $|\lambda| \le 1$ is in the approximate point spectrum for R witnessed by the vectors $x_n = \left(\frac{1}{n}, \frac{\lambda^{-1}}{n}, \ldots\right)$ for a window of size n. These also work to show $|\lambda| = 1$ is in the approximate point spectrum for L. To further classify (contiunuous or residual?) use the fact that L and R are adjoints to each other $\langle \phi, Lx\rangle = \langle R\phi, x\rangle$ where $\phi \in \ell^\infty(\mathbb{N})$ and $x \in \ell^1(\mathbb{N})$. Claim: $|\lambda| = 1$ is continuous spectrum for L. Pf: Suppose by contradiction its residual. Then $\overline{\operatorname{ran}(L - \lambda)}$ is a proper subset, so can find ℓ^∞ so that $\langle \phi, (\lambda - L)\, x\rangle = 0$ for all $x \in \ell^1$. But then $\langle (\lambda - R)\phi, x\rangle = 0$ for all x shows that $(\lambda - R)x = 0$ but this is impossible since λ is not in the point spectrum of R! Claim: $|\lambda| \le 1$ is residual spectrum for R. (Rmk: its obvious at $\lambda = 0$) Pf: Show that a ball around $c = \left(1, \lambda, \lambda^2, \ldots\right)$ is missed.
- Calculate the spectrum of the bilateral left/right shift operators on $\ell^2(\mathbb{Z})$,$\ell^\infty(\mathbb{Z})$? (Answer: Since the operators are invertible, the spectrum must be a subset of the unit disc $\{\lambda : |\lambda| = 1\}$. (Check that $\sigma(A) \subset \{\lambda : |\lambda| \le \|A\|\}$ and when A invertable $\{\lambda : |\lambda| < 1\} \subset \rho(A)$ by Banach algebra methods). On ℓ^∞ we see that all of the unit disc are eigenvalues. On ℓ^2 the whole unit circle are approximate eigenvalues (use $x_n = \left(\ldots, \frac{\lambda^n}{n}, \ldots\right)$ for a window of size n.)
- Application to the Dirhlet problem and to the eigenvalues of the Laplacian ***

3. Real

3.1. **Stuff.**

- Compactness - Arzela Ascoli type results
  - A-A: use sub-sub-sequences diagonal trick to get it converging at rationals. Then prove that the resulting subsequnce is uniformly cauchy.
- Cantor set

3.2. **Measure and Integration.**

- Definition of measure/integration
- Monotone convergengence theorem/Fatou/LDCT
- Radon Nikodym
- Lusin/Egoroff

3.3. L^p **spaces.**

- Holder's Inequality; Minkowski Inequality; Completness
- $(L^p)^* = L^q$, $\left(L^1\right)^* = L^\infty$ (Hint: define $\nu(A) = \phi(1_A)$ and use Radon-Nikodym)
- Inclusions: on finite probability spaces, $L^{p_2} \subset L^{p_1}$ for $p_2 > p_1$. On $\mathbb{R}$, have $L^{p_2} \not\subset L^{p_1}$ for $p_2 > p_1$ by choosing a function like $\frac{1}{x}$ for $x > 1$ that decays at ∞. Also $L^{p_1} \not\subset L^{p_2}$ by choosing a function like $x^{-1/2}$ for $0 < x < 1$ that blows up at 0. By stitiching these together you can get functions that are in a single L^p.

3.4. **Theory of differentiation.**

- Vitali Cover: a collection of sets so that every point is covered by an arbitarily small interval
- Vitali Cover lemma: Given an $\epsilon > 0$ can find a finite disjoint subset of the the Vitali cover that misses at most size ϵ.
- Increasing Functions are differentiable a.e.: Check that the "four different derivatives, limsup/liminf/left/right are all equal by looking at the set $D_- < u < v < D^+$, u, v rational to constuct a Vitali cover.
- By Fatou: $\int_a^b f'(x)\mathrm{d}x \leq f(b) - f(a)$
- Corr: Bounded variation functions are differentiable a.e.
- f integrable gives $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x)$ (Hint: First check for f bounded. Notice the integral is bounded variation, so the derivative exists a.e. USe the lemma $\int_a^x g(t)\mathrm{d}t = 0 \; \forall a \implies g \equiv 0$ to see the two are actually equal)$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t - f(x) \equiv 0$.)
- Absolute Continuity: $\forall \epsilon > 0, \exists \delta > 0$ so that $\sum_{i=1}^n |x_i' - x_i| < \delta \implies \sum_{i=1}^n |f(x_i') - f(x_i)| < \epsilon$.
- Abs continuity $\implies$ bounded variation. (Hint: choose $\epsilon = 1$ and let Let $K = 1 + \frac{b-a}{\delta}$ be the largest number of intervals of size δ one could cram into a, b.)
- Every abs. cont. function can be written as $f(x) = f(a) + \int_a^x f'(x)\mathrm{d}x$

3.5. **Fourier Series/Transform.**

- Fejer Kernal: $\sigma_n(f) \xrightarrow{\|\cdot\|_\infty} f$ for $f \in C(\mathbb{T})$, $\sigma_n(f) \xrightarrow{L^p} f$ for $f \in L^p(\mathbb{T})$
- Corr: Trig functions are dense, e^{int} is a basis for L^2, Uniqueness for fourier sereis, Riemann-Lebesgue lemma
- Fourier series of continuous functions may diverge - Banach-Steinhouse proof
- On Holder contunous functions space, Fourier series do converge: $S_n(f) \xrightarrow{\|\cdot\|_\infty} f$ for $f \in C^\alpha(\mathbb{T})$
- Decay at infinity, general relation between smoothness/behaviour of f vs smoothness/behaviour of $\hat{f}$

4. Complex

4.1. **Basics.** Facts to know:

- Cauchy Goursat
- Cauchy Integral formula
- Louiville's Theorem and extended Louiville's theorem
- Power series expansions and Laurent expansions
- Morera's theorem
- Braches of log
- Maximum Modulus principle, open mapping theorem
- Argument principle
- Local k-to-1 mapping
- Branch of log
  - - in any simply connected region not containing 0 (Hint: define $f(z) = \int_{z_0}^z \frac{\mathrm{d}\zeta}{\zeta} + \log(z_0)$)
  - Branch of $\log f(z)$ in simply connected domain where f is analytic and unequal to 0 (Set $\log f(z) = \int_{z_0}^z \frac{f'(\zeta)}{f(\zeta)}\mathrm{d}\zeta + \log f(z_0)$)
- Restrictions on entire holomorphic functions:
  - f poly growth $\implies f$ a polynomial (Extended Louivilles)
  - $|f(z)| \to \infty$ as $|z| \to \infty \implies f$ a polynomial (Look at $\prod(z - z_i)\frac{1}{f(z)}$ in a region where $z_i$are the roots inside R where R s.t. $|f(z)| > 1$ for $|z| > R$)
  - $f(z)$ is entire with a holomorphic inverse then $f(z) = az + b$ (Show that $f(z) \to \infty$ as $z \to \infty$)
  - FLTs are the ONLY maps that map unit disk to unit disk. (Apply Schwarz lemma)
  - $h(z)$ entire and has no zeros, then $h(z) = e^{g(z)}$ for some entire function $g(z)$ (Define a branch of logarithm in the image $h(\mathbb{C})$
  - $f(z)$ has non-essential singularity at $\infty \implies f(z)$ is a polynomial.

4.2. **Contour Integrals.** Facts to know:

- Residue theorem
- Tricks for computing residues - derivative formula, $\frac{f(z)}{g'(z)}$ formula
- Tricks for estimating functions on contours - Jordan's lemma, R^{α}trick
- Choosing a contour
  - Keyhold contour (good for logs)
  - Change the curve (a contour with no poles!)
  - Half circle (good for Jordan's lemma / R^{α} trick)

4.3. **Conformal Maps.** Facts to know:

- Fractional Linear Transformations
  - Uniqueness of FLTs by the Schwarz lemma
- Joukowsky Transform
- Riemann Mapping Theorem - connection to Dirichlet problem

4.4. **Presribed Zeros and Prescribed Poles.** Facts to know:

- Prescribed zero's problem - Weierstrass factorization theorem
- Functions of finite order $|f(z)| \leq \exp\left(|z|^k\right) \implies f(z) = Q(z)e^{P(z)}$ for polynomials P, Q or f has infinitely many zeros (Hint: For Louville's theorem, it is enough that $\mathrm{Re} f(z) \leq A\,|z|^n$ by using the Poisson integral formula for f in terms of only $\mathrm{Re} f$
- Hadamard Factorization theorem: if f is a function of finite order p then $f(z) = z^{,m} e^{p(z)} \prod \left(1 - \frac{z}{z_n}\right) e^{E_n\left(\frac{z}{z_n}\right)}$ where $p(z)$ is a polynomial and z_n are the zeros of f .
- Prescribed poles - Mitag Lefler Expansions ***

Typical Questions:

- What is the product formula for $\sin \pi z$?
- What is the product formula for $\cos \pi z$?

5. Probability

5.1. **Brownian Motion.**

- Reflection principle, $M_t \sim |B_t|$
- Brownian scaling and reflection stuff
  - $B_t \sim B_1 - B_{1-t}$
  - $B_t \sim \frac{1}{\sqrt{c}} B_{ct}$
  - $B_t \sim t B_{1/t} \backslash$
- Construction by diadic normal random variables
- Holder α Continuity for $\alpha < \frac{1}{2}$: Kolomogorov continuity theorem $\mathbf{E}\left(|X_t - X_s|^{2k}\right) = [(2k-1)!!]\,(t-s)^k$ and set up a Borel cantelli looking at $A_n = \mathbf{P}\left(\cup_{k=1}^{2^n} \left|X_{k+1/2^n} - X_{k/2^n}\right| > (2^{\gamma})^{-n}\right)$, control with a Cheb estimate and the above moments.
- Not Holder continuous for $\alpha > \frac{1}{2}$ (and consequently not differenitable) look at the probability that ΔB is $\leq M\Delta t$ on three consecutive diadic intervals, and then control these probabilities to Borel Cantelli and see it can't happen.
- Non-differentiablility at fixed times t: use $\limsup \frac{B_n}{\sqrt{n}} = \infty$ and then do an inversion $\limsup_{n\to\infty} \frac{B(\frac{1}{n}) - B(0)}{\frac{1}{n}} \geq \limsup \frac{\tilde{B_n}}{\sqrt{n}} = \infty$
- Arcsine law for time the maximum is achieved:

$$
\begin{aligned}
\mathbf{P}(m \leq s) &= \mathbf{P}\left(\max_{0\leq t\leq s} B(t) > \max_{s\leq t\leq 1} B(t)\right) \\
&= \mathbf{P}\left(\max_{0\leq t\leq s} B(t) - B(s) < \max_{s\leq t\leq 1} B(t) - B(s)\right) \\
&= \mathbf{P}\left(\max_{0\leq t\leq 1} \sqrt{s} B_1(t) < \max_{0\leq t\leq 1} B_2(t)\sqrt{1-s}\right) \\
&= \mathbf{P}\left(\sqrt{s}\,|Z_1| < |Z_2|\,\sqrt{1-s}\right) \\
&= \mathbf{P}\left(\frac{|Z_2|}{\sqrt{Z_1^2 + Z_2^2}} < \sqrt{s}\right) \\
&= \frac{2}{\pi} \arcsin(\sqrt{s}) \\
&\overset{denstiy}{\sim} \frac{1}{\pi\sqrt{x(1-x)}}
\end{aligned}
$$

- There is also an arcsine law for the location of the last zero (Follows by correspondence that $M(t) - B(t)$ behaives like a reflected Brownian motion) and the amount of time spent above the axis)

5.2. **Basic Stuff.**

- Both Borel Cantelli lemmas
- Pi Lambda theorem (Hint: a lambda system is closed under increasing unions and under differences)
- Kolmogorov 0-1 law (Hint: -show T is independent of any $\sigma(X_1,\ldots,X_n)$ Hence T is independent of the algebra $\cup_n\sigma(X_1,\ldots,X_n)$ (things that depend on finetly many). Let $\mathcal{A}$ be the set of all things independent of T. Check $\mathcal{A}$ is a monotone class. Since $\cup_n\sigma(X_1,\ldots,X_n)\subset\mathcal{A}$, hence $\sigma(\cup_n\sigma(X_1,\ldots,X_n))\subset\mathcal{A}$ by $T\in\sigma(\cup_n\sigma(X_1,\ldots,X_n))$, so T is independent of itself.

5.3. **Types of convergence.**

- Convergence in probability $\iff$ every subsequence has a further subsequence that converges a.s
- Convergence in probability $\iff$ Cauchy in probabiliy (Hint: use a.s. subsequence characterization)
- Levy's theorem: If X_n independent then X_n converges in distribution iff converges in $\mathbf{P}$ iff converges a.s. (Hint: look at char functions which are a product and are convergign. Use this to see $X_n$are Cauchy in probability. Then if the estimates are done right, can improve from convergence in $\mathbf{P}$ to a.s. by seeing that it is summable)
- Assuming convergence in $\mathbf{P}$. Then L^1 convergence iff it is U.I.

5.4. **Weak convergence stuff.**

- Skohorod Representation Theorem (watch out it only works for separable spaces!) ... Take the inverse
- Helley Selection Theorem (Vague convergence)...its a sub-sequence argument
- Def'n of tightness. Tight iff every subsequential limit is a probability measure.
- $X_n\Rightarrow X$ iff every subsequence has a furhter subsequence converging to X.
- Levy-Prohorov Metric: $\mathbf{P}_\mu(A)\le\mathbf{P}_\nu(A^\epsilon)$ and vice versa for every set A.
- Converging together lemma: $X_n\Rightarrow X$ and $Y_n\Rightarrow c$ implies $X_n+Y_n\Rightarrow X_n+c$ (Hint: show limsup for closed sets)

5.5. **Fourier/Char function stuff.**

- $Y=X-\tilde{X}$ (independent copy) has $\varphi_Y(t)=|\varphi_X(t)|^2$
- $Y=CX$ where C is a ±1 coinflip has $\varphi_Y(t)=\mathrm{Re}(\varphi_X(t))$
- Inversion formula $\int_{-T}^{T}\varphi_{[a,b]}(t)\varphi_X(t)\mathrm{d}t\to\mu(a,b)+\frac{1}{2}\mu\{a\}+\frac{1}{2}\mu\{b\}$...turns into $\int\frac{\sin(\theta x)}{x}\mathrm{d}x$
- Atoms correspond to behaviour at ∞
  - Riemann-Lebesgue lemma, $\varphi_\mu\to0$ if μ has a density (hint: show it works for continuous functions which are desne in L^1)
  - $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\varphi(t)|^2\,\mathrm{d}t=\mathbf{P}(X-\tilde{X}=0)=\sum_x\mu(\{x\})^2$
- Continuity Theorem: $\varphi_n\to\varphi$ pointiwse and φ continuous at 0 means $\mu_n\Rightarrow\mu$ weakly. (Hint: show that if its tight we are ok by sub-subsequences and use $\frac{1}{u}\int_{-u}^{u}(1-\varphi(t))\,\mathrm{d}t\to0$ to get tightness)
- Moments: if X has moments, then φ is differentiable. If n even $\varphi^{(n)}(0)$ exists then $\mathbf{E}[X^n]=\varphi^{(n)}(0)$
- Estimate $\left|\mathbf{E}\left(e^{itX}\right)-\left(1+it\mathbf{E}(X)-\frac{t^2}{2}\mathbf{E}(X^2)\right)\right|\le\mathbf{E}\left(\min\left(\frac{|tX|^3}{3!},2\frac{|tX|^2}{2!}\right)\right)$
- Ordinary CLT and Lindeberg Feller CLT

5.6. **Law of Large Numbers, Law of the Iterated Log.**

5.7. **Stable Laws/Infently Divisible Distributions.**

- A distribution is stable if the sum S_n is of the same type as X. (Type means $\exists a,b$ so $aX+b\overset{d}{=}Y$). It is strictly stable if you can do it without having to add a constant term. $S_n\overset{d}{=}c_nX$
- Can prove with the above that c_n must be $n^{1/\alpha}$ for some $0<\alpha<2$
- For $0<\alpha<2$ $\varphi(t)=\exp(-|t|^\alpha)$ is a char function. For α outside this it is not (Hint: check moments)
- These are special because they are the only distributions with non-empty domains of attraction.
- Levy-Kinchine: (helps to think of it as a Levy Process - A process with stationary independent increments) Every infinetly divisible distrubution is the sum of:
  - Drift
  - Gaaussian part (Brownian motion)
  - Compound Poisson process - large hits that happen "rarely"
  - Compound Poission process - small hits that happne "a lot"
  - Thm is $\mathbf{E}\left[e^{i\theta X_t}\right]=\exp\left(ait\theta-\frac{1}{2}\sigma^2t\theta^2+t\int_{\mathbb{R}\setminus\{0\}}\left(e^{i\theta x}-1-i\theta x1_{|x|<1}\right)\Pi(\mathrm{d}x)\right)$ where Π must have $\int_{\mathbb{R}\setminus\{0\}}\min\{x^2,1\}\Pi(\mathrm{d}x)<\infty$. $\Pi(x,x+\mathrm{d}x)$ represents the intensity of jumps of size in the range $(x,x+\mathrm{d}x)$...we have to compensate the small hits that happen often (by putting in a $i\theta x$) to stop the thing from blowing up right away.
- A compound Poisson has $\mathbf{E}(\exp(itZ))=\exp(-\lambda(1-\varphi(t)))$ where φ is the c.f. of the jumps.

5.8. **Martingales.**

5.9. **Markov Chains, Recurrence.**

5.10. **Ergodic Theorem.**

- Definitions: stationary sequence, ergodic
- Birkhoff's Ergodic Theorem $\frac{1}{n}\sum_{m=0}^{n-1}X(\varphi^m\omega)\to\mathbf{E}(X|\mathcal{I})$ a.s. and in L^1 ("Pf": Maximal ergoidic lemma)