KPZ limit of non-intersecting random walks

Mihai Nica

September 20, 2019

The goal of these notes is to prove the $n \to \infty$ scaling limit for the top walker of n non-intersecting Brownian motions at time $t = n$:

$$X_1^{(n)}(n) \approx 2n + Z_{GUE} n^{1/3},$$

where $\vec{X}^{(n)}(t) = \vec{X}(t)$ (we drop the n when this is clear) is the vector $(X_1(t), \dots, X_n(t))$ consisting of n Brownian motions which are conditioned to not intersect for all time $\{X_1(s) > \dots > X_n(s) : 0 < s < \infty\}$ and which start from $\vec{X}(0) = \vec{0} = (0, 0, \dots, 0)$. Here, Z_{GUE} is a random variable which has the GUE Tracy Widom distribution, characterized by:

$$\mathbf{P}\left(Z_{GUE} < s\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_a^{\infty} \dots \int_a^{\infty} \det_{i,j=1}^{k} \left[K_{Ai}(x_i, x_j)\right] \mathrm{d}x_1 \dots \mathrm{d}x_k$$

$$K_{Ai}(x, y) := \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x - y}$$

$$Ai(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} e^{-\frac{ik^3}{3}} \mathrm{d}k = \mathcal{F}\left(e^{-i\frac{k^3}{3}}\right)$$

There major steps in the proof are listed below. (The notation in each step is defined in detail in the respective section; note also we abuse notation to use $\mathbf{P}$ to mean both probability and probability density depending on context.)

- Step 1: Define non-intersecting walks and prove the Karlin-McGregor Theorem for the probability of the non-intersecting event:
$$\mathbf{P}\left[\left\{\vec{B}(t) = \vec{y}\right\} \cap NI_t \,\middle|\, \vec{B}_i(0) = \vec{x}\right] = \det_{i,j=1}^{n} \left[\varphi_t(x_i - y_j)\right]$$
- Step 2: Get the density function for non-intersecting Brownian motions as a limit of Brownian watermelons:
$$\mathbf{P}\left(\vec{X}(t) = \vec{x}\right) = \frac{1}{\prod_{i=0}^{n-1} i!} V\left(\frac{x_1}{\sqrt{t}}, \dots, \frac{x_n}{\sqrt{t}}\right)^2 \prod_{i=1}^{n} \varphi_t(x_i)$$
- Step 3: Use the theory of random point processes to write the distribution of the top particle $X_1(t)$ in terms of the k-point correlation function $\rho_k(x_1, \dots, x_k)$ of the point process $\{X_1(t), \dots, X_n(t)\}$:
$$\mathbf{P}\left(X_1(t) < a\right) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \int_a^{\infty} \int \dots \int_a^{\infty} \rho_k(x_1, \dots x_k) \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_k$$
- Step 4: Use the theory of "bi-orthogonal ensembles" to get a determinantal formula for the k-point correlation function in terms of the Hermite polynomials $H_k(x, t)$
$$\rho_k(x_1, \dots, x_k) = \det_{i,j=1}^{k} \left[K_n(x_i, x_j)\right]$$
$$K_{n,t}(x, y) := \sum_{i=0}^{n-1} \psi_i(x, t)\psi_i(y, t)$$
$$\psi_k(x, t) := \frac{H_k(x, t)\sqrt{\varphi_t(x)}}{\sqrt{t^k \cdot k!}}$$

- Step 5: Develop the theory of the Hermite polynomials to get identieis $H_k(x,t)$ to be able to write the kernel $K_n(x,y)$ in terms of ψ_n and its derivatives only:

$$K_n(x,y) = n\frac{\psi_n(x)\psi_n'(y) - \psi_n'(x)\psi_n(y)}{x-y} -$$

- Step 6: Write the Hermite polynomials as an countour integral and then analyze the contour integral to get the limit:

$$\lim_{n\to\infty} \psi_n(2n + un^{1/3}, n) = Ai(u)$$

- Step 7: Combine all the previous results (and check some techincal details) to get the desired limit theorem in the form:

$$\lim_{n\to\infty} \mathbf{P}\left(X_1^{(n)}(t) < 2n + un^{1/3}\right) = \mathbf{P}\left(Z_{GUE} < u\right)$$

0 Notation

Symbol	What is it
$\mathbf{P}$	We abuse notation and use this to mean either probability or probability density depending on context
n	Number of walkers
$\vec{Z}(t) = (Z_1(t), \ldots, Z_n(t))$	Generic random process of n independent walks
NI_t	Non-intersecting up to time t event $NI_t := \{Z_1(s) > Z_2(s) > \ldots > Z_n(s) : 0 < s < t\}$
$\vec{B}(t) = (B_1(t), \ldots B_n(t))$	(Ordinary) Brownian motions
$\vec{X}(t) = (X_1(t), \ldots, X_n(t))$	Non-intersecting Brownian motions $\vec{X}(\cdot) := \vec{B}(\cdot)$ conditioned on NI_∞. Since NI_∞ is a measure 0 event, we will instead take the equivalent definition $\vec{X}(\cdot) = \lim_{t^* \to \infty} \vec{W}(\cdot)$ where $\vec{W}$ is a Brownian watermelon
$t^*, \vec{0}^\epsilon, \vec{W}^\epsilon(t)$	ϵ-of-room Brownian watermelon on $t \in [0, t^*]$: $\vec{0}^\epsilon := (0, \epsilon, 2\epsilon, \ldots, (n-1)\epsilon)$ $\vec{W}^\epsilon(0) := \vec{0}^\epsilon$and $\vec{W}^\epsilon(t^*) := \vec{0}^\epsilon$ $\vec{W}(t), t \in [0, t^*]$ are indep. Brownian bridges conditioned not to intersect
$t^*, \vec{W}(t)$	Brownian watermelon on $t \in [0, t^*]$ $\vec{W}(0) = \vec{0}$ and $\vec{W}(t^*) = \vec{0}$ Defined to be the $\vec{W}(\cdot) = \lim_{\epsilon \to 0} \vec{W}^\epsilon(\cdot)$
$V(x_1, \ldots, x_n)$	Vandermonde Determinant $V(x_1, \ldots, x_n) = \prod_{i<j} (x_i - x_j)$
$\varphi_t(x)$	Gaussian density $\frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}x^2}$
S	A random point process in $\mathbb{R}$ consisting of random points
μ_S	A counting measure on $\mathbb{R}$ that counts points from S, $\mu_S(A) = \lvert S \cap A \rvert$.
$\rho_k(x_1, \ldots, x_k)$	$\rho_k : \mathbb{R}^k \to [0, \infty)$ is the k point correaltion function of the walks

1 Non-intersecting walks and the Karlin-McGregor Theorem

Definition 1. (Non-intersecting Brownian motions) Given a family of n independent Brownian motions $\vec{Z}(t) = (Z_1(t), \ldots, Z_n(t))$ started from some initial condition $Z_1(0) > \ldots > Z_n(0)$, define the non-crossing event:

$$NI_t := \{Z_1(s) > Z_2(s) > \ldots > Z_n(s) \forall 0 < s < t\}$$

Non-intersecting Brownian motions is the process one gets by conditioning $\vec{Z}(t)$ conditioned on the event NI_∞.

Remark 2. It is not obvious that the process $\vec{Z}(t)$ defined by this conditioning is a Markov process; however in many cases it turns out that it is!

Theorem 3. *[Lindstrom-Gessell-Viennot/Karlin-McGregor]*
Suppose $Z_1(\cdot), \ldots, Z_n(\cdot)$ are i.i.d. random processes on a state space Σ with transition probabilities:

$$\phi_t(x,y) := \mathbf{P}\left(Z(t) = y \,|Z(0) = x\right)$$

Let $\vec{x} = (x_1, \ldots, x_n)$ be a vector of starting points and $\vec{y} = (y_1, \ldots, y_n)$ be vector of end points. Suppose the random process has the following property:

$$\forall \textit{non-identity permutations } \sigma \in S_n, \textit{ there are no non-intersecting walks from } \vec{x} \textit{ to } \left(y_{\sigma(1)}, \ldots, y_{\sigma(n)}\right)$$

More precisely, we require $\forall \sigma \in S_n$, $\sigma \neq Id$ that:

$$\mathbf{P}\left(\left\{\vec{Z}(t) = \left(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}\right)\right\} \cap NI_t \,\middle|\, \vec{Z}(0) = \vec{x}\right) = 0$$

Then for any $\vec{x} \in \mathbb{Z}^n$ and $\vec{y} \in \mathbb{Z}^n$, we have:

$$\mathbf{P}\left(\left\{\vec{Z}(t) = \vec{y}\right\} \cap NI_t \,\middle|\, \vec{Z}(0) = \vec{x}\right) = \det\left[\phi_t(x_i, y_j)\right]_{i,j=1}^n$$

Proof. (We prove it in the case that $\vec{X}$ and t are discrete: this is known as the Lindstrom-Gessel-Viennot Lemma. The proof for processes in continuous time is called the Karlin-McGregor Theorem and is similar.)

The idea of the proof is to do the permutation expansion of the determinant. In this expansion, there is a factor of $\text{sgn}(\sigma) \in \{-1, +1\}$ which will conspire to exactly cancel out *all* the configurations which have at least one crossing, leaving only a non-zero contribution from non-intersecting configurations. Expand out:

$$\begin{aligned}
\det\left[\phi_t(x_i, y_j)\right]_{i,j=1}^n &= \sum_{\sigma \in S_n} \text{sgn}(\sigma)\phi_t(x_1, y_{\sigma(1)}) \cdot \ldots \cdot \phi_t(x_n, y_{\sigma(n)}) \\
&= \sum_{\sigma \in S_n} \text{sgn}(\sigma)\mathbf{P}\left(\bigcap_{i=1}^n \left\{X_i(t) = y_{\sigma(i)} \,|X_i(0) = x_i\right\}\right) \\
&= \sum_{\sigma \in S_n} \sum_{\vec{\pi} \in \{\text{Paths } \vec{x} \to \vec{y}_\sigma\}} \text{sgn}(\sigma)\mathbf{P}\left(\bigcap_{i=1}^n \left\{X_i(\cdot) = \pi_i(\cdot)\right\}\right)
\end{aligned}$$

where $\{\text{Paths } \vec{x} \to \vec{y}_\sigma\}$ is the set of all configurations of possible paths from $\vec{x} = (x_1, \ldots, x_n)$ to $\vec{y} = (y_{\sigma(1)}, \ldots, y_{\sigma(n)})$:

$$\{\text{Paths } \vec{x} \to \vec{y}_\sigma\} := \left\{\text{Paths } \pi_1(\cdot), \ldots, \pi_n(\cdot) \;:\; \pi_i(0) = x_i \text{ and } \pi_i(t) = y_{\sigma(i)}\right\}$$

define also the sets

$$\begin{aligned}
\{\text{Non-Intersecting Paths } \vec{x} \to \vec{y}_\sigma\} &:= \{\text{Paths } \vec{x} \to \vec{y}_\sigma\} \cap \{\pi_1(s) > \pi_2(s) > \ldots > \pi_n(s) \forall 0 < s < t\} \\
\{\text{Intersecting Paths } \vec{x} \to \vec{y}_\sigma\} &:= \{\text{Paths } \vec{x} \to \vec{y}_\sigma\} \cap \{\exists 0 < s < t \;\pi_i(s) = \pi_j(s) \;i \neq j\}
\end{aligned}$$

Now divide the sum into two parts:

$$\begin{aligned}
\det\left[\phi_t(x_i, y_j)\right]_{i,j=1}^n = &\sum_{\sigma \in S_n} \sum_{\vec{\pi} \in \{\text{Non-intersecting paths } \vec{x} \to \vec{y}_\sigma\}} \text{sgn}(\sigma)\mathbf{P}\left(\bigcap_{i=1}^n \left\{X_i(\cdot) = \pi_i(\cdot)\right\}\right) \\
&+ \sum_{\sigma \in S_n} \sum_{\vec{\pi} \in \{\text{Intersecting paths } \vec{x} \to \vec{y}_\sigma\}} \text{sgn}(\sigma)\mathbf{P}\left(\bigcap_{i=1}^n \left\{X_i(\cdot) = \pi_i(\cdot)\right\}\right) \qquad (1)
\end{aligned}$$

The first term is exactly $\mathbf{P}\left(\left\{\vec{X}(t) = \vec{y}\right\} \cap NI_t \,\middle|\, \vec{X}(0) = \vec{x}\right)$ by the condition about $\sigma = Id$ being the only path with no intersections. It remains to show that the second term is 0 to get the desired equality. Consider as follows:

Given a $(\sigma, \vec{\pi})$ with $\sigma \in S_n, \vec{\pi} \in \{\text{Intersecting paths } \vec{x} \to \vec{y}_\sigma\}$ define its *doppleganger* $\left(\tilde{\sigma}, \tilde{\vec{\pi}}\right)$ which has $\tilde{\sigma} \in S_n, \vec{\pi} \in \{\text{Intersecting paths } \vec{x} \to \vec{y}_{\tilde{\sigma}}\}$as follows:

1. Find the *first* intersection in $\vec{\pi}$, i.e. the first time $s^* \in (0,t)$ and indices i,j such that $\pi_i(s^*) = \pi_j(s^*)$ with i,j,s^* minimal

2. Define $\tilde{\vec{\pi}}$by *swapping paths* $\pi_i(s)$ and $\pi_j(s)$ for $s > s^*$ and leaving the paths unchanged beforehand.

3. Define $\tilde{\sigma}$ by *interchanging* i and j in the permutation σ, i.e. $\tilde{\sigma} = (i\ j)\sigma$.

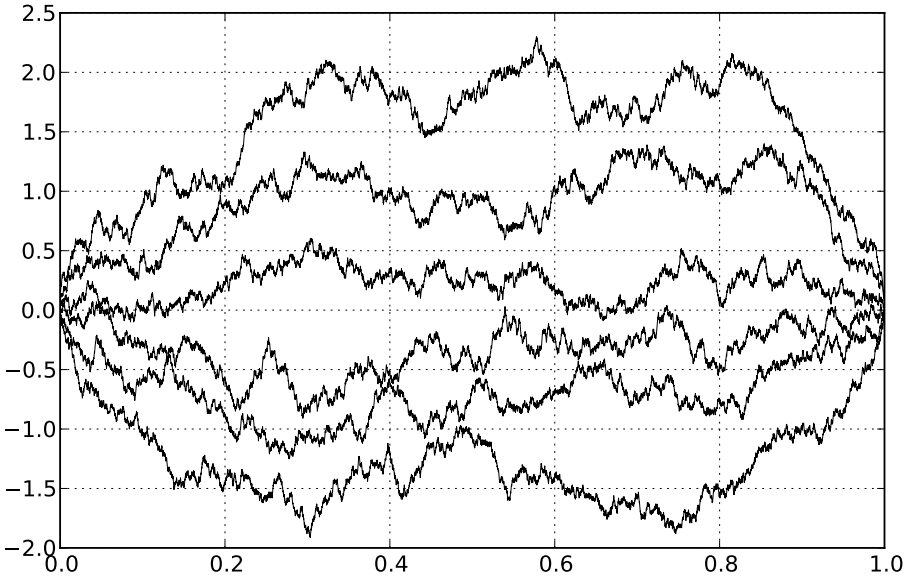
Note that since $\mathrm{sgn}(\sigma) = -\mathrm{sgn}(\tilde{\sigma})$, the corresponding terms appearing in (1) have:

$$\mathrm{sgn}(\sigma)\mathbf{P}\left(\bigcap_{i=1}^{n}\{X_i(\cdot) = \pi_i(\cdot)\}\right) = -\mathrm{sgn}(\tilde{\sigma})\mathbf{P}\left(\bigcap_{i=1}^{n}\{X_i(\cdot) = \tilde{\pi}_i(\cdot)\}\right)$$

i.e. the contribution from every $(\sigma, \vec{\pi})$ and its doppleganger $(\tilde{\sigma}, \tilde{\vec{\pi}})$ are exactly opposite in sign! This means we can pair every term in the sum over $\sigma \in S_n, \vec{\pi} \in \{\text{Intersecting paths } \vec{x} \to \vec{y}_\sigma\}$ with its doppleganger to see that the whole sum(1) must be zero! □

2 Limits of Brownian watermelons

Definition 4. A Brownian watermelon on the time interval $t \in [0,t^*]$ consists of n non-intersecting Brownian bridges $\vec{W}(t) = (W_1(t), \ldots, W_n(t))$. It is called a watermelon because it looks like the stripes of the eponymous fruit:



We would like to define the true Brownian watermelon which has starting and ending point equal to $\vec{W}(0) = \vec{0}$ and $\vec{W}(t^*) = \vec{0}$. However, from this initial condition, the non-intersecting probability is 0, which causes an issue with the conditioning. To get around this, we will first define the ϵ-of-room Brownian watermelon which starts and ends from $\vec{W}^\epsilon(0) := \vec{0^\epsilon}$ and $\vec{W}^\epsilon(0) := \vec{0^\epsilon}$, where $\vec{0^\epsilon} := ((n-1)\epsilon, (n-2)\epsilon, \ldots, \epsilon, 0)$. We will then define $\vec{W}$ by declaring it to be the $\epsilon \to 0$ limit of $\vec{W}^\epsilon$. We use this definition to obtain the following formula:

Proposition 5. *The distribution $\vec{W}(t)$ at some time $t \in [0,t^*]$ obeys:*

$$\mathbf{P}\left(\vec{W}(t) = \vec{x}\right) = c_{n,t,t^*} V(x_1, \ldots, x_n)^2 \prod_{i=1}^{n} \exp\left(-\frac{x^2}{2t}\right) \exp\left(-\frac{x^2}{2(t^*-t)}\right)$$

where

$$c_{n,t,t^*} := \frac{1}{\sqrt{2\pi}^n} \left(\frac{t^*}{t(t^*-t)}\right)^{\frac{1}{2}n^2} \frac{1}{V(0,1,\ldots,n-1)}$$

and $V(x_1, \ldots, x_n)$ *is the* Vandermonde Determinant*:*

$$V(x_1, \ldots, x_n) := \det_{i,j=1}^{n} \left[x_i^{j-1} \right]_{i,j=1}^{n} = \prod_{i<j} (x_i - x_j)$$

The proof uses the following lemma:

Lemma 6. *Let* $\vec{B}(\cdot)$ *be a Brownian motion. For any fixed* $\vec{x}$ *and* t*, we have in the limit* $\epsilon \to 0$*:*

$$\mathbf{P}\left(\left\{ \vec{B}(t) = \vec{x} \right\} \cap NI_t \,\middle|\, \vec{B}(0) = \vec{0}^{\epsilon} \right) = \left(\frac{\epsilon}{t} \right)^{\binom{n}{2}} V(x_1, \ldots, x_n) \prod_{i=1}^{n} \varphi_t(x_i) + o(\epsilon^{\binom{n}{2}})$$

Proof. By the Karlin-Macgregor theorem, the probability is given by a determinant involving $\varphi_t(x-y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$:

$$\mathbf{P}\left(\left\{ \vec{X}(t) = \vec{x} \right\} \cap NI_t \,\middle|\, \vec{X}(0) = \vec{0}^{\epsilon} \right) = \det_{i,j=1}^{n} \left[\varphi_t \left(x_i - \epsilon(j-1) \right) \right]$$

Now expand:

$$\varphi_t \left(x_i - \epsilon(j-1) \right) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x_i - (j-1)\epsilon)^2}{2t} \right) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x_i^2}{2t} \right) \exp\left(-\frac{(j-1)^2\epsilon^2}{2t} \right) \exp\left(\frac{x_i(j-1)\epsilon}{t} \right)$$

By the rules for determinants when every row/column is multiplied by a constant factor, we have:

$$\begin{aligned} \mathbf{P}\left(\left\{ \vec{X}(t) = \vec{x} \right\} \cap NI_t \,\middle|\, \vec{X}(0) = \vec{0}^{\epsilon} \right) &= \det_{i,j=1}^{n} \left[\varphi_t \left(x_i - \epsilon(j-1) \right) \right] \\ &= \left(\frac{1}{\sqrt{2\pi t}} \right)^n \prod_{i=1}^{n} \exp\left(-\frac{x_i^2}{2t} \right) \prod_{j=1}^{n} \exp\left(-\frac{(j-1)^2\epsilon^2}{2t} \right) \det\left[\exp\left(\frac{x_i(j-1)\epsilon}{t} \right) \right] \end{aligned}$$

Since $\lim_{\epsilon \to 0} \exp\left(-\frac{(j-1)^2\epsilon^2}{2t} \right) = 1$, this term will be negligable in the limit. It remains to expand out the remaining determinant. By serendipity, this determinant is exactly a Vandermonde determinant!

$$\begin{aligned} \det_{i,j=1}^{n} \left[\exp\left(\frac{x_i(j-1)\epsilon}{t} \right) \right] &= \det_{i,j=1}^{n} \left[\exp\left(\frac{x_i\epsilon}{t} \right)^{j-1} \right] \\ &= V\left(\exp\left(\frac{x_1\epsilon}{t} \right), \ldots, \exp\left(\frac{x_n\epsilon}{t} \right) \right) \\ &= \prod_{i<j} \left(\exp\left(\frac{x_i\epsilon}{t} \right) - \exp\left(\frac{x_j\epsilon}{t} \right) \right) \\ &= \prod_{i<j} \left(\frac{\epsilon}{t} x_i - \frac{\epsilon}{t} x_j + O\left(\frac{\epsilon^2}{t^2} \right) \right) \\ &= \left(\frac{\epsilon}{t} \right)^{\binom{n}{2}} V(x_1, \ldots, x_n) + o(\epsilon^{\binom{n}{2}}) \end{aligned}$$

from which the result follows. □

Proof. (Of Proposition 5) As described above, we compute this probability for the $\epsilon-$of-room watermelons $\vec{W}^{\epsilon}$ and take the limit $\epsilon \to 0$. The probability distribution of the $\epsilon-$of-room watermelon $\vec{W}^{\epsilon}$ can be decomposed into non-intersecting events for Brownian motion: it must be non-intersecting in the left interval $[0, t]$ starting at $\vec{0}^{\epsilon}$ ending at $\vec{x}$, and also in the right interval $[t, t^* - t]$ going from $\vec{x}$ to $\vec{0}^{\epsilon}$. Finally, because we are conditioning, we should divide by the non-intersecting probability to go the whole interval $[0, t^*]$ going from $\vec{0}^{\epsilon}$ to $\vec{0}^{\epsilon}$ without intersection. That is to say we have:

$$\begin{aligned} \mathbf{P}\left(\vec{W}(t) = \vec{x} \right) &= \lim_{\epsilon \to 0} \mathbf{P}\left(\vec{W}^{\epsilon}(t) = \vec{x} \right) \\ &= \lim_{\epsilon \to 0} \frac{\mathbf{P}\left(\left\{ \vec{B}(t) = \vec{x} \right\} \cap NI_t \,\middle|\, \vec{B}(0) = \vec{0}^{\epsilon} \right) \mathbf{P}\left(\left\{ \vec{B}(t) = \vec{x} \right\} \cap NI_{t^*-t} = \vec{x} \,\middle|\, \vec{B}(0) = \vec{0}^{\epsilon} \right)}{\mathbf{P}\left(\left\{ \vec{B}(t^*) = \vec{0}^{\epsilon} \right\} \cap NI_{t^*} \,\middle|\, \vec{B}(0) = \vec{0}^{\epsilon} \right)} \end{aligned}$$

Lemma 6 gives the formula for all three factors! Plugging in the correct values of $\vec{x}$ gives:

$$\mathbf{P}\left(\vec{W}(t)=\vec{x}\right)=\lim_{\epsilon\to 0}\frac{\left(\frac{\epsilon}{t}\right)^{\binom{n}{2}}V(x_1,\ldots,x_n)\prod_{i=1}^n\varphi_t(x_i)\cdot\left(\frac{\epsilon}{t^*-t}\right)^{\binom{n}{2}}V(x_1,\ldots,x_n)\prod_{i=1}^n\varphi_{t^*-t}(x_i)}{\left(\frac{\epsilon}{t^*}\right)^{\binom{n}{2}}V(0,\epsilon\ldots,\epsilon(n-1))\prod_{i=1}^n\varphi_{t^*}(\vec{0}^\epsilon)}$$

Finally, notice that $V(0,\epsilon\ldots,\epsilon(n-1))=\epsilon^{\binom{n}{2}}V(0,1,\ldots,n-1)=\epsilon^{\binom{n}{2}}\prod_{k=0}^{n-1}k!$. This makes the powers of ϵ in the top and bottom to exactly cancel! Taking $\epsilon\to 0$ and simplifying gives the desired result. □

Remark 7. In Proposition 5 we applied Lemma 6 with $\vec{x}=\vec{0}^\epsilon$, but the Lemma is only proven for fixed $\vec{x}$ (not $\vec{x}$ depending on ϵ). It is a simple exercise to adapt the proof of the lemma to compute the asymptotic form of $\mathbf{P}\left(\left\{\vec{X}(t)=\vec{0}^\epsilon\right\}\cap NI_t\,\middle|\,\vec{X}(0)=\vec{0}^\epsilon\right)$.

We can now make sense of non-intersecting Brownian motions $\vec{X}(\cdot)$ conditioned not to intersect for all time (i.e. $\vec{B}(\cdot)$ conditioned on NI_∞) by taking the $t^*\to\infty$ limit of the watermelon $\vec{W}(\cdot)$. That is to say, we *define* the process $\vec{X}(\cdot)$ by:

$$\vec{X}(\cdot):=\lim_{t^*\to\infty}\vec{W}(\cdot)$$

Proposition 8. *Non-intersecting Brownian motions have the probability density function:*

$$\mathbf{P}\left(\vec{X}(t)=\vec{x}\right)=\frac{1}{\prod_{k=1}^{n-1}k!}V\left(\frac{x_1}{\sqrt{t}},\ldots,\frac{x_n}{\sqrt{t}}\right)^2\prod_{i=1}^n\varphi_t(x_i)$$

Proof. This follows by taking the $t^*\to\infty$ limit of the watermelon formula, $\mathbf{P}\left(\vec{X}(t)=\vec{x}\right)=\lim_{t^*\to\infty}\mathbf{P}\left(\vec{W}(t)=\vec{x}\right)$. □

Remark 9. Let $\vec{\lambda}=(\lambda_1,\ldots,\lambda_n)$ be the eigenvalues of the $n\times n$ GUE. One can show that:

$$\mathbf{P}\left(\vec{\lambda}_{GUE}=\vec{x}\right)=\mathbf{P}\left(\vec{W}(1)=\vec{x}\right)$$

They obey the *exact same* formula! One can also show that the GOE obeys,

$$\begin{aligned}\mathbf{P}\left(\vec{\lambda}_{GOE}=\vec{x}\right)&=c_n\left|V\left(x_1,\ldots,x_n\right)\right|^1\prod_{i=1}^n\varphi_1(x_i)\\&=\mathbf{P}\left(\vec{B}(t)=\vec{x}\,|NI_t\right)\end{aligned}$$

This is a very similar formula where the power of the Vandermonde determinant is 1 instead of 2.

3 Random point processes and k-point correlation functions

Definition 10. A **(simple) point process** is a collection of random points: $S\subset\mathbb{R}$. "Simple" means that S has no accumulation points and there is never two points at the same location in $\mathbb{R}$.

A point process S can be thought of as a a random measure on $\mathbb{R}$: i.e. if $S=\{X_1,X_2,\ldots\}$ then we can think of the counting measure μ_S on $\mathbb{R}$ by:

$$\mu_S=\delta_{X_1}+\delta_{X_2}+\ldots$$

with this notation, $\mu_S(A)=|S\cap A|\in\mathbb{N}$ is the number of points in $S\cap A$.

Example 11. The Poisson point process on an interval $[a,b]$ is the point process where the number of points is distributed like a Poisson random variable, $|S|\sim Poisson(b-a)$ and, conditionally on the number of points $|S|$, all the points are uniformly chosen from the interval $[a,b]$.

Note that defined in this way the Poisson point process is somewhat difficult to make sense of on all of $\mathbb{R}$: it doesn't make sense to set $|S|\sim Poisson(\infty)$ and it doesn't make sense to choose uniformly from the interval $(-\infty,+\infty)$. However, the Poisson point process on $\mathbb{R}$ is perfectly well defined! The process is easier to make sense of using another tool, the k-point correlation functions.

Definition 12. The k-point correlation functions for a point process S are functions (if they exist) $\rho_k : \mathbb{R}^k \to \mathbb{R}^+$ defined so that for every colelction of k disjoint sets $A_1, \ldots, A_k \subset \mathbb{R}$ we have:

$$\mathbf{E}\left[\prod_{i=1}^{k} \mu_S(A_i)\right] = \int_{A_1}\int_{A_2}\ldots\int_{A_k} \rho_k(x_1, \ldots, x_k)\mathrm{d}x_1 \ldots \mathrm{d}x_k$$

More informally, $\rho_k(x_1, \ldots, x_k)$ is the probability of finding k points from the point process S at each of the points $x_1, \ldots, x_k$. This is made precise by taking a limit of tiny intervals around each of the points x_i:

$$\rho_k(x_1, \ldots, x_k) = \lim_{\epsilon \to 0} \frac{\mathbf{P}\left(\bigcap_{i=1}^{k} \left\{\mu_S\left([x_i, x_i + \epsilon]\right) = 1\right\}\right)}{\epsilon^k}$$

Remark 13. The k point correlation functions are often confused with a probability density; don't get them mixed up! You should think of ρ_k more like a "particle density" than a "probability density". One good exercise is to check for a point process with $n > 1$ points that:

$$\int_{\mathbb{R}^n} \rho_n\left(\vec{x}\right) \mathrm{d}\vec{x} \neq 1$$

Lemma 14. *For a point process with exactly n points, the n point correlation function can be used to yield the k point correlation function for $k \leq n$ by the following integration:*

$$\rho_k(x_1, \ldots, x_k) = \frac{1}{(n-k)!} \underbrace{\int \ldots \int}_{n-k} \rho_n(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)\mathrm{d}x_{k+1} \ldots \mathrm{d}x_n$$

Proof. Exercise! □

Example 15. For the Poisson point process on $[a, b]$, one can show that $\rho_k\left(x_1, \ldots, x_k\right) = \mathbf{1}_{[a,b]}$ for every k and for every $x_1, \ldots, x_k$. From here, the limit $a \to -\infty$, $b \to +\infty$ makes perfect sense: the Poisson point process on all of $\mathbb{R}$ would be to simply set it to be the point process that has: $\rho_k\left(x_1, \ldots, x_k\right) \equiv 1$ for all k and for all $x_1, \ldots, x_k$. (It remains to check that the k point correlation functions uniquely determine a process; this can be done too!)

Remark 16. The positions of the non-intersecting Brownian motions can be thought of as a point process on $\mathbb{R}$ with exactly n points. Since the $\vec{X}$ always has its points sorted in order, the n-point correlation functions are related to the probability density of the points by:

$$\rho_n(x_1, \ldots, x_n) = \mathbf{P}\left(\vec{X} = \left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\right)\right)$$

where σ is the permutation that orders the points so that $x_{\sigma(1)} > \ldots > x_{\sigma(n)}$. Remarkably, the formula for the density of $\vec{X}$ we found is still valid no matter what σ is! That is to say:

$$\rho_n(x_1, \ldots, x_n) = \frac{1}{\prod_{k=1}^{n-1} k!} V\left(\frac{x_1}{\sqrt{t}}, \ldots, \frac{x_n}{\sqrt{t}}\right)^2 \prod_{i=1}^{n} \varphi_t(x_i)$$

We are interested in using the k point correlation functions to access the distribution of the top particle. The following proposition does this for us:

Proposition 17. *Let S be a point process that has a top particle, and let X_1 be the location of this top particle. Then:*

$$\mathbf{P}\left(X_1(t) < a\right) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \int_a^\infty \int_a^\infty \ldots \int_a^\infty \rho_k(x_1, \ldots x_k)\, dx_1\, dx_2 \ldots dx_k \tag{2}$$

The proof is essentially an application of the following more general lemma:

Lemma 18. *Let S be a simple point process all of whose correlation functions exist. Let $\phi : \mathbb{R} \to \mathbb{R}$ be any bounded test function with bounded support B. Suppose that:*

$$\sum_{k=0}^{\infty} \frac{\|\phi\|_\infty^k}{k!} \int_{B^k} \rho_k(x_1, \ldots, x_k) dx_1 \ldots dx_k < \infty \tag{3}$$

Then:

$$\mathbf{E}\left[\prod_{x \in S} (1 + \phi(x))\right] = \sum_{k=0}^{\infty} \frac{1}{k!} \int_B \ldots \int_B \prod_{j=1}^{k} \phi(x_j) \cdot \rho_k(x_1, \ldots, x_k) dx_1 \ldots dx_k$$

Remark 19. Note that the condition that equation (3) is finite is automatically satified if the point process S has finitely many points in B, since then $\rho_k \equiv 0$ for $k > |S|$. (Since its impossible to find more than $|S|$ points, and ρ_k measures the probability to have k points at specified locations.)

Proof. (Note we can restrict down to the support of ϕ, so S has finitely many points)

Expanding out $\prod_{X \in S} (1 + \phi(X))$, we see that it can be written as:

$$\begin{aligned}\prod_{X \in S} (1 + \phi(X)) &= \sum_{k=0}^{\infty} \sum_{\substack{A \subset S \\ |A|=k}} \prod_{X \in A} \phi(X) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{B^k} \prod_{i=1}^{k} \phi(x_i) \, \mathrm{d}\Xi_k(\vec{x})\end{aligned}$$

where Ξ_k is defined to be the random measrue on $B^k \subset \mathbb{R}^k$ which keeps track of subsets of size k from S:

$$\Xi_k := \sum_{\substack{A \subset S \\ |A|=k}} \sum_{\sigma \in S^n} \delta_{(X_{\sigma(1)}, \ldots, X_{\sigma(k)})}$$

So to prove the result we have only to verify that for any set $U \subset \mathbb{R}^k$ that:

$$\mathbf{E}\left[\Xi_k(U)\right] = \int_U \rho_k(x) \mathrm{d}x \tag{4}$$

But when $U = U_1 \times \ldots \times U_k$ is a disjoint products of sets, it is easily verified from the defintion of the k point correlation function and of Ξ_k that

$$\mathbf{E}\left[\Xi_k(U_1 \times \ldots \times U_k)\right] = \mathbf{E}\left[\prod_{i=1}^{k} \mu_S(U_i)\right] = \int_{U_1 \times \ldots \times U_k} \rho_k(x_1, \ldots, x_k) \mathrm{d}x_1 \ldots, \mathrm{d}x_k \tag{5}$$

equation (4) follows by suitably upgrading the equality in equation (5) from disjoint product sets to arbitary sets U, (e.g. by the monotone class theorem.) □

Proof. (Of Prop 17) For any interval $[a, b]$, by using the test function $\phi = -1_{[a,b]}$, note that $\prod_{x \in S}(1 + \phi(x)) = 1\{\mu_S([a, b]) = 0\}$. Hence, applying the lemma we have:

$$\mathbf{P}\left[\{\mu_S([a, b]) = 0\}\right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_a^b \cdots \int_a^b \rho_k(x_1, \ldots, x_k) \mathrm{d}x_1 \ldots \mathrm{d}x_k$$

Now, by the hypothesis that the sum converges absolutley, we can take the limit $b \to \infty$. The RHS converges to the desired infinite sum in (2) by the Lebesgue dominated convergence theorem, and the LHS converges to $\mathbf{P}\left(\mu_S\left([a, \infty)\right) = 0\right) = \mathbf{P}\left(X_1 < a\right)$ as desired. □

Example 20. (Poisson Process on $[0, 1]$) For a Poisson point process on $[0, 1]$, the k-point correlation functions are $\rho_k = 1_{[0,1]}$, so we can compute that the top particle satisfies:

$$\mathbf{P}\left(X_1 < a\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_a^1 \ldots \int_a^1 1 \mathrm{d}x_1 \ldots \mathrm{d}x_k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (1 - a)^k = e^{a-1}$$

This precisly the exponential distribution!

4 Bi-orthogonal Ensembles and Determinantal Point Processes

In order to apply the technology about point processes developed in the previous section, we must first compute the k-point correlation function ρ_k for all k. To do this, we must perform the integration:

$$\rho_k(x_1, \ldots, x_k) = \frac{1}{(n-k)!} \underbrace{\int \ldots \int}_{n-k} \rho_n(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \mathrm{d}x_{k+1} \ldots \mathrm{d}x_n \tag{6}$$

This is normally quite a feat, but the following theorem allows us to use some algebraic trickery to compute it when the n point correlation functions have a particular nice form.

Theorem 21. *Suppose that $\psi_1, \ldots, \psi_n : \mathbb{R} \to \mathbb{R}$ and $\phi_1, \ldots, \phi_n : \mathbb{R} \to \mathbb{R}$ are functions which are bi-orthogonal and normalized, in the sense that:*

$$\langle \psi_i, \phi_j \rangle = \int \psi_i(x)\phi_j(x)dx = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Suppose also we have a random point process S which has exactly n points whose n-point correlation function is given by:

$$\rho_n(x_1, \ldots, x_N) = \det_{i,j=1}^{n} [\psi_i(x_j)] \cdot \det_{i,j=1}^{n} [\phi_i(x_j)] \tag{7}$$

Then, for <u>any</u> $1 \leq k \leq n$ the k point correlation functions for the point process are <u>also</u> determinants:

$$\rho_k(x_1, \ldots, x_k) = \det \left[K(x_i, x_j) \right]_{1 \leq i,j \leq k} \tag{8}$$

where $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the "determinantal kernel" given by

$$K(x, y) := \sum_{i=1}^{n} \psi_i(x)\phi_i(y)$$

Remark 22. The situation given in equation (7) is called a "bi-orthogonal ensemble". A point process which has all its k-point correlation functions for any k given by the $k \times k$ determinant as in equation (8) is called a "determinantal point process. Therefore this theorem, tersely stated, says that "every bi-orthogonal ensemble gives a determinatal point process.

The heart of Theorem 21 is the Cauchy-Binet formula, which allows one to see one determinant as a sum of minors of other determinants. This algebraic lemma is essentially what is allowing us to compute the integral in (6).

Theorem 23. *[Cauchy-Binet Formula] If A is an $k \times n$ matrix and B is an $n \times k$ matrix, then the $k \times k$ determinant of the matrix AB can be written as:*

$$\det(AB) = \sum_{\substack{S \subset \{1,2,\ldots,n\} \\ |S|=k}} \det\left(A_{\{1,2\ldots,k\},S}\right) \det\left(B_{S,\{1,2\ldots,k\}}\right)$$

where $A_{\{1,2\ldots,k\},S}$ and $B_{S,\{1,2\ldots,k\}}$ denote the $k \times k$ matrices which are given by taking the subset of columns of A and rows of B respectively indexed by the set S.

Remark 24. You should think of determinants as volumes, so the Cauchy-Binet formula is relating a volume to the sum of the projections onto the coordinate directions. This is some kind of generalization of the Pythagorean identity:

$$\left\langle \vec{a}, \vec{b} \right\rangle = \sum_{i=1}^{n} \langle \vec{a}, \vec{e}_i \rangle \left\langle \vec{e}_i, \vec{b} \right\rangle$$

Proof. (Of Theorem 21) Fix k and $x_1, \ldots, x_k$. Let Ψ, Φ be the $k \times n$ matrices with entries:

$$\Psi_{i,j} = \psi_j(x_i) \quad \Phi_{i,j} = \phi_j(x_i)$$

Then

$$\det_{i,j=1}^{k}\left[K(x_i,x_j)\right]=\det\left(\Psi\Phi^T\right)$$

By the Cauchy-Binet formula, this is equal to:

$$\det_{i,j=1}^{n}\left[K(x_i,x_j)\right]=\sum_{\substack{S\subset\{1,2\ldots,n\}\\|S|=k}}\det\left(\Psi_{\{1,\ldots,k\},S}\right)\det\left(\Phi_{S,\{1,\ldots,k\}}\right)\tag{9}$$

This is sum over the k element subsets of $x_1,\ldots,x_n$, so its not surprising that this might have something to do with ρ_k. We will now check, using the orthogonality of the ψ's, that actually precisely the same as $\rho_k(x_1,\ldots,x_k)$. By expanding the determinants using the permutation expansion, we have:

$$\begin{aligned}\rho_k(x_1,\ldots,x_k)&=\frac{1}{(n-k)!}\underbrace{\int\ldots\int}_{n-k}\det_{i,j=1}^{n}\left[\psi_i(x_j)\right]\det_{i,j=1}^{n}\left[\phi_i(x_j)\right]\mathrm{d}x_{k+1}\ldots\mathrm{d}x_n\\&=\frac{1}{(n-k)!}\underbrace{\int\ldots\int}_{n-k}\sum_{\sigma,\tau\in S_n}\mathrm{sgn}(\sigma\tau)\prod_{i=1}^{n}\psi_{\sigma(i)}(x_i)\phi_{\tau(i)}(x_i)\mathrm{d}x_{k+1}\ldots\mathrm{d}x_n\\&=\frac{1}{(n-k)!}\sum_{\sigma,\tau\in S_n}\mathrm{sgn}(\sigma\tau)\prod_{i=1}^{k}\psi_{\sigma(i)}(x_i)\phi_{\tau(i)}(x_i)\cdot\prod_{i=k+1}^{n}\left(\int\psi_{\sigma(i)}(x_i)\phi_{\tau(i)}(x_i)\mathrm{d}x_i\right)\end{aligned}$$

The last integral vanishes unless $\sigma(i)=\tau(i)$ for all $i=\{k+1,\ldots,n\}$. Hence we have:

$$\rho_k(x_1,\ldots,x_k)=\frac{1}{(n-k)!}\sum_{\substack{\sigma,\tau\in S_n\\\sigma(i)=\tau(i),\forall i>k}}\mathrm{sgn}(\sigma\tau)\prod_{i=1}^{k}\psi_{\sigma(i)}(x_i)\phi_{\tau(i)}(x_i)$$

□

But now, each σ,τ are permutation where the action on the first k elements are unreated, but the action on the last $(n-k)$ elements are equal. If we label the image of the first k elements as:

$$\tau(\{1,\ldots,k\})=\sigma(\{1,\ldots,k\})=:\{j_1,\ldots,j_k\}$$

with $1\leq j_1<,\ldots,<j_k\leq n$, we have that for every for every such choice, there are $(n-k)!$ ways to arrange the rest of the permutation (i.e. for a fixed $\{j_1,\ldots,j_k\}$, there are $(n-k)!$ possible permutations for the values of $\sigma(i)=\tau(i)$ $\forall k<i\leq n$) . Hence, σ and τ can effectively be though of as permutations of k elements onto the set $\{j_1,\ldots,j_k\}$. We have:

$$\begin{aligned}\rho_k(x_1,\ldots,x_k)&=\sum_{1\leq j_1<\ldots<j_k}\sum_{\sigma,\tau\in S_k}\mathrm{sgn}(\sigma\tau)\prod_{i=1}^{k}\psi_{j_{\sigma(i)}}(x_i)\phi_{j_{\tau(i)}}(x_i)\\&=\sum_{1\leq j_1<\ldots<j_k\leq n}\det_{\substack{i\in\{1,\ldots,k\}\\j\in\{j_1,\ldots,j_k\}}}\left[\psi_j(x_i)\right]\det_{\substack{i\in\{1,\ldots,k\}\\j\in\{j_1,\ldots,j_k\}}}\left[\phi_j(x_i)\right]\end{aligned}$$

Where we have recognized the permutation expansion of the determinant! Note now that this is precisely the same as equation (9), which is precisely the desired result!

Example 25. You can make any example by choosing your favourite orthognal functions. On the interval $[0,2\pi]$, one choice with $n=2$ is $\psi_1(x)=\phi_1(x)=\frac{1}{\sqrt{\pi}}\cos(x)$ and $\psi_2(x)=\phi_2(x)=\frac{1}{\sqrt{\pi}}\sin(x)$. In this case the kernel K is:

$$K(x,y)=\frac{1}{\pi}\cos(x)\cos(y)+\frac{1}{\pi}\sin(x)\sin(y)$$

In this case, it happens that the kernel K is actually an inner product of function of x,y:

$$\begin{aligned}K(x,y)&=\frac{1}{\pi}\left\langle\begin{pmatrix}\cos(x)\\\sin(x)\end{pmatrix},\begin{pmatrix}\cos(y)\\\sin(y)\end{pmatrix}\right\rangle\\&=\frac{1}{\pi}\left\langle f(x),f(y)\right\rangle\end{aligned}$$

where $f(x) : \mathbb{R} \to \mathbb{R}^2$ maps the interval $[0, 2\pi]$ to the unit circle. The determinant of an inner product is the volume of the points, i.e.:

$$\det_{i,j=1}^{n} [K(x_i, x_j)] = \det \left[\frac{1}{\pi} \langle f(x_i), f(x_j) \rangle \right]$$
$$\propto \text{Volume of the parreleliped spanned by } f(x_1), \ldots, f(x_n)$$

In our case, this volume is proportional to $\cos(x - y)$ (which could also have been seen by a cosine identity on the kernel K). This means the most likely configuration of the two points is that the random points are chosen to be orthogonal. It is unlikely the two points are close together because then the area of the parrelelipied spanned by the vectors $f(x_1), f(x_2)$ is small. The intuition from this example is true in general. By the theory of Reproducing Kernel Hilbert spaces (And in particular the Moore-Aronszajn Theorem), every positive definite symmetric kernel $K(x, y)$ can be realized as $K(x, y) = \langle f(x), f(y) \rangle_{\mathcal{H}}$ for some (possibly very complicated) Hilbert space $\mathcal{H}$, some function $f : \mathbb{R} \to \mathcal{H}$, and some inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.

This intuition about "choosing points according the volume" is one explanation why determinatal processes exhibit repulsion between points: Points that are close together generate small "volume", while points that are far away have more "volume". Below is an example showing a Poisson point process vs a typical determinantal point processs in the plane $\mathbb{R}^2$ (the image is from Figure 1 in Hough, Krishnapur, Peres and Viráag's excellent survey article "Determinantal Processes and Independence", published in *Probability Surveys* 2006). Another way to see this repulsion is to use algebraic identites for the determinant. e.g. the identity

$$\det_{i,j=1}^{2} [K(x_i, x_j)] = K(x_1, x_1)K(x_2, x_2) - K(x_1, x_2)^2 \leq K(x_1, x_1)K(x_2, x_2)$$

shows that it is less likely to have *both* points x_1 and x_2 occupied, than what it would be if the point were independent.

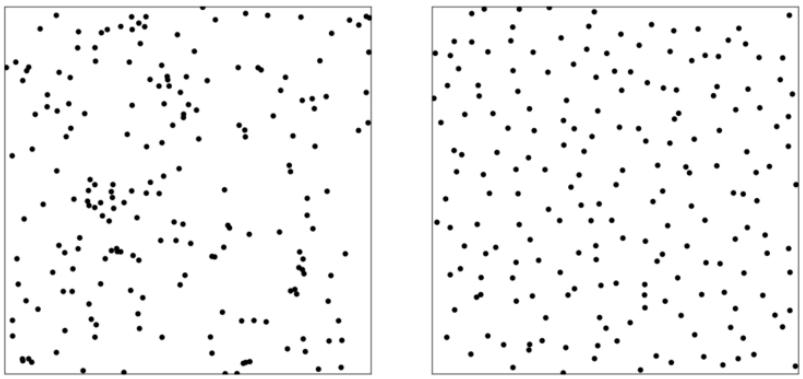


FIG 1. *Samples of translation invariant point processes in the plane: Poisson (left), determinantal for* $K(z, w) = \frac{1}{\pi} e^{z\overline{w} - \frac{1}{2}(|z|^2 + |w|^2)}$ *Determinantal processes exhibit repulsion.*

Proposition 26. *The n points of the non-intersecting Brownian motions form a determinantal point process with kernel given by:*

$$K(x, y) = \sum_{i=0}^{n-1} \psi_i(x)\psi_j(x)$$

where the functions ψ_i *are the normalized Hermite polynomials:*

$$\psi_i(x) = \frac{H_i(x, t)\sqrt{\varphi_t(x)}}{\sqrt{t^i i!}}$$

and $H_i(x, t)$ *are the monic polynomials of degree i, (i.e.* $H_i(x, t) = x^i + \ldots$*) which are orthogonal w.r.t to* $\varphi_t(x)$ *:*

$$\int H_i(x, t)H_j(x, t)\varphi_t(x)dx = 0 \text{ if } i \neq j$$

Proof. By the discussion in Remark 16 , we know the n point correlation functions satisfy:

$$\rho_n(x_1,\dots,x_n) = \frac{1}{\prod_{k=0}^{n-1} t^k k!} V(x_1,\dots,x_n)^2 \prod_{i=1}^{n} \varphi_t(x_i) \tag{10}$$

□

We will now massage this into the form:

$$\rho_n(x_1,\dots,x_n) = \left(\det\left[\psi_i(x_j)\right]_{i,j=1}^{n}\right)^2$$

so that we can apply Theorem 21 to get the desired result. To do this we will write $V(x_1,\dots,x_n)$ as the determinant of the Vandermonde matrix:

$$V(x_1,\dots,x_n) = \det\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}$$

By doing column operations, in particular adding linear combinations of the first $k-1$ columns to the k-th column, we can actually rewrite:

$$V(x_1,\dots,x_n) = \det\begin{bmatrix} 1 & P_1(x_1) & P_2(x_1) & \dots & P_{n-1}(x_1) \\ 1 & P_1(x_2) & P_2(x_2) & \dots & P_{n-1}(x_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & P_1(x_n) & P_2(x_n) & \dots & P_{n-1}(x_n) \end{bmatrix}$$

where $P_i(x)$ is any monic polynomial of degree i. This can be done because row/column operations do not effect the value of the determinant! (We will find it convenient to choose $P_i(x)$ to be the Hermite polynomials, $P_i(x) = H_i(x,t)$ as we shall see below.) Using this matrix representation in equation (10), and factoring $\sqrt{\varphi_t(x_i)}$ into the i-th row of the matrix, and factoring in $\sqrt{\frac{1}{t^i i!}}$ into the i-th column of the matrix, we have

$$\begin{aligned}\rho_n(x_1,\dots,x_n) =& \frac{1}{\prod_{k=0}^{n-1} t^k k!} \det\begin{bmatrix} 1 & P_1(x_1) & P_2(x_1) & \dots & P_{n-1}(x_1) \\ 1 & P_1(x_2) & P_2(x_2) & \dots & P_{n-1}(x_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & P_1(x_n) & P_2(x_n) & \dots & P_{n-1}(x_n) \end{bmatrix}^2 \prod_{i=1}^{n} \varphi_t(x_i) \\ =& \det_{i,j=1}^{n}\left[\frac{P_{i-1}(x_j)\sqrt{\varphi_t(x_j)}}{\sqrt{t^i i!}}\right]^2\end{aligned}$$

which is exactly of the desired form with $\psi_i(x) = \frac{P_{i-1}(x)\sqrt{\varphi_t(x)}}{\sqrt{t^i i!}}$. The choice $P_i(x) = H_i(x,t)$, the monic Hermite polynomials, ensures that $\langle \psi_i, \psi_j \rangle = 0$ if $i \neq j$ by the definition of the Hermite polynomials. Finally, one can prove that $\langle \psi_i, \psi_j \rangle = 1$ when $i = j$ with this definition by induction knowing that there are exactly n points in the particle process, or by the more explicit computation for the Hermite polynomials in the following section.

5 The Hermite Polynomials

Definition 27. The Hermite polynomials $H_k(x,t)$ are the orthogonal polynomials for the Gaussian measure $\varphi_t(x)$ such that:

- $H_k(x,t) = x^k + \dots$ is a monic polynomial of degree k
- $\int H_k(x,t) H_\ell(x,t) \varphi_t(x) \mathrm{d}x = 0$ if $k \neq \ell$

Here is a list with the first three for reference:

- $H_0(x,t) = 1$

- $H_1(x,t) = x$
- $H_2(x,t) = x^2 - t$
- $H_3(x,t) = x^3 - 3xt$
- $H_4(x,t) = x^4 - 6x^2t + 3$

Remark 28. Most authors leave out the dependence on a variable t, and often define the Hermite polynomials $H_k(x)$ as the orthogonal polynomials for the measure $\varphi_1(x)$ or else for $\varphi_{\frac{1}{2}}(x)$ or else for $\sqrt{2\pi}\varphi_1(x)$ or else for $\sqrt{\pi}\varphi_{\frac{1}{2}}(x)$.

Remark 29. Note that $H_0 \equiv 1$, so by orthogonality we have that for any k that $\mathbf{E}[H_k(B_t,t)] = 0$ where B_t is a standard Brownian motion. It actually turns out that $H_k(B_t,t)$ are *martingales*. This gives us a convenient formula for finding them in terms of the exponential martingale $\exp\left(\alpha B_t - \frac{1}{2}\alpha^2 t\right)$ as follows:

Lemma 30. *The Hermite polynomials are the derivatives of the Brownian exponential martingale at $\alpha = 0$:*

$$H_k(x,t) = \frac{\partial^k}{\partial \alpha^k}\,|_{\alpha=0} \exp\left(\alpha x - \frac{1}{2}\alpha^2 t\right)$$

and moreover $\int H_k(x,t)H_k(x,t)\varphi_t(x)dx = t^k k!$.

Proof. It is clear e.g. by induction that $H_k(x,t)$ defined in this way is a monic polynomial starting with x^k. To see that these are orthogonal, we use $\mathbf{E}[\exp(\beta B_t)] = \exp\left(\frac{1}{2}\beta^2 t\right)$ to compute (we use α as the variable for H_ℓ and β as the variable for H_k)

$$\begin{aligned}
\mathbf{E}[H_k(B_t,t)\cdot H_k(B_t,t)] &= \frac{\partial^k}{\partial\alpha^k}\,|_{\alpha=0}\ \frac{\partial^\ell}{\partial\beta^\ell}\,|_{\beta=0}\ \mathbf{E}\left[\exp\left(\alpha B_t - \frac{1}{2}\alpha^2 t\right)\exp\left(\beta B_t - \frac{1}{2}\beta^2 t\right)\right] \\
&= \frac{\partial^k}{\partial\alpha^k}\,|_{\alpha=0}\ \frac{\partial^\ell}{\partial\beta^\ell}\,|_{\beta=0} \exp\left(\frac{1}{2}(\alpha+\beta)^2 t - \frac{1}{2}\alpha^2 t - \frac{1}{2}\beta^2 t\right)] \\
&= \frac{\partial^k}{\partial\alpha^k}\,|_{\alpha=0}\ \frac{\partial^\ell}{\partial\beta^\ell}\,|_{\beta=0} \exp(\alpha\beta t) \\
&= \frac{\partial^k}{\partial\alpha^k}\,|_{\alpha=0}\ (\alpha t)^\ell \\
&= t^k k! 1\{\ell = k\}
\end{aligned}$$

the last line is 0 when $k > \ell$ because we have k derivative of the polynomials α^ℓ, and is 0 when $k < \ell$ because we are plugging in $\alpha = 0$. □

Remark 31. By Brownain scaling and changing variables, we can also verify from the definition that H_k has the following scaling:

$$H_k(bx,t) = b^k H_k\left(x, \frac{1}{b^2}t\right)$$
$$H_k(x,a^2t) = a^k H_k\left(\frac{1}{a}x, t\right)$$

Lemma 32. *Have that the x-derivative of the Hermite polynomials satisfies:*

$$\partial_x H_n(x,t) = nH_{n-1}(x,t)$$

Proof. By using the binomial formula for the multi-derivative product rule $\partial_x^{(n)} gf = g\left(\partial_x^{(n)} f\right) + n\left(\partial_x g\right)\left(\partial_x^{(n-1)} f\right) +$

... and changing the order of derivatives in the formula for H_k, we have

$$\begin{aligned}
\partial_x H_n(x,t) &= \frac{\partial}{\partial x}\frac{\partial^k}{\partial \alpha^k}\,|_{\alpha=0}\,\exp\left(\alpha x - \frac{1}{2}\alpha^2 t\right) \\
&= \frac{\partial^k}{\partial \alpha^k}\,|_{\alpha=0}\,\frac{\partial}{\partial x}\exp\left(\alpha x - \frac{1}{2}\alpha^2 t\right) \\
&= \frac{\partial^k}{\partial \alpha^k}\,|_{\alpha=0}\,\alpha\exp\left(\alpha x - \frac{1}{2}\alpha^2 t\right) \\
&= \alpha\frac{\partial^n}{\partial \alpha^n}\exp\left(\alpha x - \frac{1}{2}\alpha^2 t\right) + n\frac{\partial^{n-1}}{\partial \alpha^{n-1}}\exp\left(\alpha x - \frac{1}{2}\alpha^2 t\right)|_{\alpha=0} \\
&= nH_{n-1}(x,t)
\end{aligned}$$

□

Lemma 33. *An alternative formula for the Hermite polynomials is in terms of n derivatives of the function $e^{-x^2/2t}$.*

$$H_n(x,t) = e^{+\frac{x^2}{2t}}(-t)^n\partial_x^{(n)}e^{-\frac{x^2}{2t}}$$

Proof. This comes from completing the square and doing a change of variable $\beta = x - \alpha t$

$$\begin{aligned}
\frac{\partial^k}{\partial \alpha^k}\,|_{\alpha=0}\,\exp\left(\alpha x - \frac{1}{2}\alpha^2 t\right) &= e^{\frac{x^2}{2t}}\frac{\partial^k}{\partial \alpha^k}\,|_{\alpha=0}\,\exp\left(-\frac{1}{2t}(x-\alpha t)^2\right) \\
&= e^{\frac{x^2}{2t}}(-t)^k\frac{\partial}{\partial \beta^k}\,|_{\beta=x}\,\exp\left(-\frac{1}{2t}\beta^2\right)
\end{aligned}$$

□

Lemma 34. *The Hermite polynomials obey the three term recurrence:*

$$\begin{aligned}
H_{n+1}(x,t) &= xH_n(x,t) - t\partial_x H_n(x,t) \\
&= xH_n(x,t) - tnH_{n-1}(x,t)
\end{aligned}$$

Proof. Use the alternative formula for the Hermite polynomials 33 and the product rule:

$$\begin{aligned}
H_{n+1}(x,t) &= e^{+\frac{x^2}{2t}}(-t)^{n+1}\partial_x^{(n+1)}e^{-\frac{x^2}{2t}} \\
&= e^{+\frac{x^2}{2t}}(-t)\partial_x\left[e^{-\frac{x^2}{2t}}H_n(x,t)\right] \\
&= e^{+\frac{x^2}{2t}}(-t)\left[-\frac{x}{t}e^{-\frac{x^2}{2t}}H_n(x,t) + e^{-\frac{x^2}{2t}}\partial_x H_n(x,t)\right] \\
&= xH_n(x,t) - t\partial_x H_n(x,t)
\end{aligned}$$

□

Lemma 35. *Another formula for the Hermite polynomials in terms of the backwards heat equation:*

$$H_k(x,t) = e^{-t\frac{1}{2}\partial_x^{(2)}}[x^k]$$

where the backwards heat equation operator $e^{-t\frac{1}{2}\partial_x^{(2)}}$ can be defined by its Taylor series $e^{-t\frac{1}{2}\partial_x^{(2)}} = 1 - t\frac{1}{2}\partial_x^{(2)} + t^2\frac{1}{8}\partial_x^{(4)} + \ldots$ which yields a finite series when applied to a polynomial such as x^k. e.g. $H_2(x,t) = e^{-t\frac{1}{2}D^2}[x^2] = x^2 - t$, $H_3(x,t) = e^{-t\frac{1}{2}D^2}[x^3] = x^3 - 3xt$

Proof. By the fact that $H_k(x,t)$ is a martingale, we know that:

$$\mathbf{E}_{B_0=x}\left[H_k(B_t,t)\right] = H_k(x,0) = x^k$$

Written in terms of the generator for Brownian motion $e^{\frac{1}{2}t\partial_x^{(2)}}$, this says:

$$e^{+\frac{1}{2}t\partial_x^{(2)}}H_k(x,t) = x^k$$

□

From here, we can formally apply $e^{-t\frac{1}{2}\partial_x^{(2)}}$ to both sides and the convergence is guaranteed by the fact that the Taylor series expansion is finite.

Lemma 36. *(Christoffel-Darboux formula):*

$$\sum_{k=0}^{n-1} \frac{H_k(x,t)H_k(y,t)}{t^k k!} = \frac{H_n(x,t)H_{n-1}(y,t) - H_n(y,t)H_{n-1}(x,t)}{t^{n-1}(n-1)!(x-y)}$$
$$= \frac{H_n(x,t)\partial_x H_n(y,t) - H_n(y,t)\partial_x H_n(x,t)}{t^{n-1}n!(x-y)}$$

Proof. This can be proven using the three term recurrence (e.g. by induction or by arranging the LHS into a telescoping sum) □

Lemma 37. *With the definition,*

$$\psi_k(x) = \frac{H_k(x,t)\sqrt{\varphi_t(x)}}{\sqrt{t^k k!}}$$

we have

$$K(x,y) = \sum_{k=0}^{n-1} \psi_k(x)\psi_k(y) = t\frac{\psi_n(x)\psi_n'(y) - \psi_n(y)\psi_n'(x)}{x-y} - \frac{1}{2t}\psi_n(x)\psi_n(y)$$

Proof. Follows by the Christoffel-Darboux formula and by directly computing that:

$$\psi_k'(x) = \frac{\partial_x H_k(x,t)\sqrt{\varphi_t(x)}}{\sqrt{t^k k!}} - \frac{x}{2t}\frac{H_k(x,t)\sqrt{\varphi_t(x)}}{\sqrt{t^k k!}}$$

□