



Differential Forms, the Early Days; or the Stories of Deahna's Theorem and of Volterra's Theorem

Author(s): Hans Samelson

Source: *The American Mathematical Monthly*, Vol. 108, No. 6 (Jun. - Jul., 2001), pp. 522-530

Published by: Mathematical Association of America

Stable URL: <http://www.jstor.org/stable/2695706>

Accessed: 25/11/2009 12:44

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/action/showPublisher?publisherCode=maa>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.



Mathematical Association of America is collaborating with JSTOR to digitize, preserve and extend access to *The American Mathematical Monthly*.

<http://www.jstor.org>

Differential Forms, the Early Days; or the Stories of Deahna's Theorem and of Volterra's Theorem

Hans Samelson

This is a short informal history of the beginning of differential forms, up to the time of de Rham's work. It started with my being curious about how Poincaré actually stated Poincaré's Lemma. There were some surprises, mainly that Poincaré's Lemma is due to another person, and the same for the well known Frobenius Theorem.

I did look up everything that I quote. But of course I haven't looked up everything; when I say that something appeared first at such and such a place, I mean that I haven't come across an earlier reference.

Let us recall briefly what differential forms are and do: They are generalizations of the well known expressions $Pdx + Qdy$ and $Adydz + Bdzdx + Cdx dy$ that function as integrands of line or surface integrals and represents things like work done moving along a curve in a force field or the flux of a vector field through a surface. Thus a differential form ω , of degree r , defined in some open set in n -space \mathbb{R}^n (coordinates x_1, x_2, \dots, x_n), is an expression of the form $\sum a_{i_1 i_2 \dots i_r} dx_{i_1} dx_{i_2} \dots dx_{i_r}$, where the "coefficients" a_{\dots} are (suitably differentiable) functions, and where the "differentials" dx_i are symbols associated with the coordinates x_i (they are actually 1-forms, with coefficient 1).

The basic rule for multiplication of the dx_i is skew-symmetry: $dx_i dx_j = -dx_j dx_i$. In particular one has $dx_i dx_i = 0$. (This makes the appearance of Jacobians automatic: e.g., if x and y are functions of u and v , then $dx dy = (x_u du + x_v dv)(y_u du + y_v dv) = x_u y_u du du + x_u y_v du dv + x_v y_u dv du + x_v y_v dv dv = 0 + x_u y_v du dv - x_v y_u du dv + 0 = \frac{\partial(x,y)}{\partial(u,v)} du dv$. The algebraic context is exterior algebra.) In line with this the a_{\dots} are usually taken skew-symmetric in their indices; for the above example $Adydz + \dots$ this would mean rewriting it as $1/2 Adydz - 1/2 Adz dy + \dots$.

For a function f of the x_i one defines the "differential" df as the 1-form $\sum f_{x_i} dx_i$. (Thus the dx_i are the differentials of the functions x_i .)

For integration of an r -form over any oriented r -manifold in the domain of definition of ω one represents the manifold locally parametrically by writing the x_i as functions of r variables u_α that are adapted to the orientation of the manifold. The differentials dx_i now become 1-forms in the du_α , and the form ω reduces (by "multiplying out") to a single term $A du_1 du_2 \dots du_r$, where A is a function of the u_α . (This is the "restriction" of the form to the manifold.) One forms the usual integral of this over the appropriate region in u_α -space and combines the local contributions (using a partition of unity) to get the integral of ω over the whole manifold. (Parenthetically, there is a slightly different way of integrating something over a manifold, namely when the manifold carries some kind of distribution—matter or electricity or ...—with a density ρ . On the manifold an area-element dA is given, and finds the total mass or charge as $\int \rho dA$. This reduces of course again to local integrals, using local parametrizations. An orientation of the manifold is here usually not required.)

The next thing is to extend the operation d (as in df) to all forms: One forms the "exterior derivative" $d\omega$ by replacing each coefficient a_{\dots} by its differential da_{\dots} (and again "multiplying out"), resulting in an $(r + 1)$ -form. It turns out, quite formally, that the operation dd (d applied twice) is always 0 (from $f_{xy} = f_{yx}$).

Finally, the main fact, *Stokes's theorem*: If N is an oriented $(r + 1)$ -manifold, with boundary manifold $\delta N = M$ (appropriately oriented), then the integral of ω over M equals the integral of $d\omega$ over N : $\int_N d\omega = \int_{\delta N} \omega$. (Note: the boundary δN is closed; its boundary is empty.)

This concludes our very short overview of what differential forms are and do. All of it makes good sense in any manifold instead of just \mathbb{R}^n .

The beginning of the idea of differential forms is certainly what was later called the “total differential” of a function f of two (real) variables x, y : the expression $df = f_x dx + f_y dy$, interpreted as giving the change in f if one changes x and y by (small) amounts dx and dy . I don't know where this originated, but it was well known in Euler's days.

Around 1740 A.C. Clairaut [7, pp. 294–297], Euler [14, pp. 176–179], and (reportedly—see [7, p. 294]) A. Fontaine [15] [16], apparently independently, had the idea to investigate when an expression $Pdx + Qdy$, where P, Q are functions of x and y , is the differential of a function. (Actually Euler's work was earlier, 1734–1735, but it appeared only in 1740.) First they establish the commutativity relation $f_{xy} = f_{yx}$, then the necessary condition $P_y = Q_x$, and go on to show that the latter is sufficient: They assume f in the form $\int Pdx + r$ where r is a function of y only, thus assuring $f_x = P$, and then show that one can determine r so that $f_y = Q$ (here the necessary condition comes in, via $\int P_y dx = \int Q_x dx$). This is not really an existence proof, since the integrals are indefinite and there is no assurance of their existence.

Clairaut in [7], in a footnote on p. 294, says that Euler and Fontaine had the same result, Euler's work just appearing in [14] and Fontaine presenting his work to the Royal Academie in Paris the same day that Clairaut was lecturing about it there. According to [12] Fontaine's work appears in [15] (I could not trace this book, and consulted [16] which may well be the same book); all I could find there however, on p. 26, are the two statements: $\frac{d}{dy} \int \mu dx = \int \frac{\partial \mu}{\partial y} dx$ and the necessary condition: If $d\phi = Adx + Bdy$, then $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$.

Later Cauchy noted that one can interpret the argument as giving the formula $f = \int_{x_0}^x P(x, y_0) dx + \int_{y_0}^y Q(x, y) dy$, where now the integrals are definite, proving existence (modulo appropriate hypotheses); he also gave the corresponding formula for \mathbb{R}^n . All this is in [23, vol. 2, pp. 339–341, 488–490] (the book is based on Cauchy's lectures plus some other material).

But it took some time before it was realized that, if the integrability conditions hold, the line integral depends only on the endpoints of the curve (assuming simple connectedness of the domain—nobody worried about that at the time); and only late in the eighteen hundreds did Morera notice that one can find the function f by integrating along the straight segment from a fixed point p_0 to the variable point p (or indeed any other curve).

In fact, Clairaut, Euler, and Fontaine went further. They studied the question when a form $Pdx + Qdy + Rdz$ is “completely integrable” (we discuss the concept below). It amounts here to the existence of a “multiplier”, a function M such that $M(Pdx + Qdy + Rdz)$ is differential of a function. Euler gave the necessary (and as it turned out later, also sufficient) “integrability conditions” in [13], Clairaut had them in an implicit form [8] and (according to Clairaut) Fontaine had them as equations for M . (Incidentally, Euler seems to have considered forms that do not satisfy the integrability conditions as illegal. Some time later Monge noted that such expressions make some sense, because one can satisfy them by taking the variables as suitable functions of one variable [13, *Op. Omn.*, preface, p. IX].)

The next step was taken in [25], 1814–15 by J. Pfaff, a well known German mathematician (a contemporary French mathematician—I have forgotten who (Laplace?)—was asked about the best mathematician in Germany and answered “J. Pfaff”; upon “What about Gauss?”, he said “Gauss is the best mathematician in all of Europe”). He introduced the idea of finding integral manifolds, submanifolds (of as high a dimension as possible) of the space, on which a given differential form $a_1 dx_1 + a_2 dx_2 + \cdots + a_n dx_n$ vanishes (reduces to the form 0). Here the a_i are given functions (C^∞ , say) of the real variables x_1, x_2, \dots, x_n .

This became known as *Pfaff’s problem* [18], [21].

In this context Pfaff looked for a normal form for such expressions under change of variables. His answer was that one can find (locally, of course) either systems of functions $(f_1, \dots, f_r, g_1, \dots, g_r)$ or systems $(f, f_1, \dots, f_r, g_1, \dots, g_r)$ with *independent* differentials, such that the 1-form can be written as $\sum_1^r f_i dg_i$ or $df + \sum_1^r f_i dg_i$. (One could then take these functions as part of a new coordinate system.) The number r is called the *class* of the 1-form. A special case is that where the form can be written as df , i.e., where it is a total differential. (In \mathbb{R}^{2n+1} a form of maximal class n , thus essentially $dz + y_1 dx_1 + \cdots + y_n dx_n$ with coordinates x_i, y_i, z , is called a *contact form*.)

Pfaff’s work was continued mostly in the direction of what now is called the Frobenius theorem, the basic and constantly used theorem about the solution manifolds of a system of 1-forms. This is the question of “complete integrability of a system of r 1-forms ω_i ”: What are the conditions under which these forms have $(n - r)$ -dimensional integral manifolds, in the sense that one can find r independent (i.e., with everywhere independent differentials) functions f_1, f_2, \dots, f_r such that the integral manifolds are given by putting the f_i equal to arbitrary constants? (It looks like the—almost too simple—case where the ω_i are just dx_1, dx_2, \dots, dx_r and the f_i can be taken as the x_1, \dots, x_r .) This is, at least at first, a local problem; one asks for a solution in sufficiently small open sets.

There is a “dual” version of the problem, in terms of vector fields: A vector X_x at a point x is given by its components ξ_1, \dots, ξ_n . It operates on functions defined near x by $X_x f = \lim_{t \rightarrow 0} (f(x + tX_x) - f(x))/t = \sum \xi_i f_{x_i}(x)$; this is “the derivative of f along X_x ”. (Operating on the coordinate x_i gives the component ξ_i .) We should also mention that a differential form acts at each point x as a linear function on the space of vectors at x , by $\omega(X_x) = \sum a_i(x) \cdot \xi_i$.

A vector field X assigns to each x a vector X_x , in a C^∞ way (the components are C^∞ functions); and then for any function f the derivative “along X ” is a new function Xf with $Xf(x) = X_x f$. A vector field X also defines a *flow*, along its “integral curves”, in its region of definition; this amounts to solving the ordinary differential equations $dx/dt = X$.)

Dual to the differential d for forms is the operation $[]$ (“bracket”), which assigns to two vector fields X, Y the vector field $XY - YX$. This is to be understood by operating on functions, with XYf meaning the result of applying X to the function Yf . This seems to involve second derivatives, but they drop out because $f_{xy} = f_{yx}$. Exercise: find the components of $[XY]$ in terms of those of X and Y (apply $[XY]$ to the coordinate functions x_i).

The complete integrability question now reads as follows: Let X_1, \dots, X_r be r vector fields (assumed independent at each x). Under what conditions can one find $n - r$ independent functions f_1, \dots, f_{n-r} such that the X_i nullify the f_j ? (Geometrically, the X_i are tangent to the manifolds given by setting all f_j equal to constants.)

Back to history.

Jacobi in [21] treated the case of a system of commuting vector fields (satisfying $[XY] = XY - YX = 0$ for any two fields in the system). This is actually pretty straightforward: one takes a small $(n - r)$ -surface that is transversal to the vectors at a point, and then lets the flows of the fields act on the surface. A. Clebsch attempted and maybe succeeded (see comments in [17]) in reducing the general case to Jacobi's. The work was continued and extended by many people; see ([21], [24], [8], [17], [20], [10], [22], [11]). In the long paper [17] Frobenius reviewed what had been done before and gave his own proof, introducing the concept of the "bilinear covariant" of a 1-form (it amounts to the exterior derivative). It is from here that Frobenius's name got attached to the theorem; he himself didn't give it any name. In the paper Frobenius notes that much earlier, in 1840, the paper [11] by F. Deahna stated and gave a proof for the full Frobenius theorem (Frobenius goes through and simplifies the proof). The paper seems to have been completely overlooked until Frobenius referred to it.

Thus Frobenius's Theorem is really Deahna's Theorem (unless some earlier author appears).

The way Deahna stated his condition is quite different from that of the other authors and might have been difficult to understand at the time, although from today's point of view it makes good sense and is to my mind the one form that really tells what is going on:

The now customary form of the conditions (the $d\omega_i$ are in the ideal generated by the ω_i , or the commutators $[X_i X_j]$ are linear combinations $\sum_1^r f_k X_k$ of the X_k with functions f_k as coefficients) are short and clean, but not very intuitive. Deahna's version, pushed a little, says: A system of independent 1-forms $\{\omega_i\}_1^{n-r}$, looked at (equivalently) as a field of r -planes (an r -distribution in the sense of Chevalley), is completely integrable, if and only if it is invariant under the flow generated by any vector field that lies everywhere in the distribution. (He states the condition in terms of the "variation" of the forms under such vector fields, without much explaining what this variation is; it is in fact the Lie derivative.) Later E. Cartan took up Pfaff's problem again in his paper [4], where he treats it in the language of (first order) differential forms and their "covariants" (exterior derivatives), which had been introduced by Frobenius and Darboux. In this paper he also introduces, quite formally, differential forms of higher order and their exterior derivatives (without mentioning Poincaré); actually the only higher forms he uses are what he calls the *higher derivatives*, i.e., the powers of the exterior derivative of the given form ω and their products with ω itself.

A couple of years later Cartan investigated a more general problem, which apparently had been considered only once earlier, by O. Biermann [1], namely that of finding integrals for a (not completely integrable) system of several Pfaffian (first order) forms ω_i . Put differently, one is given a distribution \mathcal{D} of p -planes and wants to find "integral" manifolds of maximal possible dimension whose tangent planes at each point are contained in the plane of \mathcal{D} , in other words, such that the restriction of the given 1-forms to the manifold vanish. In his very innovative approach he first notices that the exterior derivatives $d\omega_i$ also restrict to 0 on an integral manifold (his description is difficult to read; using the language of his time, he says that at each point the integral element must belong to several "complexes", a complex in \mathbb{R}^n or projective space being defined by a skew symmetric bilinear form; the confusion with complex numbers is what made Weyl introduce the greek version "symplectic"), and accordingly introduces the notion of an integral element (consisting of a point and a subspace of the space assigned to the point by \mathcal{D} , i.e., on which the ω_i vanish, and on which also the $d\omega_i$ restrict to 0) and of increasing chains (flags) of such elements of dimensions 1, 2, ... at a point; the main strategy is then inductive, trying to extend an integral el-

ement and integral manifolds of some dimension to one of the next higher dimension. (He assumes that everything is real-analytic.)

Higher dimensional forms (without the name) had of course been around for a while in the guise of multiple integrals, extended over suitable surfaces, although usually one considered only a fixed surface with a mass or charge distribution, so that the presence of a form was not easily detected. There is a quite remarkable paper by Cartan [6] in which he (re)discovered integral geometry (obviously he hadn't heard of Crofton's work, e.g., expressing the length of a curve in the plane as an integral of the number of points of intersection with a variable straight line [9]). In it differential forms, as integrands of multiple integrals, defined in all of space and capable of being integrated over any "surface" (of the right dimension), are in the forefront. The manifolds in which he operates are those of interest for integral geometry: the (two-dimensional) space of lines in the plane, the spaces of lines or planes in 3-space (of dimension 4 or 3). Cartan finds the integrands that are invariant under the groups of motion in these spaces, and derives Crofton's formula and similar results.

The next big step comes with Poincaré. In his relatively early paper [26], whose main purpose is a discussion of residues of (complex) double integrals, he quite casually, with not much definition, introduces the general notion of a p -form for any p and immediately derives the "integrability conditions" (in essence the vanishing of the exterior derivative).

Integrability means for him that the integral of the form over a p -manifold depends only on the boundary of the manifold, and the integrability conditions are derived by considering a one-parameter family of manifolds with the same boundary.

In [27] he again takes up the matter. He states, again quite casually, that there is a general Stokes theorem, i.e., that to any p -form ω there is an associated $(p + 1)$ -form, defined by the integrability conditions (and which we now call the exterior derivative $d\omega$) such that the integral of the form over any closed p -manifold equals the integral of the associated form over any $(p + 1)$ -manifold that has the p -manifold as boundary. He states, again casually, that iterating this process "gives nothing" (in our terms, dd is 0), deduced either from formal computation or from the fact that for a form $d\psi$ the integral over any closed manifold vanishes (via Stokes's theorem for ψ), and so the integrability conditions hold for it.

Next comes, with more emphasis, but without proof or even any comment that a proof is necessary or possible, the converse: given a p -form ω whose integral over any closed manifold is 0, then there is a $(p - 1)$ -form, let's say ψ , that stands to ω in the relation described by Stokes's theorem (so that $\omega = d\psi$; he calls such an ω *exact*). Thus we have here the non-trivial half of what today one calls the *Poincaré Lemma*: ω is $d\psi$ for some ψ (ω is "exact") if and only if $d\omega = 0$ (ω is "closed"). (The trivial half is the relation $dd = 0$.) For $p = 1$ this is the Clairaut-Euler-Fontaine theorem about $Pdx + Qdy$ described earlier (note $d(Pdx + Qdy) = (Q_x - P_y)dxdy$).

Actually there are two versions here: The Lemma as now understood is local; ω is given in some neighborhood of a point, and ψ has to exist only in a smaller neighborhood of the point. For Poincaré, ω seems to have been defined in all of \mathbb{R}^n and he seems to assert the existence of ψ also in all of \mathbb{R}^n . That is a stronger statement; however the proofs for the two are pretty much identical.

The Lemma expresses an important property of differential forms; it can be regarded as the basis for their usefulness in topology and other places.

The presentation of all this is made more complicated and more difficult to understand by the fact that Poincaré's concern is actually integral invariants, i.e., differential forms that are invariant under the flow generated by a given vector field X (the interest

coming from mechanics); the converse of Poincaré's lemma occurs within an argument about integral invariants.

Apparently it took quite a while for people to realize that the converse does need a real proof. In [18, p. 335] Goursat “deduces” the existence of ψ with $d\psi = \omega$ if $d\omega$ vanishes by saying that this amounts to some differential equations for the coefficients of ψ and that $d\omega = 0$ means that “the integrability conditions are satisfied”. It is only in 1922 that both Cartan and Goursat in their books [2, p. 71] and [19, p. 105] state and give a detailed proof for the existence. NB: Goursat uses the term “exacte” for a form that today one calls *closed*, i.e., one whose exterior derivative vanishes.

Cartan's proof is for all of \mathbb{R}^n as domain. He adds that the result is not necessarily true for other domains. As an example he gives the case of a sphere $\sum_1^n x_i^2 = 1$ as “domain”, where for an $(n - 1)$ -form (automatically closed) one has an additional necessary condition, namely that the integral of the form over the sphere must be 0 (by Stokes: the integral of $d\psi$ over S^{n-1} equals the integral of ψ over the boundary of the sphere, which is empty, and so the integral is 0). This seems to be the first time that a global fact or a connection between differential forms and the topology of manifolds was noted; it is the first hint of De Rham's Theorem.

Some time after I had written all this, Ted Frankel pointed out to me a reference that he had gotten from de Rham's book [29, pp. 83 and 105], which changes the story radically. It brings in a new player, Vito Volterra, and the upshot is that Poincaré's Lemma is really Volterra's Theorem. His work is contained in several Notes published in the *Rendiconti* of the Accademia dei Lincei [30]. It came out of his idea of “funzione delle linee”, functions of curves (functionals), meaning (mainly real or complex valued) functions on the space of either all curves, closed or not, or all closed curves, and later on the space of either all (closed or not) or all closed r -dimensional manifolds (always with a given orientation), in \mathbb{R}^n . He seems to have had in mind embedded C^∞ -manifolds. Also, curves are allowed to have corners; for r -manifolds angles (locally of the type of two half-spaces with the same boundary $(r - 1)$ -space) are allowed. Continuity and differentiability of functionals is defined; the latter via infinitesimal variations of the manifold given by a vector field in \mathbb{R}^n defined on the manifold.

His interest in this topic came from physics, in particular electromagnetism: things like the force from a magnetic field or a current flowing through a wire.

In practice such functionals are of course mostly given by (multiple) integrals, but Volterra thought of them as “arbitrary” functions. However he singled out a special class: An r -functional Φ for closed r -manifolds is called *simple* or *of first degree*, if it satisfies the following additivity condition: Suppose two closed r -manifolds S_1, S_2 “overlap”, i.e., have in common a r -submanifold (with boundary) on which they induce opposite orientations (they might make an angle along the boundary of the submanifold). (Think of two spheres in 3-space that touch along a flattened disk.) Write S_{12} for the closed manifold obtained as the union of S_1 and S_2 with the common submanifold removed; then $\Phi(S_{12})$ equals $\Phi(S_1) + \Phi(S_2)$.

For such a Φ he defines a $(r + 1)$ -functional for arbitrary $(r + 1)$ -manifolds S , which one might well call $d\Phi$, by the Stokes rule $d\Phi(S) = \Phi(\delta S)$ (with δ meaning boundary) and then proves (it is not clear how rigorous the proofs are) that $d\Phi$ is what amounts to an $(r + 1)$ -form: $d\Phi(S)$ is given by an integral over S with skew-symmetric coefficients. For such systems of coefficients he immediately defines what amounts to the exterior derivative, states the Stokes Formula, and says that it can be established as in the standard case. He notes that $d\Phi$ is a closed form, i.e., that the coefficients satisfy the relevant integrability condition.

In the third of the notes [30] he states and proves explicitly, as Theorem 1, both parts of the Poincaré Lemma, of course in terms of the skew-symmetric coefficients of what would be a differential form and its exterior derivative; he never mentioned forms as such. The existence proof (for all of \mathbb{R}^n or some “rectangle” in it) is by induction on dimension, reducing it to the problem of writing a function f on \mathbb{R}^r as a divergence $\sum_1^r \partial g_i / \partial x_i$, which is trivial (one simply prescribes $r - 1$ of the g_i and initial values for the last one).

Thus Poincaré’s Lemma is really Volterra’s Theorem (unless some earlier author appears).

There is quite a lot more. First, since Volterra has shown already that $d\Phi$ “is” a closed $(r + 1)$ -form, it follows now that any simple Φ itself is an r -form. Next he defines the Hodge star operator $*$ (and the associated operators, which have become so very important in the theory of harmonic forms, Yang-Mills fields, etc.): it assigns to any r -form an $(n - r)$ -form by $*dx_{i_1} dx_{i_2} \cdots dx_{i_r} = dx_{j_1} dx_{j_2} \cdots dx_{j_{n-r}}$, where the i_u and the j_v together give an even permutation of $\{1, 2, \dots, n\}$. He does not mention forms of course, but writes it out in terms of a system of coefficients. Then comes the “co- d ” operator d^* or δ , defined as $*d*$ (up to a sign), again not introduced explicitly, but written down within a corollary to Theorem 1 as the operation on the system of coefficients (corresponding to an $(r + 1)$ -form) yielding $\sum_{i_t} \partial a_{i_1 i_2 \dots i_r i_t} / \partial x_{i_t}$, and the “Poincaré Lemma” for this operator. (There is a small mistake in his formula; he has the sum over t , which would give 0 by skewness.) Then come Theorems 2 and 3, which say in effect $d d^* + d^* d = \Delta$ (i.e., equal to the standard Laplacean in \mathbb{R}^n), which relation has become important in the theory of harmonic forms. All quite amazing.

His reason for developing all this is his idea of generalizing the notion of harmonic functions and of conjugate functions from the theory of functions of a complex variable. In effect he calls two forms π, κ of degrees $r - 1$ and $n - r - 1$ *conjugate*, if $*d\pi = d\kappa$. It follows at once that then κ and $-\pi$ are also conjugate, and that both $d\pi$ and $d\kappa$ are closed ($d = 0$), coclosed ($d^* = 0$), and harmonic ($\Delta = 0$). It also follows that, given a harmonic r -form μ , the two forms $d * \mu$ and $*d\mu$ are conjugate (up to some sign). He then proves the converse: two conjugate forms always come in this way from a harmonic form. It doesn’t seem to follow that π and κ are themselves harmonic. He thought this generalization of the notion of conjugate functions very interesting, but I don’t know what came of it.

A note on Volterra’s life, as reported in his obituary in [30]: In 1931 he refused to take the loyalty oath that the Fascists in Italy required of all state employees. He was dismissed from his position and gradually had to relinquish all his other offices and activities. He died in Rome in 1940. In 1943 an SS detachment appeared at his house, with orders to arrest him, for transport to one of the German concentration-extinction camps in Eastern Europe. . .

Next in the story comes a quantum jump: de Rham’s thesis. Apparently in the meantime Cartan and maybe others had formed the idea that there was indeed a connection between forms and topology. In [3] Cartan explicitly stated, and in [28] de Rham proved (among other things) the basic conjecture, now known as de Rham’s theorem: In a closed (sufficiently) differentiable manifold a p -form whose integral over all p -cycles vanishes is exact, i.e., is an exterior derivative, and, given n independent p -cycles z_i and n real numbers r_i , there exists a p -form whose integral over z_i is r_i . Thus de Rham cohomology, the vector space of closed forms modulo exact forms is the dual or transpose of real homology, the vector space of cycles (with real coefficients) modulo boundaries.)

This is the beginning of the “modern” phase where the appearance of differential forms in topology is ubiquitous, and is the end of our story.

Two more remarks:

1. It is probably not widely known that de Rham in his thesis also computed the compact cohomology of \mathbb{R}^n (which uses only forms with compact support, i.e., 0 outside some compact set), with the result that it is 0 in all dimensions less than n , and that an n -form with compact support is derivative of an $(n - 1)$ -form of compact support exactly if its integral over \mathbb{R}^n is 0; [28, Lemme II, p. 170 = p. 56].
2. Although de Rham's thesis topic came from E. Cartan's investigations, his thesis advisor was H. Lebesgue or at any rate the thesis is dedicated to Lebesgue. The "Rapport sur la Thèse" is by E. Cartan.

As a final note we must mention an area that in a sense anticipated, by a considerable time span, the theory of differential forms and that only fairly recently was recognized as an equivalent of the theory of forms for the case of Euclidean 3-space (with its customary metric), namely the vector calculus, developed by Stokes, Maxwell, and others, with its "·" and "×" products, its differential operators grad, curl, and div, which correspond to the exterior derivative d on 0-, 1-, and 2-forms), its identities, and its basic theorems (Gauss's divergence theorem, Stokes's theorem for surfaces, and $\int_a^b f'(x) dx = f(b) - f(a)$ (one half of the fundamental theorem of Calculus), which are instances of "the" Stokes theorem for differential forms.

REFERENCES

1. O. Biermann, Ueber n simultane Differentialgleichungen der Form $\sum X_\mu dx_\mu = 0$, *Schlömilch's Zeitschrift für Mathematik = Zeitschrift für Mathematik und Physik* XXX (1885).
2. E. Cartan, *Leçons sur les invariants intégraux*, Hermann & fils, Paris 1922.
3. E. Cartan, Sur les nombres de Betti des groupes de Lie clos, *C. R. Acad. Sci. Paris* **187** (1928) 196–197.
4. E. Cartan, Sur certaines expressions différentielles et le problème de Pfaff, *Ann. Ec. Normale* **16** (1899) 239–332.
5. E. Cartan, Sur l'intégration des systèmes d'équations aux différentielles totales, *Ann. Ec. Normale* **18** (1901) 241–311.
6. E. Cartan, Le principe de dualité et certaines intégrales multiples de l'espace tangentiel et de l'espace réglé, *Bull. Soc. Math. France* **24** (1896) 140–177.
7. A.C. Clairaut, Recherches Générales sur le Calcul Intégral, *Mém. de math. et phys., Acad. Royale des Sciences* (1739), pp. 425–436; Sur l'Intégration ou la Construction des Équations différentielles du première ordre, *ibid* (1740) pp. 293–323.
8. A. Clebsch, Ueber die simultane Integration linearer partieller Differentialgleichungen, *J. Reine Angew. Math. (Crelle)* **65** (1866) 257–268.
9. M.W. Crofton, On the Theory of Local Probability, applied to Straight Lines drawn at random in the plane, etc., *Phil. Trans. Royal Soc.* **158** (1868) 181–199.
10. G. Darboux, Sur le problème de Pfaff, *Bull. Sci. Math. 2nd series* VI (1882) 14–36, 49–68.
11. F. Deahna, Ueber die Bedingungen der Integrabilität . . . , *J. Reine Angew. Math.* **20** (1840) 340–350.
12. H. Burckhardt et al., *Enzyklopädie der Mathematischen Wissenschaften*, vol. 2, Analysis, part 3, second half, Teubner, Leipzig 1922–1927 p. 1125.
13. L. Euler, Calculi Integralis III, pt. I, *Acad. Scient. Borussiae*, St Petersburg (1770), p. 5 = *Opera Omnia*, series 1, vol. XIII, Teubner, Leipzig and Berlin, 1914, p. 9.
14. L. Euler, De Infinitis Curvis etc, *Commentarii Acad. Sci. Petropolitanae* 7 (1734, 1735 [printed 1740]) 174–183, 184–200.
15. A. Fontaine, *Mémoires de mathématiques recueillis et publiés avec quelques pièces inédites*, Paris 1764.
16. A. Fontaine, *Mémoires données à l'Acad. royale des Sciences, non imprimés dans leur temps*, Paris 1764.
17. G. Frobenius, Über das Pfaffsche Problem, *J. Reine Angew. Math.* (1875) 230–315 = *Gesammelte Abhandlungen*, Springer, Berlin, 1966.
18. E. Goursat, Sur les invariants intégraux, *J. Math. Pures Appl.* IV (1908) 331–365.
19. E. Goursat, *Leçons sur le Problème de Pfaff*, Hermann, Paris, 1922.
20. H. Grassmann, *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik*, O. Wigand, Leipzig, 1844.

21. C. Jacobi, Über die Integration der partiellen Differentialgleichungen erster Ordnung and Über die Pfaff'sche Methode, eine gewöhnliche lineare Differentialgleichung ... *J. Reine Angew. Math. (Crelle)* 2 (1827), 317–329, 347–357 = *Gesammelte Werke*, G.Reimer, Berlin, IV, 1–15, 17–29.
22. S. Lie, Theorie des Pfaff'schen Problems, *Arch. for Math. og Nat.* II (1877) 338–378.
23. F. Moigno, *Leçons de calcul différentiel et intégral* ... , Bachelier, Paris, 1844.
24. L. Natani, Ueber totale und partielle Differentialgleichungen, *J. Reine Angew. Math.* 58 (1861) 301–328.
25. J. Pfaff, Methodus Generalis ... , *Königl. Preuss. Akad. der Wissensch. Berlin* (1814-1815) 76–136.
26. H. Poincaré, Sur les residus des intégrales doubles, *Acta Math.* 9 (1886/7) 321–380.
27. H. Poincaré, *Les Méthodes nouvelles da la Mécanique céleste*, vol. 3, Gauthier-Villars, Paris, 1899, pp. 9–15 = *New Methods of Celestial Mechanics, History of Modern Physics and Astronomy*, American Institute of Physics, 3 (1993) 729-734 = *New Methods of Celestial Mechanics*, 3, translated from the French, NASA TT I-452, vol. 3 (1966), pp. 9–17.
28. G. de Rham, Sur l'Analysis situs des variétés à n dimensions, *J. Math. Pures Appl.* X (1931) 115–200 = *Œuvres Mathématiques*, L'Enseignement Mathématique, Université de Genève, 1981, pp. 25–100.
29. G. de Rham, *Differentiable Manifolds*, Grundlehren der mathematischen Wissenschaften 266, Springer-Verlag, Berlin, 1986.
30. V. Volterra, Delle variabili complesse negli iperspazii, *Rend. Accad. dei Lincei*, ser. IV, vol. V₁ (1889₁), Nota I, pp. 158–165, Nota II, pp. 291–299; Sulle funzione conjugate, *ibidem*, ser. I V^a, vol. V₁ (1889₁), p. 158–169 = *Opere Matematiche*, Accad. Nazion. dei Lincei, Roma (1954), vol. 1 , pp. 403–410, 411–419, 420–432.

HANS SAMELSON received his Dr. sc. math. in 1940 in Zürich at the ETH under Heinz Hopf. In 1941 he emigrated to the United States of America. After a year at the Institute for Advanced Study he held various positions, first as instructor at the University of Wyoming, then at Syracuse University, on the The University of Michigan, and finally settled at Stanford University, where he as Professor Emeritus is now resident mathematical fossil. His main interests have been and are topology and Lie groups. He enjoys playing bassoon, recorder, and CD.

Stanford University, Stanford, CA 94305
samelson@math.stanford.edu