Geometry and Topology I

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Contents

1 Manifolds

A manifold is a space which looks like $\mathbb{R}^n$ at small scales (i.e. "locally"), but which may be very different from this at large scales (i.e. "globally"). In other words, manifolds are made by gluing pieces of $\mathbb{R}^n$ together to make a more complicated whole. We want to make this precise.

1.1 Topological manifolds

Definition 1.1. A real, n-dimensional *topological manifold* is a Hausdorff, second countable topological space which is locally homeomorphic to $\mathbb{R}^n$.

"Locally homeomorphic to $\mathbb{R}^n$" simply means that each point p has an open neighbourhood U for which we can find a homeomorphism $\varphi : U \longrightarrow V$ to an open subset $V \in \mathbb{R}^n$. Such a homeomorphism φ is called a *coordinate chart* around p. A collection of charts which cover the manifold is called an *atlas*.

Remark 1.2. Without the Hausdorff assumption, we would have examples such as the following: take the disjoint union $\mathbb{R}_1 \sqcup \mathbb{R}_2$ of two copies of the real line, and form the quotient by the equivalence relation

$$\mathbb{R}_1 \setminus \{0\} \ni x \sim \varphi(x) \in \mathbb{R}_2 \setminus \{0\}, \tag{1}$$

where φ is the identification $\mathbb{R}_1 \to \mathbb{R}_2$. The resulting quotient topological space is locally homeomorphic to $\mathbb{R}$ but the points $[0 \in \mathbb{R}_1], [0 \in \mathbb{R}_2]$ cannot be separated by open neighbourhoods.

Second countability is not as crucial, but will be necessary for the proof of the Whitney embedding theorem, among other things.

We now give examples of topological manifolds. The simplest is, technically, the empty set. Then we have a countable set of points (with the discrete topology), and $\mathbb{R}^n$ itself, but there are more:

Example 1.3 (Circle)**.** Define the circle $S^1 = \{z \in \mathbb{C} \ : \ |z| = 1\}$. Then for any fixed point $z \in S^1$, write it as $z = e^{2\pi i c}$ for a unique real number $0 \leq c < 1$, and define the map

$$\begin{aligned} \mathbb{R} &\xrightarrow{\tilde{\nu}_z} S^1 \\ t &\longmapsto e^{2\pi i t} \end{aligned} \tag{2}$$

Let $I_c = (c - \frac{1}{2}, c + \frac{1}{2})$, and note that $\nu_z = \tilde{\nu}_z|_{I_c}$ is a homeomorphism from I_c to the neighbourhood of z given by $S^1 \backslash \{-z\}$. Then $\varphi_z = \nu_z^{-1}$ is a coordinate chart near z.

By taking products of coordinate charts, we obtain charts for the Cartesian product of manifolds. Hence the Cartesian product is a manifold.

Example 1.4 (n-torus)**.** $S^1 \times \cdots \times S^1$ is a topological manifold (of dimension given by the number n of factors), with charts $\{\varphi_{z_1} \times \cdots \times \varphi_{z_n} \ : \ z_i \in S^1\}$.

Example 1.5 (open subsets)**.** Any open subset $U \subset M$ of a topological manifold is also a topological manifold, where the charts are simply restrictions $\varphi|_U$ of charts φ for M. For instance, the real $n \times n$ matrices $\mathrm{Mat}(n, \mathbb{R})$ form a vector space isomorphic to $\mathbb{R}^{n^2}$, and contain an open subset

$$GL(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n, \mathbb{R}) \ : \ \det A \neq 0\}, \tag{3}$$

known as the general linear group, which is a topological manifold.

Example 1.6 (Spheres)**.** The n-sphere is defined as the subspace of unit vectors in $\mathbb{R}^{n+1}$:

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \ : \ \sum x_i^2 = 1\}.$$

Let $N = (1, 0, \ldots, 0)$ be the north pole and let $S = (-1, 0, \ldots, 0)$ be the south pole in S^n. Then we may write S^n as the union $S^n = U_N \cup U_S$, where $U_N = S^n \backslash \{S\}$ and $U_S = S^n \backslash \{N\}$ are equipped with coordinate charts φ_N, φ_S into $\mathbb{R}^n$, given by the "stereographic projections" from the points S, N respectively

$$\varphi_N : (x_0, \vec{x}) \mapsto (1 + x_0)^{-1}\vec{x}, \tag{4}$$

$$\varphi_S : (x_0, \vec{x}) \mapsto (1 - x_0)^{-1}\vec{x}. \tag{5}$$

Remark 1.7. We have endowed the sphere S^n with a certain topology, but is it possible for another topological manifold $\tilde{S}^n$ to be homotopy equivalent to S^n *without* being homeomorphic to it? The answer is no, and this is known as the topological Poincaré conjecture, and is usually stated as follows: any homotopy n-sphere is homeomorphic to the n-sphere. It was proven for $n > 4$ by Smale, for $n = 4$ by Freedman, and for $n = 3$ is equivalent to the smooth Poincaré conjecture which was proved by Hamilton-Perelman. In dimensions $n = 1, 2$ it is a consequence of the classification of topological 1- and 2-manifolds.

Example 1.8 (Projective spaces)**.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $\mathbb{K}P^n$ is defined to be the space of lines through $\{0\}$ in $\mathbb{K}^{n+1}$, and is called the projective space over $\mathbb{K}$ of dimension n.

More precisely, let $X = \mathbb{K}^{n+1} \backslash \{0\}$ and define an equivalence relation on X via $x \sim y$ iff $\exists \lambda \in \mathbb{K}^* = \mathbb{K} \backslash \{0\}$ such that $\lambda x = y$, i.e. x, y lie on the same line through the origin. Then

$$\mathbb{K}P^n = X/\sim,$$

and it is equipped with the quotient topology.

The projection map $\pi : X \longrightarrow \mathbb{K}P^n$ is an *open* map, since if $U \subset X$ is open, then tU is also open $\forall t \in \mathbb{K}^*$, implying that $\cup_{t \in \mathbb{K}^*} tU = \pi^{-1}(\pi(U))$ is open, implying $\pi(U)$ is open. This immediately shows, by the way, that $\mathbb{K}P^n$ is second countable.

To show $\mathbb{K}P^n$ is Hausdorff (which we must do, since Hausdorff is preserved by subspaces and products, but *not* quotients), we show that the graph of the equivalence relation is closed in $X \times X$ (this, together with the openness of π, gives us the Hausdorff property for $\mathbb{K}P^n$). This graph is simply

$$\Gamma_\sim = \{(x, y) \in X \times X \ : \ x \sim y\},$$

and we notice that $\Gamma_\sim$ is actually the common zero set of the following continuous functions

$$f_{ij}(x,y) = (x_i y_j - x_j y_i) \quad i \neq j.$$

An atlas for $\mathbb{K}P^n$ is given by the open sets $U_i = \pi(\tilde{U}_i)$, where

$$\tilde{U}_i = \{(x_0, \dots, x_n) \in X \ : \ x_i \neq 0\},$$

and these are equipped with charts to $\mathbb{K}^n$ given by

$$\varphi_i([x_0, \dots, x_n]) = x_i^{-1}(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \tag{6}$$

which are indeed invertible by $(y_1, \dots, y_n) \mapsto (y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n)$.

Sometimes one finds it useful to simply use the "coordinates" $(x_0, \dots, x_n)$ for $\mathbb{K}P^n$, with the understanding that the x_i are well-defined only up to overall rescaling. This is called using "projective coordinates" and in this case a point in $\mathbb{K}P^n$ is denoted by $[x_0 : \dots : x_n]$.

Example 1.9 (Connected sum)**.** Let $p \in M$ and $q \in N$ be points in topological manifolds and let (U, φ) and (V, ψ) be charts around p, q such that $\varphi(p) = 0$ and $\psi(q) = 0$.

Choose ϵ small enough so that $B(0, 2\epsilon) \subset \varphi(U)$ and $B(0, 2\epsilon) \subset \varphi(V)$, and define the map of annuli

$$\begin{aligned} B(0,2\epsilon)\backslash\overline{B(0,\epsilon)} &\xrightarrow{\ \phi\ } B(0,2\epsilon)\backslash\overline{B(0,\epsilon)} \\ x &\longmapsto \tfrac{2\epsilon^2}{|x|^2}x \end{aligned} \tag{7}$$

This is a homeomorphism of the annulus to itself, exchanging the boundaries. Now we define a new topological manifold, called the *connected sum* $M\#N$, as the quotient $X/\sim$, where

$$X = (M\backslash\overline{\varphi^{-1}(B(0,\epsilon))}) \sqcup (N\backslash\overline{\psi^{-1}(B(0,\epsilon))}),$$

and we define an identification $x \sim \psi^{-1}\phi\varphi(x)$ for $x \in \varphi^{-1}(B(0, 2\epsilon))$. If $\mathcal{A}_M$ and $\mathcal{A}_N$ are atlases for M, N respectively, then a new atlas for the connect sum is simply

$$\mathcal{A}_M|_{M\backslash\overline{\varphi^{-1}(B(0,\epsilon))}} \cup \mathcal{A}_N|_{N\backslash\overline{\psi^{-1}(B(0,\epsilon))}}$$

Remark 1.10. The homeomorphism type of the connected sum of connected manifolds M, N is independent of the choices of p, q and φ, ψ, except that it may depend on the two possible orientations of the gluing map $\psi^{-1}\phi\varphi$. To prove this, one must appeal to the so-called *annulus theorem.*

Remark 1.11. By iterated connect sum of S^2 with T^2 and $\mathbb{R}P^2$, we can obtain all compact 2-dimensional manifolds.

Example 1.12. Let F be a topological space. A fiber bundle with fiber F is a triple (E, p, B), where E, B are topological spaces called the "total space" and "base", respectively, and $p : E \longrightarrow B$ is a continuous surjective map called the "projection map", such that, for each point $b \in B$, there is a neighbourhood U of b and a homeomorphism

$$\Phi : p^{-1}U \longrightarrow U \times F,$$

such that $p_U \circ \Phi = p$, where $p_U : U \times F \longrightarrow U$ is the usual projection. The submanifold $p^{-1}(b) \cong F$ is called the "fiber over b".

When B, F are topological manifolds, then clearly E becomes one as well. We will often encounter such manifolds.

Example 1.13 (General gluing construction)**.** To construct a topological manifold "from scratch", we glue open subsets of $\mathbb{R}^n$ together using homeomorphisms, as follows.

Begin with a countable collection of open subsets of $\mathbb{R}^n$: $\mathcal{A} = \{U_i\}$. Then for each i, we choose finitely many open subsets $U_{ij} \subset U_i$ and gluing maps

$$U_{ij} \xrightarrow{\varphi_{ij}} U_{ji} \ , \tag{8}$$

which we require to satisfy $\varphi_{ij}\varphi_{ji} = \mathrm{Id}_{U_{ji}}$, and such that $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all k, and most important of all, φ_{ij} must be *homeomorphisms*.

Next, we want the pairwise gluings to be consistent (transitive) and so we require that $\varphi_{ki}\varphi_{jk}\varphi_{ij} = \mathrm{Id}_{U_{ij} \cap U_{jk}}$ for all i, j, k. This will ensure that the equivalence relation in (10) is well-defined.

Second countability of the glued manifold is guaranteed since we started with a countable collection of opens, but the Hausdorff property is not necessarily satisfied without a further assumption: we require that the graph of φ_{ij}, namely

$$\{(x, \varphi_{ij}(x)) \ : x \in U_{ij}\} \tag{9}$$

is a closed subset of $U_i \times U_j$.

The final glued topological manifold is then

$$M = \frac{\bigsqcup U_i}{\sim}, \tag{10}$$

for the equivalence relation $x \sim \varphi_{ij}(x)$ for $x \in U_{ij}$, for all i, j. This space has a distinguished atlas $\mathcal{A}$, whose charts are simply the inclusions of the U_i in $\mathbb{R}^n$.

1.2 Smooth manifolds

Given coordinate charts (U_i, φ_i) and (U_j, φ_j) on a topological manifold, we can compare them along the intersection $U_{ij} = U_i \cap U_j$, by forming the map

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij}). \tag{11}$$

This is a homeomorphism, since it is a composition of homeomorphisms. In this sense, topological manifolds are glued together by homeomorphisms.

This means that we may be able to differentiate a function in one coordinate chart but not in another – there is no way to make sense of calculus on topological manifolds. This is why we introduce smooth manifolds, where the gluing maps are *smooth*.

Remark 1.14 (Aside on smooth maps of vector spaces)**.** Let $U \subset V$ be an open set in a finite-dimensional vector space, and let $f : U \longrightarrow W$ be a function with values in another vector space W. We say f is differentiable at $p \in U$ if there is a linear map $Df(p) : V \longrightarrow W$ which approximates f near p, meaning that

$$\lim_{\substack{x\to 0\\ x\neq 0}} \frac{||f(p+x) - f(p) - Df(p)(x)||}{||x||} = 0. \tag{12}$$

Notice that $Df(p)$ is uniquely characterized by the above property.

We have implicitly chosen inner products, and hence norms, on V and W in the above definition, though the differentiability of f is independent of this choice, since all norms are equivalent in finite dimensions. This is no longer true for infinite-dimensional vector spaces, where the norm or topology must be clearly specified and $Df(p)$ is required to be a continuous linear map. Most of what we do in this course can be developed in the setting of Banach spaces, i.e. complete normed vector spaces.

A basis for V has a corresponding dual basis $(x_1, \ldots, x_n)$ of linear functions on V, and we call these "coordinates". Similarly, let $(y_1, \ldots, y_m)$ be coordinates on W. Then the vector-valued function f has m scalar components $f_j = y_j \circ f$, and then the linear map $Df(p)$ may be written, relative to the chosen bases for V, W, as an $m \times n$ matrix, called the *Jacobian matrix* of f at p.

$$Df(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{13}$$

We say that f is differentiable in U when it is differentiable at all $p \in U$, and we say it is continuously differentiable when

$$Df : U \longrightarrow \mathrm{Hom}(V, W) \tag{14}$$

is continuous. The vector space of continuously differentiable functions on U with values in W is called $C^1(U, W)$.

Notice that the first derivative Df is itself a map from U to a vector space $\mathrm{Hom}(V, W)$, so if its derivative exists, we obtain a map

$$D^2 f : U \longrightarrow \mathrm{Hom}(V, \mathrm{Hom}(V, W)), \tag{15}$$

and so on. The vector space of k times continuously differentiable functions on U with values in W is called $C^k(U, W)$. We are most interested in C^∞ or "smooth" maps, all of whose derivatives exist; the space of these is denoted $C^\infty(U, W)$, and so we have

$$C^\infty(U, W) = \bigcap_k C^k(U, W). \tag{16}$$

Note: for a C^2 function, $D^2 f$ actually has values in a smaller subspace of $V^* \otimes V^* \otimes W$, namely in $\mathrm{Sym}^2(V^*) \otimes W$, since "mixed partials are equal".

Definition 1.15. A *smooth manifold* is a topological manifold equipped with an equivalence class of smooth atlases, as explained next.

Definition 1.16. An atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ for a topological manifold is called *smooth* when all gluing maps

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij}) \tag{17}$$

are smooth maps, i.e. lie in $C^\infty(\varphi_i(U_{ij}), \mathbb{R}^n)$. Two atlases $\mathcal{A}, \mathcal{A}'$ are *equivalent* if $\mathcal{A} \cup \mathcal{A}'$ is itself a smooth atlas.

Remark 1.17. Instead of requiring an atlas to be smooth, we could ask for it to be C^k, or real-analytic, or even holomorphic (this makes sense for a $2n$-dimensional topological manifold when we identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$). This is how we define C^k, real-analytic, and complex manifolds, respectively.

We may now verify that all the examples from §1.1 are actually smooth manifolds:

Example 1.18 (Circle)**.** For Example 1.3, only two charts, e.g. $\varphi_{\pm 1}$, suffice to define an atlas, and we have

$$\varphi_{-1} \circ \varphi_1^{-1} = \begin{cases} t+1 & -\frac{1}{2} < t < 0 \\ t & 0 < t < \frac{1}{2}, \end{cases} \tag{18}$$

which is clearly C^∞. In fact all the charts φ_z are smoothly compatible. Hence the circle is a smooth manifold.

The Cartesian product of smooth manifolds inherits a natural smooth structure from taking the Cartesian product of smooth atlases. Hence the n-torus, for example, equipped with the atlas we described in Example 1.4, is smooth. Example 1.5 is clearly defining a smooth manifold, since the restriction of a smooth map to an open set is always smooth.

Example 1.19 (Spheres)**.** The charts for the n-sphere given in Example 1.6 form a smooth atlas, since

$$\varphi_N \circ \varphi_S^{-1} : \vec{z} \mapsto \tfrac{1-x_0}{1+x_0}\vec{z} = \tfrac{(1-x_0)^2}{|\vec{x}|^2}\vec{z} = |\vec{z}|^{-2}\vec{z} \tag{19}$$

is a smooth map $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$, as required.

Example 1.20 (Projective spaces)**.** The charts for projective spaces given in Example 1.8 form a smooth atlas, since

$$\varphi_1 \circ \varphi_0^{-1}(z_1, \ldots, z_n) = (z_1^{-1}, z_1^{-1}z_2, \ldots, z_1^{-1}z_n), \tag{20}$$

which is smooth on $\mathbb{R}^n \backslash \{z_1 = 0\}$, as required, and similarly for all φ_i, φ_j.

The two remaining examples were constructed by gluing: the connected sum in Example 1.9 is clearly smooth since ϕ is a smooth map, and any topological manifold from Example 1.13 will be endowed with a natural smooth atlas as long as the gluing maps φ_{ij} are chosen to be C^∞.

1.3 Manifolds with boundary

Manifolds with boundary relate manifolds of different dimension. Since manifolds are not defined as subsets of another topological space, the notion of boundary is not the usual one from point set topology. To introduce boundaries, we change the local model for manifolds to

$$H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_n \geq 0\}, \tag{21}$$

with the induced topology from $\mathbb{R}^n$.

Definition 1.21. A topological manifold with boundary M is a second countable Hausdorff topological space which is locally homeomorphic to H^n. Its *boundary* ∂M is the $(n-1)$ manifold consisting of all points mapped to $x_n = 0$ by a chart, and its *interior* $\operatorname{Int} M$ is the set of points mapped to $x_n > 0$ by some chart. It follows that $M = \partial M \sqcup \operatorname{Int} M$.

A smooth structure on such a manifold *with boundary* is an equivalence class of smooth atlases, with smoothness as defined below.

Definition 1.22. Let V, W be finite-dimensional vector spaces, as before. A function $f : A \longrightarrow W$ from an arbitrary subset $A \subset V$ is smooth when it admits a smooth extension to an open neighbourhood $U_p \subset W$ of every point $p \in A$.

Example 1.23. The function $f(x, y) = y$ is smooth on H^2 but $f(x, y) = \sqrt{y}$ is not, since its derivatives do not extend to $y \leq 0$.

Remark 1.24. If M is an n-manifold with boundary, then $\operatorname{Int} M$ is a usual n-manifold (without boundary). Also, ∂M is an $n-1$-manifold without boundary. This is sometimes phrased as the equation

$$\partial^2 = 0. \tag{22}$$

Example 1.25 (Möbius strip)**.** Consider the quotient of $\mathbb{R} \times [0, 1]$ by the identification $(x, y) \sim (x + 1, 1 - y)$. The result E is a manifold with boundary. It is also a fiber bundle over S^1, via the map $\pi : [(x, y)] \mapsto e^{2\pi i x}$. The boundary, ∂E, is isomorphic to S^1, so this provides us with our first example of a non-trivial fiber bundle, since the trivial fiber bundle $S^1 \times [0, 1]$ has disconnected boundary.

1.4 Cobordism

Compact $(n+1)$-Manifolds with boundary provide us with a natural equivalence relation on compact n-manifolds, called *cobordism*.

Definition 1.26. Compact n-manifolds M_1, M_2 are *cobordant* when there exists N, a compact $n+1$-manifold with boundary, such that ∂N is isomorphic to the disjoint union $M_1 \sqcup M_2$. All manifolds cobordant to M form the *cobordism class* of M. We say that M is null-cobordant if $M = \partial N$ for N a compact $n + 1$–manifold with boundary.

Remark 1.27. It is important to assume compactness, otherwise all manifolds are null-cobordant, by taking Cartesian product with the noncompact manifold with boundary $[0, 1)$.

Let Ω^n be the set of cobordism classes of compact n-manifolds, including the empty set $\varnothing$. Using the disjoint union operation $[M_1] + [M_2] = [M_1 \sqcup M_2]$, we see that Ω^n is an abelian group with identity $[\varnothing]$.

The direct sum $\Omega^\bullet = \oplus_{n\geq 0}\Omega^n$ is then endowed with another operation,

$$[M_1] \cdot [M_2] = [M_1 \times M_2], \tag{23}$$

rendering $\Omega^\bullet$ into a commutative ring, called the *cobordism ring*. It has a multiplicative unit $[*]$, the class of the 0-manifold consisting of a single point. It is also graded by dimension.

Proposition 1.28. *The cobordism ring is 2-torsion, i.e.* $x + x = 0 \quad \forall x$.

Proof. For any manifold M, the manifold with boundary $M \times [0, 1]$ has boundary $M \sqcup M$. Hence $[M] + [M] = [\varnothing] = 0$, as required. ☐

Example 1.29. The n-sphere S^n is null-cobordant (i.e. cobordant to $\varnothing$), since $\partial \overline{B_{n+1}(0, 1)} \cong S^n$, where $B_{n+1}(0, 1)$ denotes the unit ball in $\mathbb{R}^{n+1}$.

Example 1.30. Any oriented compact 2-manifold is null-cobordant: we may embed it in $\mathbb{R}^3$ and the "inside" is a 3-manifold with boundary.

We now state an amazing theorem of Thom, which is a complete description of the cobordism ring of smooth compact n-manifolds.

Theorem 1.31. *The cobordism ring is a (countably generated) polynomial ring over* $\mathbb{F}_2$ *with generators in every dimension* $n \neq 2^k - 1$, *i.e.*

$$\Omega^\bullet = \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \ldots]. \tag{24}$$

This theorem implies that there are 3 cobordism classes in dimension 4, namely x_2^2, x_4, and $x_2^2 + x_4$. Can you find 4-manifolds representing these classes? Can you find *connected* representatives?

1.5 Smooth maps

For topological manifolds M, N of dimension m, n, the natural notion of morphism from M to N is that of a continuous map. A continuous map with continuous inverse is then a homeomorphism from M to N, which is the natural notion of equivalence for topological manifolds. Since the composition of continuous maps is continuous, we obtain a "category" of topological manifolds and continuous maps.

A category is a class of objects $\mathcal{C}$ (in our case, topological manifolds) and a class of arrows $\mathcal{A}$ (in our case, continuous maps). Each arrow goes from an object (the source) to another object (the target), meaning that there are "source" and "target" maps from $\mathcal{A}$ to $\mathcal{C}$:

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & \mathcal{C} \end{tikzcd} \tag{25}$$

Also, a category has an identity arrow for each object, given by a map $\mathrm{id} : \mathcal{C} \longrightarrow \mathcal{A}$ (in our case, the identity map of any manifold to itself). Furthermore, there is an associative composition operation on arrows.

Conventionally, we write the set of arrows from X to X as $\mathrm{Hom}(X, Y)$, i.e.

$$\mathrm{Hom}(X, Y) = \{a \in \mathcal{A} \ : \ s(a) = X \text{ and } t(a) = Y\}. \tag{26}$$

Then the associative composition of arrows mentioned above becomes a map

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z). \tag{27}$$

We have described the category of topological manifolds; we now describe the category of smooth manifolds by defining the notion of a smooth map.

Definition 1.32. A map $f : M \to N$ is called *smooth* when for each chart (U, φ) for M and each chart (V, ψ) for N, the composition $\psi \circ f \circ \varphi^{-1}$ is a smooth map, i.e. $\psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n)$.

The set of smooth maps (i.e. morphisms) from M to N is denoted $C^\infty(M, N)$. A smooth map with a smooth inverse is called a *diffeomorphism*.

Proposition 1.33. *If $g : L \to M$ and $f : M \to N$ are smooth maps, then so is the composition $f \circ g$.*

Proof. If charts φ, χ, ψ for L, M, N are chosen near $p \in L$, $g(p) \in M$, and $(fg)(p) \in N$, then $\psi \circ (f \circ g) \circ \varphi^{-1} = A \circ B$, for $A = \psi f \chi^{-1}$ and $B = \chi g \varphi^{-1}$ both smooth mappings $\mathbb{R}^n \to \mathbb{R}^n$. By the chain rule, $A \circ B$ is differentiable at p, with derivative $D_p(A \circ B) = (D_{g(p)}A)(D_p B)$ (matrix multiplication). $\square$

Now we have a new category, the category of smooth manifolds and smooth maps; two manifolds are considered isomorphic when they are diffeomorphic. In fact, the definitions above carry over, word for word, to the setting of manifolds with boundary. Hence we have defined another category, the category of smooth manifolds with boundary.

In defining the arrows for the category of manifolds with boundary, we may choose to consider all smooth maps, or only those smooth maps which send the boundary to the boundary, i.e. boundary-preserving maps.

The operation ∂ of "taking the boundary" sends a manifold with boundary to a usual manifold. Furthermore, if $\psi : M \to N$ is a boundary-preserving smooth map, then we can "take its boundary" by restricting it to the boundary, i.e. $\partial\psi = \psi|_{\partial M}$. Since ∂ takes objects to objects and arrows to arrows in a manner which respects compositions and identity maps, it is called a "functor" from the category of manifolds with boundary (and boundary-preserving smooth maps) to the category of smooth manifolds.

Example 1.34. Let φ_z be a chart for S^1, and let $j : S^1 \to \mathbb{C}$ be the inclusion map of S^1. We see that j is smooth since $j \circ \varphi^{-1}$ is the map

$$t \mapsto e^{2\pi i t} = (\cos(2\pi t), \sin(2\pi t)), \tag{28}$$

which is a smooth map from $I_c \subset \mathbb{R}$ to $\mathbb{R}^2$.

Example 1.35. The complex projective line $\mathbb{C}P^1$ is diffeomorphic to the 2-sphere S^2: consider the maps $f_+(x_0, x_1, x_2) = [1 + x_0 : x_1 + ix_2]$ and $f_-(x_0, x_1, x_2) = [x_1 - ix_2 : 1 - x_0]$. Since $f_\pm$ is continuous on $x_0 \neq \pm 1$, and since $f_- = f_+$ on $|x_0| < 1$, the pair (f_-, f_+) defines a continuous map $f : S^2 \longrightarrow \mathbb{C}P^1$. To check smoothness, we compute the compositions

$$\varphi_0 \circ f_+ \circ \varphi_N^{-1} : (y_1, y_2) \mapsto y_1 + iy_2, \tag{29}$$

$$\varphi_1 \circ f_- \circ \varphi_S^{-1} : (y_1, y_2) \mapsto y_1 - iy_2, \tag{30}$$

both of which are obviously smooth maps.

Example 1.36. The smooth inclusion $j : S^1 \to \mathbb{C}$ induces a smooth inclusion $j \times j$ of the 2-torus $T^2 = S^1 \times S^1$ into $\mathbb{C}^2$. The image of $j \times j$ does not include zero, so we may compose with the projection $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$ and the diffeomorphism $\mathbb{C}P^1 \to S^2$, to obtain a smooth map

$$\pi \circ (j \times j) : T^2 \to S^2. \tag{31}$$

Remark 1.37 (Exotic smooth structures)**.** The topological Poincaré conjecture, now proven, states that any topological manifold homotopic to the n-sphere is in fact homeomorphic to it. We have now seen how to put a differentiable structure on this n-sphere. Remarkably, there are other differentiable structures on the n-sphere which are not diffeomorphic to the standard one we gave; these are called *exotic* spheres.

Since the connected sum of spheres is homeomorphic to a sphere, and since the connected sum operation is well-defined as a smooth manifold, it follows that the connected sum defines a *monoid* structure on the set of smooth n-spheres. In fact, Kervaire and Milnor showed that for $n \neq 4$, the set of (oriented) diffeomorphism classes of smooth n-spheres forms a finite abelian group under the connected sum operation. This is not known to be the case in four dimensions. Kervaire and Milnor also compute the order of this group, and the first dimension where there is more than one smooth sphere is $n = 7$, in which case they show there are 28 smooth spheres, which we will encounter later on.

The situation for spheres may be contrasted with that for the Euclidean spaces: any differentiable manifold homeomorphic to $\mathbb{R}^n$ for $n \neq 4$ must be diffeomorphic to it. On the other hand, by results of Donaldson, Freedman, Taubes, and Kirby, we know that there are uncountably many non-diffeomorphic smooth structures on the topological manifold $\mathbb{R}^4$; these are called *fake* $\mathbb{R}^4$s.

Remark 1.38. The maps $\alpha : x \mapsto x$ and $\beta : x \mapsto x^3$ are both homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$. Each one defines, by itself, a smooth atlas on $\mathbb{R}$. These two smooth atlases are not compatible (why?), so they do not define the same smooth structure on $\mathbb{R}$. Nevertheless, the smooth structures are equivalent, since there is a diffeomorphism taking one to the other. What is it?

Example 1.39 (Lie groups)**.** A group is a set G with an associative multiplication $G \times G \xrightarrow{m} G$, an identity element $e \in G$, and an inversion map $\iota : G \longrightarrow G$, usually written $\iota(g) = g^{-1}$.

If we endow G with a topology for which G is a topological manifold and m, ι are continuous maps, then the resulting structure is called a *topological group*. If G is a given a smooth structure and m, ι are smooth maps, the result is a *Lie group*.

The real line (where m is given by addition), the circle (where m is given by complex multiplication), and their Cartesian products give simple but important examples of Lie groups. We have also seen the general linear group $GL(n, \mathbb{R})$, which is a Lie group since matrix multiplication and inversion are smooth maps.

Since $m : G \times G \longrightarrow G$ is a smooth map, we may fix $g \in G$ and define smooth maps $L_g : G \longrightarrow G$ and $R_g : G \longrightarrow G$ via $L_g(h) = gh$ and $R_g(h) = hg$. These are called *left multiplication* and *right multiplication*. Note that the group axioms imply that $R_g L_h = L_h R_g$.

2 The tangent bundle

The tangent bundle of a manifold is an absolutely central topic in differential geometry. In this section, we describe the tangent bundle intrinsically, without reference to any embedding of the manifold in a vector space. By way of motivation, however, we briefly discuss this case.

The definition of the tangent bundle is simplest for an open subset $U \subset V$ of a finite-dimensional vector space V. In this case, a tangent vector to a point $p \in U$ is simply a vector in V, and so the tangent bundle, which consists of all tangent vectors to all points in U, is simply given by

$$TU = U \times V. \tag{32}$$

The tangent bundle TU of U is then equipped with a projection map $\pi : TU \to U$, and a vector field on U is nothing but a *section* of this projection, i.e. a smooth map $X : U \to TU$ such that $\pi \circ X = \mathrm{id}_U$. We now investigate the problem of generalizing the tangent bundle to other manifolds, where the convenience of being an open set in a vector space is not available.

2.1 Submanifolds

There are several ways to define the notion of submanifold. We will use a definition which works for topological and smooth manifolds, based on the local model of inclusion of a vector subspace. These are sometimes called *regular* or *embedded* submanifolds.

Definition 2.1. A subspace $L \subset M$ of an m-manifold is called a submanifold of codimension k when each point $x \in L$ is contained in a chart (U, φ) for M such that

$$L \cap U = f^{-1}(0), \tag{33}$$

where f is the composition of φ with the projection $\mathbb{R}^m \to \mathbb{R}^k$ to the last k coordinates $(x_{m-k+1}, \ldots, x_m)$. A submanifold of codimension 1 is usually called a *hypersurface*.

Now suppose that $L \subset \mathbb{R}^m$ is a submanifold of codimension k, and let φ be a diffeomorphism which "rectifies" a neighbourhood $U \subset \mathbb{R}^n$ of a point $p \in L$, sending U to an open set in $\mathbb{R}^m$ in which the image of $L \cap U$ is a linear subspace, given by $x_{m-k+1} = \cdots = x_m = 0$. Then we say that $u \in \mathbb{R}^m$ is tangent to L at p when the derivative $D\varphi(p)$ takes u to that same linear subspace.

The tangent bundle TL of L is the set of all pairs (p, u), where $p \in L$ and $u \in \mathbb{R}^m$ is tangent to L at p. It is a subset of $T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m$, and it is itself a submanifold of $\mathbb{R}^{2m}$ of codimension $2k$.

2.2 General construction

The tangent bundle of an n-manifold M is a $2n$-manifold, called TM, naturally constructed in terms of M. As a set, it is fairly easy to describe, as simply the disjoint union of all tangent spaces. However we must explain precisely what we mean by the tangent space T_pM to $p \in M$.

Definition 2.2. Let $(U, \varphi), (V, \psi)$ be coordinate charts around $p \in M$. Let $u \in T_{\varphi(p)}\varphi(U)$ and $v \in T_{\psi(p)}\psi(V)$. Then the triples (U, φ, u), (V, ψ, v) are called equivalent when $D(\psi \circ \varphi^{-1})(\varphi(p)) : u \mapsto v$. The chain rule for derivatives $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ guarantees that this is indeed an equivalence relation.

The set of equivalence classes of such triples is called the tangent space to p of M, denoted T_pM. It is a real vector space of dimension $\dim M$, since both $T_{\varphi(p)}\varphi(U)$ and $T_{\psi(p)}\psi(V)$ are, and $D(\psi \circ \varphi^{-1})$ is a linear isomorphism.

As a set, the tangent bundle is defined by

$$TM = \bigsqcup_{p \in M} T_pM, \tag{34}$$

and it is equipped with a natural surjective map $\pi : TM \longrightarrow M$, which is simply $\pi(X) = x$ for $X \in T_xM$.

We now give it a manifold structure in a natural way.

Proposition 2.3. *For an n-manifold M, the set TM has a natural topology and smooth structure which make it a $2n$-manifold, and make $\pi : TM \longrightarrow M$ a smooth map.*

Proof. Any chart (U, φ) for M defines a bijection

$$T\varphi(U) \cong U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U) \tag{35}$$

via $(p, v) \mapsto (U, \varphi, v)$. Using this, we induce a smooth manifold structure on $\pi^{-1}(U)$, and view the inverse of this map as a chart $(\pi^{-1}(U), \Phi)$ to $\varphi(U) \times \mathbb{R}^n$.

given another chart (V, ψ), we obtain another chart $(\pi^{-1}(V), \Psi)$ and we may compare them via

$$\Psi \circ \Phi^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n, \tag{36}$$

which is given by $(p, u) \mapsto ((\psi \circ \varphi^{-1})(p), D(\psi \circ \varphi^{-1})_p u)$, which is smooth. Therefore we obtain a topology and smooth structure on all of TM (by defining W to be open when $W \cap \pi^{-1}(U)$ is open for every U in an atlas for M; all that remains is to verify the Hausdorff property, which holds since points x, y are either in the same chart (in which case it is obvious) or they can be separated by the given type of charts. $\square$

Remark 2.4. This is a more constructive way of looking at the tangent bundle: We choose a countable, locally finite atlas $\{(U_i, \varphi_i)\}$ for M and glue together $U_i \times \mathbb{R}^n$ to $U_j \times \mathbb{R}^n$ via an equivalence

$$(x, u) \sim (y, v) \Leftrightarrow y = \varphi_j \circ \varphi_i^{-1}(x) \text{ and } v = D(\varphi_j \circ \varphi_i^{-1})_x u, \tag{37}$$

and verify the conditions of the general gluing construction 1.13. The choice of a different atlas yields a canonically diffeomorphic manifold.

2.3 The derivative

A description of the tangent bundle is not complete without defining the derivative of a general smooth map of manifolds $f : M \longrightarrow N$. Such a map may be defined locally in charts (U_i, φ_i) for M and (V_α, ψ_α) for N as a collection of vector-valued functions $\psi_\alpha \circ f \circ \varphi_i^{-1} = f_{i\alpha} : \varphi_i(U_i) \longrightarrow \psi_\alpha(V_\alpha)$ which satisfy

$$(\psi_\beta \circ \psi_\alpha^{-1}) \circ f_{i\alpha} = f_{j\beta} \circ (\varphi_j \circ \varphi_i^{-1}). \tag{38}$$

Differentiating, we obtain

$$D(\psi_\beta \circ \psi_\alpha^{-1}) \circ Df_{i\alpha} = Df_{j\beta} \circ D(\varphi_j \circ \varphi_i^{-1}). \tag{39}$$

Equation 39 shows that $Df_{i\alpha}$ and $Df_{j\beta}$ glue together to define a map $TM \longrightarrow TN$. This map is called the derivative of f and is denoted $Df : TM \longrightarrow TN$. Sometimes it is called the "push-forward" of vectors and is denoted f_*. The map fits into the commutative diagram

$$\begin{tikzcd} TM \arrow[r, "Df"] \arrow[d, "\pi"'] & TN \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & N \end{tikzcd} \tag{40}$$

Each fiber $\pi^{-1}(x) = T_xM \subset TM$ is a vector space, and the map $Df : T_xM \longrightarrow T_{f(x)}N$ is a linear map. In fact, (f, Df) defines a homomorphism of vector bundles from TM to TN.

The usual chain rule for derivatives then implies that if $f \circ g = h$ as maps of manifolds, then $Df \circ Dg = Dh$. As a result, we obtain the following category-theoretic statement.

Proposition 2.5. *The mapping T which assigns to a manifold M its tangent bundle TM, and which assigns to a map $f : M \longrightarrow N$ its derivative $Df : TM \longrightarrow TN$, is a functor from the category of manifolds and smooth maps to itself*[1]*.*

For this reason, the derivative map Df is sometimes called the "tangent mapping" Tf.

[1] We can also say that it is a functor from manifolds to the category of smooth vector bundles.

2.4 Vector fields

A vector field on an open subset $U \subset V$ of a vector space V is what we usually call a vector-valued function, i.e. a function $X : U \to V$. If $(x_1, \ldots, x_n)$ is a basis for V^*, hence a coordinate system for V, then the constant vector fields dual to this basis are usually denoted in the following way:

$$\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right). \tag{41}$$

The reason for this notation is that we may identify a vector v with the operator of directional derivative in the direction v. We will see later that vector fields may be viewed as derivations on functions. A derivation is a linear map D from smooth functions to $\mathbb{R}$ satisfying the Leibniz rule $D(fg) = fDg + gDf$.

The tangent bundle allows us to make sense of the notion of vector field in a global way. Locally, in a chart (U_i, φ_i), we would say that a vector field X_i is simply a vector-valued function on U_i, i.e. a function $X_i : \varphi(U_i) \longrightarrow \mathbb{R}^n$. Of course if we had another vector field X_j on (U_j, φ_j), then the two would agree as vector fields on the overlap $U_i \cap U_j$ when $D(\varphi_j \circ \varphi_i^{-1}) : X_i \mapsto X_j$. So, if we specify a collection $\{X_i \in C^\infty(U_i, \mathbb{R}^n)\}$ which glue together on overlaps, it defines a global vector field.

Definition 2.6. A smooth vector field on the manifold M is a smooth map $X : M \longrightarrow TM$ such that $\pi \circ X = \mathrm{id}_M$. In words, it is a smooth assignment of a unique tangent vector to each point in M.

Such maps X are also called *cross-sections* or simply *sections* of the tangent bundle TM, and the set of all such sections is denoted $C^\infty(M, TM)$ or, better, $\Gamma^\infty(M, TM)$, to distinguish them from all smooth maps $M \longrightarrow TM$. The space vector fields is also sometimes denoted by $\mathfrak{X}(M)$.

Example 2.7. From a computational point of view, given an atlas $(\tilde{U}_i, \varphi_i)$ for M, let $U_i = \varphi_i(\tilde{U}_i) \subset \mathbb{R}^n$ and let $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$. Then a global vector field $X \in \Gamma^\infty(M, TM)$ is specified by a collection of vector-valued functions

$$X_i : U_i \longrightarrow \mathbb{R}^n, \tag{42}$$

such that

$$D\varphi_{ij}(X_i(x)) = X_j(\varphi_{ij}(x)) \tag{43}$$

for all $x \in \varphi_i(\tilde{U}_i \cap \tilde{U}_j)$. For example, if $S^1 = U_0 \sqcup U_1 / \sim$, with $U_0 = \mathbb{R}$ and $U_1 = \mathbb{R}$, with $x \in U_0 \backslash \{0\} \sim y \in U_1 \backslash \{0\}$ whenever $y = x^{-1}$, then $\varphi_{01} : x \mapsto x^{-1}$ and $D\varphi_{01}(x) : v \mapsto -x^{-2} v$. Then if we define (letting x be the standard coordinate along $\mathbb{R}$)

$$\begin{aligned} X_0 &= \frac{\partial}{\partial x} \\ X_1 &= -y^2 \frac{\partial}{\partial y}, \end{aligned}$$

we see that this defines a global vector field, which does not vanish in U_0 but vanishes to order 2 at a single point in U_1. Find the local expression in these charts for the rotational vector field on S^1 given in polar coordinates by $\frac{\partial}{\partial \theta}$.

Remark 2.8. While a vector $v \in T_pM$ is mapped to a vector $(Df)_p(v) \in T_{f(p)}N$ by the derivative of a map $f \in C^\infty(M,N)$, there is no way, in general, to transport a vector field X on M to a vector field on N. If f is invertible, then of course $Df \circ X \circ f^{-1} : N \to TN$ defines a vector field on N, which can be called f_*X, but if f is not invertible this approach fails.

Definition 2.9. We say that $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are f–related, for $f \in C^\infty(M,N)$, when the following diagram commutes

$$\begin{tikzcd} TM \arrow[r, "Df"] & TN \\ M \arrow[u, "X"] \arrow[r, "f"'] & N \arrow[u, "Y"'] \end{tikzcd} . \tag{44}$$

2.5 Local structure of smooth maps

In some ways, smooth manifolds are easier to produce or find than general topological manifolds, because of the fact that smooth maps have linear approximations. Therefore smooth maps often behave like linear maps of vector spaces, and we may gain inspiration from vector space constructions (e.g. subspace, kernel, image, cokernel) to produce new examples of manifolds.

In charts (U,φ), (V,ψ) for the smooth manifolds M,N, a smooth map $f : M \longrightarrow N$ is represented by a smooth map $\psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n)$. We shall give a general local classification of such maps, based on the behaviour of the derivative. The fundamental result which provides information about the map based on its derivative is the *inverse function theorem.*

Theorem 2.10 (Inverse function theorem)**.** *Let $U \subset \mathbb{R}^m$ an open set and $f : U \longrightarrow \mathbb{R}^m$ a smooth map such that $Df(p)$ is an invertible linear operator. Then there is a neighbourhood $V \subset U$ of p such that $f(V)$ is open and $f : V \longrightarrow f(V)$ is a diffeomorphism. furthermore, $D(f^{-1})(f(p)) = (Df(p))^{-1}$.*

Proof. Without loss of generality, assume that U contains the origin, that $f(0) = 0$ and that $Df(p) = \mathrm{Id}$ (for this, replace f by $(Df(0))^{-1} \circ f$. We are trying to invert f, so solve the equation $y = f(x)$ uniquely for x. Define g so that $f(x) = x + g(x)$. Hence $g(x)$ is the nonlinear part of f.

The claim is that if y is in a sufficiently small neighbourhood of the origin, then the map $h_y : x \mapsto y - g(x)$ is a contraction mapping on some closed ball; it then has a unique fixed point $\phi(y)$, and so $y - g(\phi(y)) = \phi(y)$, i.e. ϕ is an inverse for f.

Why is h_y a contraction mapping? Note that $Dh_y(0) = 0$ and hence there is a ball $B(0,r)$ where $||Dh_y|| \le \frac{1}{2}$. This then implies (mean value theorem) that for $x, x' \in B(0,r)$,

$$||h_y(x) - h_y(x')|| \le \tfrac{1}{2}||x - x'||.$$

Therefore h_y does look like a contraction, we just have to make sure it's operating on a complete metric space. Let's estimate the size of $h_y(x)$:

$$||h_y(x)|| \le ||h_y(x) - h_y(0)|| + ||h_y(0)|| \le \tfrac{1}{2}||x|| + ||y||.$$

Therefore by taking $y \in B(0, \frac{r}{2})$, the map h_y is a contraction mapping on $\overline{B(0,r)}$. Let $\phi(y)$ be the unique fixed point of h_y guaranteed by the contraction mapping theorem.

To see that ϕ is continuous (and hence f is a homeomorphism), we compute

$$\begin{aligned}||\phi(y) - \phi(y')|| &= ||h_y(\phi(y)) - h_{y'}(\phi(y'))|| \\ &\leq ||g(\phi(y)) - g(\phi(y'))|| + ||y - y'|| \\ &\leq \tfrac{1}{2}||\phi(y) - \phi(y')|| + ||y - y'||,\end{aligned}$$

so that we have $||\phi(y) - \phi(y')|| \leq 2||y - y'||$, as required.

To see that ϕ is differentiable, we guess the derivative $(Df)^{-1}$ and compute. Let $x = \phi(y)$ and $x' = \phi(y')$. For this to make sense we must have chosen r small enough so that Df is nonsingular on $\overline{B(0,r)}$, which is not a problem.

$$\begin{aligned}||\phi(y) - \phi(y') - (Df(x))^{-1}(y - y')|| &= ||x - x' - (Df(x))^{-1}(f(x) - f(x'))|| \\ &\leq ||(Df(x))^{-1}||||(Df(x))(x - x') - (f(x) - f(x'))||.\end{aligned}$$

Now note that $||(Df(x))^{-1}||$ is bounded and $||x - x'|| \leq 2||y - y'||$ as shown before. Dividing by $||y - y'||$, taking the limit $y \to y'$, and using the differentiability of f, we get that ϕ is differentiable, and with derivative $(Df)^{-1}$. That is,

$$D\phi = (Df)^{-1}. \tag{45}$$

Since inversion is C^∞, ϕ has as many derivatives as f, hence ϕ is C^∞. □

This theorem provides us with a local normal form for a smooth map with $Df(p)$ invertible: we may choose coordinates on sufficiently small neighbourhoods of $p, f(p)$ so that f is represented by the identity map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

In fact, the inverse function theorem leads to a normal form theorem for a more general class of maps:

Theorem 2.11 (Constant rank theorem)**.** *Let $f : M^m \to N^n$ be a smooth map such that Df has constant rank k in a neighbourhood of $p \in M$. Then there are charts (U, φ) and (V, ψ) containing $p, f(p)$ such that*

$$\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0). \tag{46}$$

Proof. Begin by choosing charts so that without loss of generality M is an open set in $\mathbb{R}^m$ and N is $\mathbb{R}^n$.

Since rk $Df = k$ at p, there is a $k \times k$ minor of $Df(p)$ with nonzero determinant. Reorder the coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$ so that this minor is top left, and translate coordinates so that $f(0) = 0$. label the coordinates $(x_1, \ldots, x_k, y_1, \ldots y_{m-k})$ on the domain and $(u_1, \ldots u_k, v_1, \ldots, v_{n-k})$ on the codomain.

Then we may write $f(x, y) = (Q(x, y), R(x, y))$, where Q is the projection to $u = (u_1, \ldots, u_k)$ and R is the projection to v. with $\frac{\partial Q}{\partial x}$ nonsingular. First we wish to put Q into normal form. Consider the map $\phi(x, y) = (Q(x, y), y)$, which has derivative

$$D\phi = \begin{pmatrix} \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \\ 0 & 1 \end{pmatrix} \tag{47}$$

As a result we see $D\phi(0)$ is nonsingular and hence there exists a local inverse $\phi^{-1}(x,y) = (A(x,y), B(x,y))$. Since it's an inverse this means $(x,y) = \phi(\phi^{-1}(x,y)) = (Q(A,B), B)$, which implies that $B(x,y) = y$.

Then $f \circ \phi^{-1} : (x,y) \mapsto (x, \tilde{R} = R(A,y))$, and must still be of rank k. Since its derivative is

$$D(f \circ \phi^{-1}) = \begin{pmatrix} I_{k\times k} & 0 \\ \frac{\partial \tilde{R}}{\partial x} & \frac{\partial \tilde{R}}{\partial y} \end{pmatrix} \tag{48}$$

we conclude that $\frac{\partial \tilde{R}}{\partial y} = 0$, meaning that

$$f \circ \phi^{-1} : (x,y) \mapsto (x, S(x)). \tag{49}$$

We now postcompose by the diffeomorphism $\sigma : (u,v) \mapsto (u, v - S(u))$, to obtain

$$\sigma \circ f \circ \phi^{-1} : (x,y) \mapsto (x,0), \tag{50}$$

as required. □

As we shall see, these theorems have many uses. One of the most straightforward uses is for defining submanifolds.

Proposition 2.12. *If $f : M \longrightarrow N$ is a smooth map of manifolds, and if $Df(p)$ has constant rank on M, then for any $q \in f(M)$, the inverse image $f^{-1}(q) \subset M$ is a regular submanifold.*

Proof. Let $x \in f^{-1}(q)$. Then there exist charts ψ, φ such that $\psi \circ f \circ \varphi^{-1}$: $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$ and $f^{-1}(q) \cap U = \{x_1 = \cdots = x_k = 0\}$. Hence we obtain that $f^{-1}(q)$ is a codimension k submanifold. □

Example 2.13. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be given by $(x_1, \ldots, x_n) \mapsto \sum x_i^2$. Then $Df(x) = (2x_1, \ldots, 2x_n)$, which has rank 1 at all points in $\mathbb{R}^n \backslash \{0\}$. Hence since $f^{-1}(q)$ contains $\{0\}$ iff $q = 0$, we see that $f^{-1}(q)$ is a regular submanifold for all $q \neq 0$. Exercise: show that this manifold structure is compatible with that obtained in Example 1.19.

The previous example leads to the following special case.

Proposition 2.14. *If $f : M \longrightarrow N$ is a smooth map of manifolds and $Df(p)$ has rank equal to $\dim N$ along $f^{-1}(q)$, then this subset $f^{-1}(q)$ is an embedded submanifold of M.*

Proof. Since the rank is maximal along $f^{-1}(q)$, it must be maximal in an open neighbourhood $U \subset M$ containing $f^{-1}(q)$, and hence $f : U \longrightarrow N$ is of constant rank. □

Definition 2.15. If $f : M \longrightarrow N$ is a smooth map such that $Df(p)$ is surjective, then p is called a *regular point*. Otherwise p is called a *critical point*. If all points in the level set $f^{-1}(q)$ are regular points, then q is called a *regular value*, otherwise q is called a critical value. In particular, if $f^{-1}(q) = \varnothing$, then q is regular.

It is often useful to highlight two classes of smooth maps; those for which Df is everywhere *injective*, or, on the other hand *surjective*.

Definition 2.16. A smooth map $f : M \longrightarrow N$ is called a *submersion* when $Df(p)$ is surjective at all points $p \in M$, and is called an *immersion* when $Df(p)$ is injective at all points $p \in M$. If f is an injective immersion which is a homeomorphism onto its image (when the image is equipped with subspace topology), then we call f an *embedding.*

Proposition 2.17. *If $f : M \longrightarrow N$ is an embedding, then $f(M)$ is a regular submanifold.*

Proof. Let $f : M \longrightarrow N$ be an embedding. Then for all $m \in M$, we have charts (U, φ), (V, ψ) where $\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)$. If $f(U) = f(M) \cap V$, we're done. To make sure that some other piece of M doesn't get sent into the neighbourhood, use the fact that $F(U)$ is open in the subspace topology. This means we can find a smaller open set $V' \subset V$ such that $V' \cap f(M) = f(U)$. Restricting the coordinates to V', we see that $f(M)$ is cut out by $(x_{m+1}, \ldots, x_n)$, where $n = \dim N$. $\square$

Example 2.18. If $\iota : M \longrightarrow N$ is an embedding of M into N, then $D\iota : TM \longrightarrow TN$ is also an embedding (hence so are $D^k\iota : T^kM \longrightarrow T^kN$), showing that TM is a submanifold of TN.

2.6 Smooth maps between manifolds with boundary

We may also use the constant rank theorem to study manifolds with boundary.

Proposition 2.19. *Let M be a smooth n-manifold and $f : M \longrightarrow \mathbb{R}$ a smooth and proper real-valued function, and let a, b, with $a < b$, be regular values of f. Then $f^{-1}([a, b])$ is a cobordism between the closed $n-1$-manifolds $f^{-1}(a)$ and $f^{-1}(b)$.*

Proof. The pre-image $f^{-1}((a, b))$ is an open subset of M and hence a submanifold. Since p is regular for all $p \in f^{-1}(a)$, we may (by the constant rank theorem) find charts such that f is given near p by the linear map

$$(x_1, \ldots, x_m) \mapsto x_m. \tag{51}$$

Possibly replacing x_m by $-x_m$, we therefore obtain a chart near p for $f^{-1}([a, b])$ into H^m, as required. Proceed similarly for $p \in f^{-1}(b)$. $\square$

Example 2.20. Using $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $(x_1, \ldots, x_n) \mapsto \sum x_i^2$, this gives a simple proof for the fact that the closed unit ball $\overline{B(0, 1)} = f^{-1}([-1, 1])$ is a manifold with boundary.

Example 2.21. Consider the C^∞ function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by $(x, y, z) \mapsto x^2 + y^2 - z^2$. Both $+1$ and -1 are regular values for this map, with pre-images given by 1- and 2-sheeted hyperboloids, respectively. Hence $f^{-1}([-1, 1])$ is a cobordism between hyperboloids of 1 and 2 sheets. In other words, it defines a cobordism between the disjoint union of two closed disks and the closed cylinder (each of which has boundary $S^1 \sqcup S^1$). Does this cobordism tell us something about the cobordism class of a connected sum?

Proposition 2.22. *Let $f : M \longrightarrow N$ be a smooth map from a manifold with boundary to the manifold N. Suppose that $q \in N$ is a regular value of f and also of $f|_{\partial M}$. Then the pre-image $f^{-1}(q)$ is a submanifold with boundary*[2]*. Furthermore, the boundary of $f^{-1}(q)$ is simply its intersection with ∂M.*

Proof. If $p \in f^{-1}(q)$ is not in ∂M, then as before $f^{-1}(q)$ is a submanifold in a neighbourhood of p. Therefore suppose $p \in \partial M \cap f^{-1}(q)$. Pick charts φ, ψ so that $\varphi(p) = 0$ and $\psi(q) = 0$, and $\psi f \varphi^{-1}$ is a map $U \subset H^m \longrightarrow \mathbb{R}^n$. Extend this to a smooth function $\tilde{f}$ defined in an open set $\tilde{U} \subset \mathbb{R}^m$ containing U. Shrinking $\tilde{U}$ if necessary, we may assume $\tilde{f}$ is regular on $\tilde{U}$. Hence $\tilde{f}^{-1}(0)$ is a submanifold of $\mathbb{R}^m$ of codimension n.

Now consider the real-valued function $\pi : \tilde{f}^{-1}(0) \longrightarrow \mathbb{R}$ given by the restriction of $(x_1, \ldots, x_m) \mapsto x_m$. $0 \in \mathbb{R}$ must be a regular value of π, since if not, then the tangent space to $\tilde{f}^{-1}(0)$ at 0 would lie completely in $x_m = 0$, which contradicts the fact that q is a regular point for $f|_{\partial M}$.

Hence, by Proposition 2.19, we have expressed $f^{-1}(q)$, in a neighbourhood of p, as a regular submanifold with boundary given by $\{\varphi^{-1}(x) : x \in \tilde{f}^{-1}(0) \text{ and } \pi(x) \geq 0\}$, as required. □

3 Transversality

We continue to use the constant rank theorem to produce more manifolds, except now these will be cut out only *locally* by functions. Globally, they are cut out by intersecting with another submanifold. You should think that intersecting with a submanifold locally imposes a number of constraints equal to its codimension.

The problem is that the intersection of submanifolds need not be a submanifold; this is why the condition of transversality is so important - it guarantees that intersections are smooth.

Two subspaces $K, L \subset V$ of a vector space V are *transverse* when $K + L = V$, i.e. every vector in V may be written as a (possibly non-unique) linear combination of vectors in K and L. In this situation one can easily see that $\dim V = \dim K + \dim L - \dim K \cap L$, or equivalently

$$\operatorname{codim}(K \cap L) = \operatorname{codim} K + \operatorname{codim} L. \tag{52}$$

We may apply this to submanifolds as follows:

Definition 3.1. Let $K, L \subset M$ be regular submanifolds such that every point $p \in K \cap L$ satisfies

$$T_pK + T_pL = T_pM. \tag{53}$$

Then K, L are said to be *transverse* submanifolds and we write $K \pitchfork L$.

Proposition 3.2. *If $K, L \subset M$ are transverse submanifolds, then $K \cap L$ is either empty, or a submanifold of codimension* $\operatorname{codim} K + \operatorname{codim} L$.

[2] i.e. locally modeled on the inclusion $H^k \subset H^n$ given by $(x_1, \ldots x_k) \mapsto (0, \ldots, 0, x_1, \ldots x_k)$.

Proof. Let $p \in K \cap L$. Then there is a neighbourhood U of p for which $K \cap U = f^{-1}(0)$ for 0 a regular value of a function $f : U \longrightarrow \mathbb{R}^{\mathrm{codim} K}$ and $L \cap U = g^{-1}(0)$ for 0 a regular value of a function $g : L \cap U \longrightarrow \mathbb{R}^{\mathrm{codim} L}$.

Then p must be a regular point for $(f, g) : L \cap M \cap U \longrightarrow \mathbb{R}^{\mathrm{codim} K + \mathrm{codim} L}$, since the kernel of its derivative is the intersection $\ker Df(p) \cap \ker Dg(p)$, which is exactly $T_pK \cap T_pL$, which has codimension $\mathrm{codim}K + \mathrm{codim}L$ by the transversality assumption, implying $D(f, g)(p)$ is surjective. Therefore $(f, g)|_{\tilde{U}}^{-1}(0, 0) = f^{-1}(0) \cap g^{-1}(0) = K \cap L \cap \tilde{U}$ is a submanifold. □

Example 3.3 (Exotic spheres)**.** Consider the following intersections in $\mathbb{C}^5 \backslash 0$:

$$S^7_k = \{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\} \cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}. \tag{54}$$

This is a transverse intersection, and for $k = 1, \ldots, 28$ the intersection is a smooth manifold homeomorphic to S^7. These exotic 7-spheres were constructed by Brieskorn and represent each of the 28 diffeomorphism classes on S^7.

We may choose to phrase the previous transversality result in a slightly different way, in terms of the embedding maps k, l for K, L in M. Specifically, we say the maps k, l are transverse in the sense that $\forall a \in K, b \in L$ such that $k(a) = l(b) = p$, we have $\mathrm{im}(Dk(a)) + \mathrm{im}(Dl(b)) = T_pM$. The advantage of this approach is that it makes sense for any maps, not necessarily embeddings.

Definition 3.4. Two maps $f : K \longrightarrow M$, $g : L \longrightarrow M$ of manifolds are called *transverse* when $\mathrm{im}(Df(a)) + \mathrm{im}(Dg(b)) = T_pM$ for all a, b, p such that $f(a) = g(b) = p$.

Proposition 3.5. *If $f : K \longrightarrow M$, $g : L \longrightarrow M$ are transverse smooth maps, then $K{}_f\times_g L = \{(a, b) \in K \times L \ : \ f(a) = g(b)\}$ is naturally a smooth manifold equipped with commuting maps*

$$\begin{tikzcd}
K \times L \arrow[drr, "p_2"] \arrow[ddr, "p_1"'] & & \\
& K{}_f\times_g L \arrow[ul, "i"'] \arrow[r] \arrow[d] \arrow[dr, "f\cap g"] & L \arrow[d, "g"] \\
& K \arrow[r, "f"'] & M
\end{tikzcd} \tag{55}$$

where i is the inclusion and $f \cap g : (a, b) \mapsto f(a) = g(b)$.

The manifold $K{}_f\times_g L$ of the previous proposition is called the *fiber product* of K with L over M, and is a generalization of the intersection product. It is often denoted simply by $K \times_M L$, when the maps to M are clear.

Proof. Consider the graphs $\Gamma_f \subset K \times M$ and $\Gamma_g \subset L \times M$. To impose $f(k) = g(l)$, we can take an intersection with the diagonal submanifold

$$\Delta = \{(k, m, l, m) \in K \times M \times L \times M\}. \tag{56}$$

Step 1. We show that the intersection $\Gamma = (\Gamma_f \times \Gamma_g) \cap \Delta$ is transverse. Let $f(k) = g(l) = m$ so that $x = (k, m, l, m) \in \Gamma$, and note that

$$T_x(\Gamma_f \times \Gamma_g) = \{((v, Df(v)), (w, Dg(w))),\ v \in T_k K, \quad w \in T_l L\} \tag{57}$$

whereas we also have

$$T_x(\Delta) = \{((v, m), (w, m))\ :\ v \in T_k K, \quad w \in T_l L, \quad m \in T_p M\} \tag{58}$$

By transversality of f, g, any tangent vector $m_i \in T_p M$ may be written as $Df(v_i) + Dg(w_i)$ for some (v_i, w_i), $i = 1, 2$. In particular, we may decompose a general tangent vector to $M \times M$ as

$$(m_1, m_2) = (Df(v_2), Df(v_2)) + (Dg(w_1), Dg(w_1)) + (Df(v_1 - v_2), Dg(w_2 - w_1)), \tag{59}$$

leading directly to the transversality of the spaces (57), (58). This shows that Γ is a submanifold of $K \times M \times L \times M$.

Step 2. The projection map $\pi : K \times M \times L \times M \to K \times L$ takes Γ bijectively to $K_f \times_g L$. Since (57) is a graph, it follows that $\pi|_\Gamma : \Gamma \to K \times L$ is an injective immersion. Since the projection π is an open map, it also follows that $\pi|_\Gamma$ is a homeomorphism onto its image, hence is an embedding. This shows that $K_f \times_g L$ is a submanifold of $K \times L$. $\square$

Example 3.6. If $K_1 = M \times Z_1$ and $K_2 = M \times Z_2$, we may view both K_i as "fibering" over M with fibers Z_i. If p_i are the projections to M, then $K_1 \times_M K_2 = M \times Z_1 \times Z_2$, hence the name "fiber product".

Example 3.7. Consider the Hopf map $p : S^3 \longrightarrow S^2$ given by composing the embedding $S^3 \subset \mathbb{C}^2 \backslash \{0\}$ with the projection $\pi : \mathbb{C}^2 \backslash \{0\} \longrightarrow \mathbb{C}P^1 \cong S^2$. Then for any point $q \in S^2$, $p^{-1}(q) \cong S^1$. Since p is a submersion, it is obviously transverse to itself, hence we may form the fiber product

$$S^3 \times_{S^2} S^3,$$

which is a smooth 4-manifold equipped with a map $p \cap p$ to S^2 with fibers $(p \cap p)^{-1}(q) \cong S^1 \times S^1$.

These are our first examples of nontrivial fiber bundles, which we shall explore later.

The following result is an exercise: just as we may take the product of a manifold with boundary K with a manifold without boundary L to obtain a manifold with boundary $K \times L$, we have a similar result for fiber products.

Proposition 3.8. *Let K be a manifold with boundary where L, M are without boundary. Assume that $f : K \longrightarrow M$ and $g : L \longrightarrow M$ are smooth maps such that both f and ∂f are transverse to g. Then the fiber product $K \times_M L$ is a manifold with boundary equal to $\partial K \times_M L$.*

3.1 Stability

Transversality is a stable condition. In other words, if transversality holds, it will continue to hold for any sufficiently small perturbation (of the submanifolds or maps involved). Not only is transversality *stable*, it is actually *generic*, meaning that even if it does not hold, it can be made to hold by a small perturbation. In a sense, stability says that transversal maps form an open set, and genericity says that this open set is dense in the space of maps. To make this precise, we would introduce a topology on the space of maps, something which we leave for another course.

Definition 3.9. We call a smooth map

$$F : M \times [0,1] \to N \tag{60}$$

a smooth homotopy from f_0 to f_1, where $f_t = F \circ j_t$ and $j_t : M \to M \times [0,1]$ is the embedding $x \mapsto (x, t)$.

Definition 3.10. A property of a smooth map $f : M \longrightarrow N$ is *stable* under perturbations when for any smooth homotopy f_t with $f_0 = f$, there exists an $\epsilon > 0$ such that the property holds for all f_t with $t < \epsilon$.

Proposition 3.11. *If M is compact, then the property of $f : M \to N$ being an immersion (or submersion) is stable under perturbations.*

Proof. If $f_t, t \in [0,1]$ is a smooth homotopy of the immersion f_0, then in any chart around the point $p \in M$, the derivative $Df_0(p)$ has a $m \times m$ submatrix with nonvanishing determinant, for $m = \dim M$. By continuity, this $m \times m$ submatrix must have nonvanishing determinant in a neighbourhood around $(p, 0) \in M \times [0,1]$. We can cover $M \times \{0\}$ by a finite number of such neighbourhoods, since M is compact. Choose ϵ such that $M \times [0, \epsilon)$ is contained in the union of these intervals, giving the result. The proof for submersions is identical. □

Corollary 3.12. *If K is compact and $f : K \to M$ is transverse to the closed submanifold $L \subset M$ (this just means that f is transverse to the embedding $\iota : L \to M$), then the transversality is stable under perturbations of f.*

Proof. Let $F : K \times [0,1] \to M$ be a homotopy with $f_0 = f$. We show that K has an open cover by neighbourhoods in which f_t is transverse for t in a small interval; we then use compactness to obtain a uniform interval.

First the points which do not intersect L: $F^{-1}(M \setminus L)$ is open in $K \times [0,1]$ and contains $(K \setminus f^{-1}(L)) \times \{0\}$. So, for each $p \in K \setminus f^{-1}(L)$, there is a neighbourhood $U_p \subset K$ of p and an interval $I_p = [0, \epsilon_p)$ such that $F(U_p \times I_p) \cap L = \varnothing$.

Now, the points which do intersect L: L is a submanifold, so for each $p \in f^{-1}(L)$, we can find a neighbourhood $V \subset M$ containing $f(p)$ and a submersion $\psi : V \to \mathbb{R}^l$ cutting out $L \cap V$. Transversality of f and L is then the statement that ψf is a submersion at p. This implies there is a neighbourhood $\tilde{U}_p$ of $(p, 0)$ in $K \times [0,1]$ where ψf_t is a submersion. Choose an open subset (containing $(p, 0)$) of the form $U_p \times I_p$, for $I_p = [0, \epsilon_p)$.

By compactness of K, choose a finite subcover of $\{U_p\}_{p \in K}$; the smallest ϵ_p in the resulting subcover gives the required interval in which f_t remains transverse to L. □

Remark 3.13. Transversality of two maps $f : M \to N$, $g : M' \to N$ can be expressed in terms of the transversality of $f \times g : M \times M' \to N \times N$ to the diagonal $\Delta_N \subset N \times N$. So, if M and M' are compact, we get stability for transversality of f, g under perturbations of both f and g.

Remark 3.14. Local diffeomorphism and embedding are also stable properties.

3.2 Sard's theorem

The fundamental idea which allows us to prove that transversality is a generic condition is a the theorem of Sard showing that critical values of a smooth map $f : M \longrightarrow N$ (i.e. points $q \in N$ for which the map f and the inclusion $\iota : q \hookrightarrow N$ fail to be transverse maps) are *rare*. The following proof is taken from Milnor, based on Pontryagin.

The meaning of "rare" will be that the set of critical values is of *measure zero*, which means, in $\mathbb{R}^m$, that for any $\epsilon > 0$ we can find a sequence of balls in $\mathbb{R}^m$, containing $f(C)$ in their union, with total volume less than ϵ. Some easy facts about sets of measure zero: the countable union of measure zero sets is of measure zero, the complement of a set of measure zero is dense.

We begin with an elementary lemma describing the behaviour of measure-zero sets under differentiable maps.

Lemma 3.15. *Let $I^m = [0,1]^m$ be the unit cube, and $f : I^m \longrightarrow \mathbb{R}^n$ a C^1 map. If $m < n$ then $f(I^m)$ has measure zero. If $m = n$ and $A \subset I^m$ has measure zero, then $f(A)$ has measure zero.*

Proof. If $f \in C^1$, its derivative is bounded on I^m, so for all $x, y \in I^m$ we have

$$||f(y) - f(x)|| \leq M||y - x||, \tag{61}$$

for a constant[3] $M > 0$ depending only on f. So, the image of a ball of radius r in I^m is contained in a ball of radius Mr, which has volume proportional to r^n.

If $A \subset I^m$ has measure zero, then for each ϵ we have a countable covering of A by balls of radius r_k with total volume $c_m \sum_k r_k^m < \epsilon$. We deduce that $f(A_i)$ is covered by balls of radius Mr_k with total volume $M^n c_n \sum_k r_k^n$; since $n \geq m$ this goes to zero as $\epsilon \to 0$. We conclude that $f(A)$ is of measure zero.

If $m < n$ then f defines a C^1 map $I^m \times I^{n-m} \longrightarrow \mathbb{R}^n$ by pre-composing with the projection map to I^m. Since $I^m \times \{0\} \subset I^m \times I^{n-m}$ clearly has measure zero, its image must also. $\square$

Remark 3.16. If we considered the case $n < m$, the resulting sum of volumes may be larger in $\mathbb{R}^n$. For example, the projection map $\mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $(x, y) \mapsto x$ clearly takes the set of measure zero $y = 0$ to one of positive measure.

A subset $A \subset M$ of a manifold is said to have measure zero when its image in each chart of an atlas has measure zero. Lemma 3.15, together with the fact that a manifold is second countable, implies that the property is independent of the choice of atlas, and that it is preserved under equidimensional maps:

[3] This is called a Lipschitz constant.

Corollary 3.17. *Let $f : M \to N$ be a C^1 map of manifolds where $\dim M = \dim N$. Then the image $f(A)$ of a set $A \subset M$ of measure zero also has measure zero.*

Corollary 3.18 (Baby Sard)**.** *Let $f : M \to N$ be a C^1 of manifolds where $\dim M < \dim N$. Then $f(M)$ (i.e. the set of critical values) has measure zero in N.*

Remark 3.19. Note that this implies that space-filling curves are not C^1.

Now we investigate the measure of the critical values of a map $f : M \to N$ where $\dim M = \dim N$. The set of critical points need not have measure zero, but we shall see that

> **The variation of f is constrained along its critical locus since this is where Df drops rank. In fact, the set of critical *values* has measure zero.**

Theorem 3.20 (Equidimensional Sard)**.** *Let $f : M \to N$ be a C^1 map of n-manifolds, and let $C \subset M$ be the set of critical points. Then $f(C)$ has measure zero.*

Proof. It suffices to show result for the unit cube mapping to Euclidean space. Let $f : I^n \longrightarrow \mathbb{R}^n$ a C^1 map, and let M be the Lipschitz constant for f on I^n, i.e.

$$||f(x) - f(y)|| \leq M||x - y||, \quad \forall x, y \in I^n. \tag{62}$$

Let c be a critical point, so that the image of $Df(c)$ is a proper subspace of $\mathbb{R}^n$. Choose a hyperplane containing this subspace, translate it to $f(c)$, and call it H. Then

$$d(f(x), H) \leq ||f(x) - (f(c) + Df(c)(x - c))||, \tag{63}$$

but by Taylor's theorem, this is bounded by $C||x - c||^2$, for a constant C, for all x in the compact set I^n.

If $||x - c|| \leq \epsilon$, then $f(x)$ is within a distance $C\epsilon^2$ from H and within a distance $M\epsilon$ of $f(c)$, so lies within a paralellepiped of volume

$$(2C\epsilon^2)(2M\epsilon)^{n-1}. \tag{64}$$

Now subdivide I^n into h^n cubes of edge length h^{-1} and apply the argument for each small cube, in which $||x - c|| \leq h^{-1}\sqrt{n}$. This gives a total volume for the image less than

$$(2^n C M^{n-1} n^{(n+1)/2} h^{-n-1})(h^n), \tag{65}$$

which is arbitrarily small as $h \to \infty$. ☐

The argument above will not work for $\dim N < \dim M$; we need more control on the function f. In particular, one can find a C^1 function $I^2 \longrightarrow \mathbb{R}$ which fails to have critical values of measure zero. (Hint: find a C^1 function $f : \mathbb{R} \to \mathbb{R}$ with critical values containing the Cantor set $C \subset [0, 1]$. Compose $f \times f$ with the sum $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and note that $C + C = [0, 2]$.) As a result, Sard's theorem in general requires more differentiability of f.

Theorem 3.21 (Big Sard's theorem)**.** *Let $f : M \longrightarrow N$ be a C^k map of manifolds of dimension m, n, respectively. Let C be the set of critical points. Then $f(C)$ has measure zero if $k > \frac{m}{n} - 1$.*

Proof. As before, it suffices to show for $f : I^m \longrightarrow \mathbb{R}^n$. We do an induction on m – note that the theorem holds for $m = 0$.

Define $C_1 \subset C$ to be the set of points x for which $Df(x) = 0$. Define $C_i \subset C_{i-1}$ to be the set of points x for which $D^j f(x) = 0$ for all $j \leq i$. So we have a descending sequence of closed sets:

$$C \supset C_1 \supset C_2 \supset \cdots \supset C_k. \tag{66}$$

We will show that $f(C)$ has measure zero by showing

1. $f(C_k)$ has measure zero,
2. each successive difference $f(C_i \backslash C_{i+1})$ has measure zero for $i \geq 1$,
3. $f(C \backslash C_1)$ has measure zero.

Step 1: For $x \in C_k$, Taylor's theorem gives the estimate

$$||f(x + t) - f(x)|| \leq c||t||^{k+1}, \tag{67}$$

where c depends only on I^m and f.

Subdivide I^m into h^m small cubes with edge h^{-1}; then any point in in the small cube I_0 containing x may be written as $x + t$ with $||t|| \leq h^{-1}\sqrt{m}$. As a result, $f(I_0)$ is contained by a cube of edge $ah^{-(k+1)}$, with $a = 2cm^{(k+1)/2}$ independent of the small cube size. At most h^m cubes are necessary to cover C_k, and their images have total volume less than

$$h^m(ah^{-(k+1)})^n = a^n h^{m-(k+1)n}. \tag{68}$$

Assuming that $k > \frac{m}{n} - 1$, this tends to 0 as we increase the number of cubes.

Step 2: For each $x \in C_i \backslash C_{i+1}$, $i \geq 1$, there is a $i + 1^{th}$ partial, say wlog $\partial^{i+1} f_1/\partial x_1 \cdots \partial x_{i+1}$, which is nonzero at x. Therefore the function

$$w(x) = \partial^i f_1/\partial x_2 \cdots \partial x_{i+1} \tag{69}$$

vanishes on C_i but its partial derivative $\partial w/\partial x_1$ is nonvanishing near x. Then

$$(w(x), x_2, \ldots, x_m) \tag{70}$$

forms an alternate coordinate system in a neighbourhood V around x by the inverse function theorem (the change of coordinates is of class C^k), and we have trapped C_i inside a hyperplane. The restriction of f to $w = 0$ in V is clearly critical on $C_i \cap V$ and so by induction on m we have that $f(C_i \cap V)$ has measure zero. Cover $C_i \setminus C_{i+1}$ by countably many such neighbourhoods V.

Step 3: Let $x \in C \backslash C_1$. Note that we won't necessarily be able to trap C in a hypersurface. But, since there is some partial derivative, wlog $\partial f_1/\partial x_1$, which is nonzero at x, so defining $w = f_1$, we have that

$$(w(x), x_2, \ldots, x_m) \tag{71}$$

is an alternative coordinate system in some neighbourhood V of x (the coordinate change is a diffeomorphism of class C^k). In these coordinates, the hyperplanes $w = t$ in the domain are sent into hyperplanes $y_1 = t$ in the codomain, and so f can be described as a family of maps f_t whose domain and codomain has dimension reduced by 1. Since $w = f_1$, the derivative of f in these coordinates can be written

$$Df = \begin{pmatrix} 1 & 0 \\ * & Df_t \end{pmatrix}, \tag{72}$$

and so a point $x' = (t, p)$ in V is critical for f if and only if p is critical for f_t. Therefore, the critical values of f consist of the union of the critical values of f_t on each hyperplane $y_1 = t$ in the codomain. Since the domain of f_t has dimension reduced by one, by induction it has critical values of measure zero. So the critical values of f intersect each hyperplane in a set of measure zero, and by Fubini's theorem this means they have measure zero. Cover $C \setminus C_1$ by countably many such neighbourhoods. □

Remark 3.22. Note that $f(C)$ is measurable, since it is the countable union of compact subsets (the set of critical values is not necessarily closed, but the set of critical points is closed and hence a countable union of compact subsets, which implies the same of the critical values.)

To show the consequence of Fubini's theorem directly, we can use the following argument. First note that for any covering of $[a, b]$ by intervals, we may extract a finite subcovering of intervals whose total length is $\leq 2|b - a|$. To see this, first choose a minimal subcovering $\{I_1, \ldots, I_p\}$, numbered according to their left endpoints. Then the total overlap is at most the length of $[a, b]$. Therefore the total length is at most $2|b - a|$.

Now let $B \subset \mathbb{R}^n$ be compact, so that we may assume $B \subset \mathbb{R}^{n-1} \times [a, b]$. We prove that if $B \cap P_c$ has measure zero in the hyperplane $P_c = \{x^n = c\}$, for any constant $c \in [a, b]$, then it has measure zero in $\mathbb{R}^n$.

If $B \cap P_c$ has measure zero, we can find a covering by open sets $R_c^i \subset P_c$ with total volume $< \epsilon$. For sufficiently small α_c, the sets $R_c^i \times [c - \alpha_c, c + \alpha_c]$ cover $B \cap \bigcup_{z \in [c - \alpha_c, c + \alpha_c]} P_z$ (since B is compact). As we vary c, the sets $[c - \alpha_c, c + \alpha_c]$ form a covering of $[a, b]$, and we extract a finite subcover $\{I_j\}$ of total length $\leq 2|b - a|$.

Let R_j^i be the set R_c^i for $I_j = [c - \alpha_c, c + \alpha_c]$. Then the sets $R_j^i \times I_j$ form a cover of B with total volume $\leq 2\epsilon|b - a|$. We can make this arbitrarily small, so that B has measure zero.

3.3 Brouwer's fixed point theorem

Corollary 3.23. *Let M be a compact manifold with boundary. There is no smooth map $f : M \longrightarrow \partial M$ leaving ∂M pointwise fixed. Such a map is called a smooth retraction of M onto its boundary.*

Proof. Such a map f must have a regular value by Sard's theorem, let this value be $y \in \partial M$. Then y is obviously a regular value for $f|_{\partial M} = \mathrm{Id}$ as well, so that $f^{-1}(y)$ must be a compact 1-manifold with boundary given by $f^{-1}(y) \cap \partial M$, which is simply the point y itself. Since there is no compact 1-manifold with a single boundary point, we have a contradiction. □

For example, this shows that the identity map $S^n \to S^n$ may not be extended to a smooth map $f : \overline{B(0,1)} \to S^n$.

Lemma 3.24. *Every smooth map of the closed n-ball to itself has a fixed point.*

Proof. Let $D^n = \overline{B(0,1)}$. If $g : D^n \to D^n$ had no fixed points, then define the function $f : D^n \to S^{n-1}$ as follows: let $f(x)$ be the point in S^{n-1} nearer to x on the line joining x and $g(x)$.

This map is smooth, since $f(x) = x + tu$, where

$$u = ||x - g(x)||^{-1}(x - g(x)), \tag{73}$$

and t is the positive solution to the quadratic equation $(x+tu)\cdot(x+tu) = 1$, which has positive discriminant $b^2 - 4ac = 4(1 - |x|^2 + (x \cdot u)^2)$. Such a smooth map is therefore impossible by the previous corollary. □

Theorem 3.25 (Brouwer fixed point theorem)**.** *Any continuous self-map of D^n has a fixed point.*

Proof. The Weierstrass approximation theorem says that any continuous function on $[0,1]$ can be uniformly approximated by a polynomial function in the supremum norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$. In other words, the polynomials are dense in the continuous functions with respect to the supremum norm. The Stone-Weierstrass is a generalization, stating that for any compact Hausdorff space X, if A is a subalgebra of $C^0(X, \mathbb{R})$ such that A separates points ($\forall x, y, \exists f \in A \ : f(x) \neq f(y)$) and contains a nonzero constant function, then A is dense in C^0.

Given this result, approximate a given continuous self-map g of D^n by a polynomial function p' so that $||p' - g||_\infty < \epsilon$ on D^n. To ensure p' sends D^n into itself, rescale it via

$$p = (1 + \epsilon)^{-1}p'. \tag{74}$$

Then clearly p is a D^n self-map while $||p - g||_\infty < 2\epsilon$. If g had no fixed point, then $|g(x) - x|$ must have a minimum value μ on D^n, and by choosing $2\epsilon = \mu$ we guarantee that for each x,

$$|p(x) - x| \geq |g(x) - x| - |g(x) - p(x)| > \mu - \mu = 0. \tag{75}$$

Hence p has no fixed point. Such a smooth function can't exist and hence we obtain the result. □

3.4 Genericity

Theorem 3.26 (Transversality theorem)**.** *Let $F : X \times S \longrightarrow Y$ and $g : Z \longrightarrow Y$ be smooth maps of manifolds where only X has boundary. Suppose that F and ∂F are transverse to g. Then for almost every $s \in S$, $f_s = F(\cdot, s)$ and ∂f_s are transverse to g.*

Proof. Due to the transversality, the fiber product $W = (X \times S) \times_Y Z$ is a submanifold (with boundary) of $X \times S \times Z$ and projects to S via the usual projection map π. We show that any $s \in S$ which is a regular value for both the projection map $\pi : W \longrightarrow S$ and its boundary map $\partial\pi$ gives rise to a f_s which is transverse to g. Then by Sard's theorem the s which fail to be regular in this way form a set of measure zero.

Suppose that $s \in S$ is a regular value for π. Suppose that $f_s(x) = g(z) = y$ and we now show that f_s is transverse to g there. Since $F(x,s) = g(z)$ and F is transverse to g, we know that

$$\mathrm{im}DF_{(x,s)} + \mathrm{im}Dg_z = T_yY.$$

Therefore, for any $a \in T_yY$, there exists $b = (w,e) \in T(X \times S)$ with $DF_{(x,s)}b - a$ in the image of Dg_z. But since $D\pi$ is surjective, there exists $(w', e, c') \in T_{(x,y,z)}W$. Hence we observe that

$$(Df_s)(w-w')-a = DF_{(x,s)}[(w,e)-(w',e)]-a = (DF_{(x,s)}b-a)-DF_{(x,s)}(w',e),$$

where both terms on the right hand side lie in $\mathrm{im}Dg_z$, since $(w', e, c') \in T_{(x,y,z)}W$ means $Dg_z(c') = DF_{(x,y)}(w', e)$.

Precisely the same argument (with X replaced with ∂X and F replaced with ∂F) shows that if s is regular for $\partial\pi$ then ∂f_s is transverse to g. This gives the result. $\square$

The previous result immediately shows that transversal maps to $\mathbb{R}^n$ are generic, since for any smooth map $f : M \longrightarrow \mathbb{R}^n$ we may produce a family of maps

$$F : M \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \tag{76}$$

via $F(x,s) = f(x) + s$. This new map F is clearly a submersion and hence is transverse to any smooth map $g : Z \longrightarrow \mathbb{R}^n$. For arbitrary target manifolds, we will imitate this argument, but we will require a (weak) version of Whitney's embedding theorem for manifolds into $\mathbb{R}^n$.

In the next section we will show that any manifold Y can be embedded via $\iota : Y \to \mathbb{R}^N$ in some large Euclidean space, and in such a way that the image has a "tubular neighbourhood" $U \subset \mathbb{R}^N$ of radius $\epsilon(y)$ (for a positive real-valued function $\epsilon : Y \to \mathbb{R}$) equipped with a projection $\pi : U \to Y$ such that $\pi\iota = \mathrm{id}_Y$.

Corollary 3.27. *Let X be a manifold with boundary and $f : X \longrightarrow Y$ be a smooth map to a manifold Y. Then there is an open ball $S = B(0,1) \subset \mathbb{R}^N$ and a smooth map $F : X \times S \longrightarrow Y$ such that $F(x,0) = f(x)$ and for fixed x, the map $f_x : s \mapsto F(x,s)$ is a submersion $S \longrightarrow Y$.*

In particular, F and ∂F are submersions, so are transverse to any $g : Z \to Y$.

Proof. Use the embedding of $\iota : Y \to \mathbb{R}^N$ and the tubular neighbourhood $\pi : U \to Y$ to define

$$F(x,s) = \pi(\iota(f(x)) + \epsilon(y)s). \tag{77}$$

$\square$

The transversality theorem then guarantees that given any smooth $g : Z \longrightarrow Y$, for almost all $s \in S$ the maps $f_s, \partial f_s$ are transverse to g. We improve this slightly to show that f_s may be chosen to be *homotopic* to f.

Corollary 3.28 (Transversality homotopy theorem)**.** *Given any smooth maps $f_0 : X \longrightarrow Y$, $g : Z \longrightarrow Y$, where only X has boundary, there exists a smooth map $f_1 : X \longrightarrow Y$ homotopic to f_0 with $f_1, \partial f_1$ both transverse to g.*

Proof. Let S, F be as in the previous corollary. Away from a set of measure zero in S, the functions $f_s, \partial f_s$ are transverse to g, by the transversality theorem. But these f_s are all homotopic to f via the homotopy $X \times [0,1] \longrightarrow Y$ given by

$$(x,t) \mapsto F(x,ts). \tag{78}$$

□

The last theorem we shall prove concerning transversality is a very useful extension result which is essential for intersection theory:

Theorem 3.29 (Homotopic transverse extension of boundary map)**.** *Let X be a manifold with boundary and $f_0 : X \longrightarrow Y$ a smooth map to a manifold Y. Suppose that ∂f_0 is transverse to the closed map $g : Z \longrightarrow Y$. Then there exists a map $f_1 : X \longrightarrow Y$, homotopic to f and with $\partial f_1 = \partial f_0$, such that f_1 is transverse to g.*

Proof. First observe that since ∂f_0 is transverse to g on ∂X, f_0 is also transverse to g there, and furthermore since g is closed, f_0 is transverse to g in a neighbourhood U of ∂X. (for example, if $x \in \partial X$ but x not in $f_0^{-1}(g(Z))$ then since the latter set is closed, we obtain a neighbourhood of x for which f_0 is transverse to g.)

Now choose a smooth function $\gamma : X \longrightarrow [0,1]$ which is 1 outside U but 0 on a neighbourhood of ∂X. (why does γ exist? exercise.) Then set $\tau = \gamma^2$, so that $d\tau(x) = 0$ wherever $\tau(x) = 0$. Recall the map $F : X \times S \longrightarrow Y$ we used in proving the transversality homotopy theorem and modify it via

$$G(x,s) = F(x, \tau(x)s). \tag{79}$$

The claim is that G and ∂G are transverse to g. This is clear for x such that $\tau(x) \neq 0$. But if $\tau(x) = 0$,

$$TG_{(x,s)}(v,w) = TF_{(x,0)}(v,0) = T(f_0)_x(v), \tag{80}$$

but $\tau(x) = 0$ means that $x \in U$, in which f is transverse to g.

Since transversality holds, there exists s such that $f_1 : x \mapsto G(x,s)$ and ∂f_1 are transverse to g (and homotopic to f_0, as before). Finally, if x is in the neighbourhood of ∂X for which $\tau = 0$, then $f_1(x) = F(x,0) = f_0(x)$. □

Corollary 3.30. *If $f_0 : X \longrightarrow Y$ and $f_1 : X \longrightarrow Y$ are homotopic smooth maps of manifolds, each transverse to the closed map $g : Z \longrightarrow Y$, then the fiber products $W_0 = X_{f_0}\times_g Z$ and $W_1 = X_{f_1}\times_g Z$ are cobordant.*

Proof. if $F : X \times [0, 1] \longrightarrow Y$ is the homotopy between f_0, f_1, then by the previous theorem, we may find a (homotopic) homotopy $G : X \times [0, 1] \longrightarrow Y$ which is transverse to g, without changing F on the boundary. Hence the fiber product $U = (X \times [0, 1])_G \times_g Z$ is a cobordism with boundary $W \sqcup W'$. $\square$

3.5 Intersection theory

The previous corollary allows us to make the following definition:

Definition 3.31. Let $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ be smooth maps with X compact, g closed, and $\dim X + \dim Z = \dim Y$. Then we define the (mod 2) intersection number of f and g to be

$$I_2(f, g) = \#(X_{f'} \times_g Z) \pmod 2,$$

where $f' : X \longrightarrow Y$ is any smooth map smoothly homotopic to f but transverse to g, and where we assume the fiber product to consist of a finite number of points (this is always guaranteed, e.g. if g is proper, or if g is a closed embedding).

Example 3.32. If C_1, C_2 are two distinct great circles on S^2 then they have two transverse intersection points, so $I_2(C_1, C_2) = 0$ in $\mathbb{Z}_2$. Of course we can shrink one of the circles to get a homotopic one which does not intersect the other at all. This corresponds to the standard cobordism from two points to the empty set.

Example 3.33. If (e_1, e_2, e_3) is a basis for $\mathbb{R}^3$ we can consider the following two embeddings of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ into $\mathbb{R}P^2$: $\iota_1 : \theta \mapsto \langle \cos(\theta/2)e_1 + \sin(\theta/2)e_2 \rangle$ and $\iota_2 : \theta \mapsto \langle \cos(\theta/2)e_2 + \sin(\theta/2)e_3 \rangle$. These two embedded submanifolds intersect transversally in a single point $\langle e_2 \rangle$, and hence $I_2(\iota_1, \iota_2) = 1$ in $\mathbb{Z}_2$. As a result, there is no way to deform ι_i so that they intersect transversally in zero points.

Example 3.34. Given a smooth map $f : X \longrightarrow Y$ for X compact and $\dim Y = 2 \dim X$, we may consider the self-intersection $I_2(f, f)$. In the previous examples we may check $I_2(C_1, C_1) = 0$ and $I_2(\iota_1, \iota_1) = 1$. Any embedded S^1 in an oriented surface has no self-intersection. If the surface is nonorientable, the self-intersection may be nonzero.

Example 3.35. Let $p \in S^1$. Then the identity map $\mathrm{Id} : S^1 \longrightarrow S^1$ is transverse to the inclusion $\iota : p \longrightarrow S^1$ with one point of intersection. Hence the identity map is not (smoothly) homotopic to a constant map, which would be transverse to ι with zero intersection. Using smooth approximation, get that Id is not continuously homotopic to a constant map, and also that S^1 is not contractible.

Example 3.36. By the previous argument, any compact manifold is not contractible.

Example 3.37. Consider $SO(3) \cong \mathbb{R}P^3$ and let $\ell \subset \mathbb{R}P^3$ be a line, diffeomorphic to S^1. This line corresponds to a path of rotations about an axis by $\theta \in [0, \pi]$ radians. Let $\mathcal{P} \subset \mathbb{R}P^3$ be a plane intersecting ℓ in one point. Since this is a transverse intersection in a single point, ℓ cannot be deformed to a point (which would have zero intersection with $\mathcal{P}$. This shows that the path of rotations is not homotopic to a constant path.

If $\iota : \theta \mapsto \iota(\theta)$ is the embedding of S^1, then traversing the path twice via $\iota' : \theta \mapsto \iota(2\theta)$, we obtain a map ι' which is transverse to $\mathcal{P}$ but with two intersection points. Hence it is possible that ι' may be deformed so as not to intersect $\mathcal{P}$. Can it be done?

Example 3.38. Consider $\mathbb{R}P^4$ and two transverse hyperplanes P_1, P_2 each an embedded copy of $\mathbb{R}P^3$. These then intersect in $P_1 \cap P_2 = \mathbb{R}P^2$, and since $\mathbb{R}P^2$ is not null-homotopic, we cannot deform the planes to remove all intersection.

Intersection theory also allows us to define the degree of a map modulo 2. The degree measures how many generic preimages there are of a local diffeomorphism.

Definition 3.39. Let $f : M \longrightarrow N$ be a smooth map of manifolds of the same dimension, and suppose M is compact and N connected. Let $p \in N$ be any point. Then we define $\deg_2(f) = I_2(f, p)$.

Example 3.40. Let $f : S^1 \longrightarrow S^1$ be given by $z \mapsto z^k$. Then $\deg_2(f) = k \pmod 2$.

Example 3.41. If $p : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$ is a polynomial of degree k, then as a map $S^2 \longrightarrow S^2$ we have $\deg_2(p) = k \pmod 2$, and hence any odd polynomial has at least one root. To get the fundamental theorem of algebra, we must consider *oriented cobordism*

Even if submanifolds C, C' do not intersect, it may be that there are more sophisticated geometrical invariants which cause them to be "intertwined" in some way. One example of this is linking number.

Definition 3.42. Suppose that $M, N \subset \mathbb{R}^{k+1}$ are compact embedded submanifolds with $\dim M + \dim N = k$, and let us assume they are transverse, meaning they do not intersect at all.

Then define $\lambda : M \times N \longrightarrow S^k$ via

$$(x, y) \mapsto \frac{x - y}{|x - y|}.$$

Then we define the $\pmod 2$ linking number of M, N to be $\deg_2(\lambda)$.

Example 3.43. Consider the standard Hopf link in $\mathbb{R}^3$. Then it is easy to calculate that $\deg_2(\lambda) = 1$. On the other hand, the standard embedding of disjoint circles (differing by a translation, say) has $\deg_2(\lambda) = 0$. Hence it is impossible to deform the circles through embeddings of $S^1 \sqcup S^1 \longrightarrow \mathbb{R}^3$, so that they are unlinked. Why must we stay within the space of embeddings, and not allow the circles to intersect?

3.6 Partitions of unity and Whitney embedding

Partitions of unity allow us to *go from local to global*, i.e. to build a global object on a manifold by building it on each open set of a cover, smoothly tapering each local piece so it is compactly supported in each open set, and then taking a sum over open sets. This is a very flexible operation which uses the properties of smooth functions—it will not work for complex manifolds, for example. Our main example of such a passage from local to global is to build a global map from a manifold to $\mathbb{R}^N$ which is an embedding, a result first proved by Whitney.

Definition 3.44. A collection of subsets $\{U_\alpha\}$ of the topological space M is called *locally finite* when each point $x \in M$ has a neighbourhood V intersecting only finitely many of the U_α.

Definition 3.45. A covering $\{V_\alpha\}$ is a *refinement* of the covering $\{U_\beta\}$ when each V_α is contained in some U_β.

Lemma 3.46. *Any open covering $\{A_\alpha\}$ of a topological manifold has a countable, locally finite refinement $\{(U_i, \varphi_i)\}$ by coordinate charts such that $\varphi_i(U_i) = B(0,3)$ and $\{V_i = \varphi_i^{-1}(B(0,1))\}$ is still a covering of M. We will call such a cover a* regular *covering. In particular, any topological manifold is* paracompact *(i.e. every open cover has a locally finite refinement)*

Proof. If M is compact, the proof is easy: choosing coordinates around any point $x \in M$, we can translate and rescale to find a covering of M by a refinement of the type desired, and choose a finite subcover, which is obviously locally finite.

For a general manifold, we note that by second countability of M, there is a countable basis of coordinate neighbourhoods and each of these charts is a countable union of open sets P_i with $\overline{P_i}$ compact. Hence M has a countable basis $\{P_i\}$ such that $\overline{P_i}$ is compact.

Using these, we may define an increasing sequence of compact sets which exhausts M: let $K_1 = \overline{P}_1$, and

$$K_{i+1} = \overline{P_1 \cup \cdots \cup P_r},$$

where $r > 1$ is the first integer with $K_i \subset P_1 \cup \cdots \cup P_r$.

Now note that M is the union of ring-shaped sets $K_i \backslash K_{i-1}^\circ$, each of which is compact. If $p \in A_\alpha$, then $p \in K_{i+1} \backslash K_i^\circ$ for some i. Now choose a coordinate neighbourhood $(U_{p,\alpha}, \varphi_{p,\alpha})$ with $U_{p,\alpha} \subset K_{i+2} \backslash K_{i-1}^\circ$ and $\varphi_{p,\alpha}(U_{p,\alpha}) = B(0,3)$ and define $V_{p,\alpha} = \varphi^{-1}(B(0,1))$.

Letting p, α vary, these neighbourhoods cover the compact set $K_{i+1} \backslash K_i^\circ$ without leaving the band $K_{i+2} \backslash K_{i-1}^\circ$. Choose a finite subcover $V_{i,k}$ for each i. Then $(U_{i,k}, \varphi_{i,k})$ is the desired locally finite refinement. $\square$

Definition 3.47. A smooth partition of unity is a collection of smooth non-negative functions $\{f_\alpha : M \longrightarrow \mathbb{R}\}$ such that

i) $\{\mathrm{supp} f_\alpha = \overline{f_\alpha^{-1}(\mathbb{R}\backslash\{0\})}\}$ is locally finite,

ii) $\sum_\alpha f_\alpha(x) = 1 \ \ \forall x \in M$, hence the name.

A partition of unity is *subordinate* to an open cover $\{U_i\}$ when $\forall \alpha$, $\mathrm{supp} f_\alpha \subset U_i$ for some i.

Theorem 3.48. *Given a regular covering $\{(U_i, \varphi_i)\}$ of a manifold, there exists a partition of unity $\{f_i\}$ subordinate to it with $f_i > 0$ on V_i and $\mathrm{supp} f_i \subset \varphi_i^{-1}(\overline{B(0,2)})$.*

Proof. A *bump function* is a smooth non-negative real-valued function $\tilde{g}$ on $\mathbb{R}^n$ with $\tilde{g}(x) = 1$ for $||x|| \leq 1$ and $\tilde{g}(x) = 0$ for $||x|| \geq 2$. For instance, take

$$\tilde{g}(x) = \frac{h(2 - ||x||)}{h(2 - ||x||) + h(||x|| + 1)},$$

for $h(t)$ given by $e^{-1/t}$ for $t > 0$ and 0 for $t < 0$.

Having this bump function, we can produce non-negative bump functions on the manifold $g_i = \tilde{g} \circ \varphi_i$ which have support $\mathrm{supp} g_i \subset \varphi_i^{-1}(\overline{B(0,2)})$ and take the value $+1$ on $\overline{V_i}$. Finally we define our partition of unity via

$$f_i = \frac{g_i}{\sum_j g_j}, \quad i = 1, 2, \ldots.$$

□

We now investigate the embedding of arbitrary smooth manifolds as regular submanifolds of $\mathbb{R}^k$. We shall first show by a straightforward argument that any smooth manifold may be embedded in some $\mathbb{R}^N$ for some sufficiently large N. We will then explain how to cut down on N and approach the optimal $N = 2 \dim M$ which Whitney showed (we shall reach $2 \dim M + 1$ and possibly at the end of the course, show $N = 2 \dim M$.)

Theorem 3.49 (Compact Whitney embedding in $\mathbb{R}^N$). *Any compact manifold may be embedded in $\mathbb{R}^N$ for sufficiently large N.*

Proof. Let $\{(U_i \supset V_i, \varphi_i)\}_{i=1}^k$ be a *finite* regular covering, which exists by compactness. Choose a partition of unity $\{f_1, \ldots, f_k\}$ as in Theorem 3.48 and define the following "zoom-in" maps $M \longrightarrow \mathbb{R}^{\dim M}$:

$$\tilde{\varphi}_i(x) = \begin{cases} f_i(x)\varphi_i(x) & x \in U_i, \\ 0 & x \notin U_i. \end{cases}$$

Then define a map $\Phi : M \longrightarrow \mathbb{R}^{k(\dim M + 1)}$ which zooms simultaneously into all neighbourhoods, with extra information to guarantee injectivity:

$$\Phi(x) = (\tilde{\varphi}_1(x), \ldots, \tilde{\varphi}_k(x), f_1(x), \ldots, f_k(x)).$$

Note that $\Phi(x) = \Phi(x')$ implies that for some i, $f_i(x) = f_i(x') \neq 0$ and hence $x, x' \in U_i$. This then implies that $\varphi_i(x) = \varphi_i(x')$, implying $x = x'$. Hence Φ is injective.

We now check that $D\Phi$ is injective, which will show that it is an injective immersion. At any point x the differential sends $v \in T_x M$ to the following vector in $\mathbb{R}^{\dim M} \times \cdots \times \mathbb{R}^{\dim M} \times \mathbb{R} \times \cdots \times \mathbb{R}$.

$$(Df_1(v)\varphi_1(x) + f_1(x)D\varphi_1(v), \ldots, Df_k(v)\varphi_k(x) + f_k(x)D\varphi_1(v), Df_1(v), \ldots, Df_k(v)$$

But this vector cannot be zero. Hence we see that Φ is an immersion.

But an injective immersion from a compact space must be an embedding: view Φ as a bijection onto its image. We must show that Φ^{-1} is continuous, i.e. that Φ takes closed sets to closed sets. If $K \subset M$ is closed, it is also compact and hence $\Phi(K)$ must be compact, hence closed (since the target is Hausdorff). □

Theorem 3.50 (Compact Whitney embedding in $\mathbb{R}^{2n+1}$)**.** *Any* compact *n-manifold may be embedded in $\mathbb{R}^{2n+1}$.*

Proof. Begin with an embedding $\Phi : M \longrightarrow \mathbb{R}^N$ and assume $N > 2n+1$. We then show that by projecting onto a hyperplane it is possible to obtain an embedding to $\mathbb{R}^{N-1}$.

A vector $v \in S^{N-1} \subset \mathbb{R}^N$ defines a hyperplane (the orthogonal complement) and let $P_v : \mathbb{R}^N \longrightarrow \mathbb{R}^{N-1}$ be the orthogonal projection to this hyperplane. We show that the set of v for which $\Phi_v = P_v \circ \Phi$ fails to be an embedding is a set of measure zero, hence that it is possible to choose v for which Φ_v is an embedding.

Φ_v fails to be an embedding exactly when Φ_v is not injective or $D\Phi_v$ is not injective at some point. Let us consider the two failures separately:

If v is in the image of the map $\beta_1 : (M \times M)\backslash\Delta_M \longrightarrow S^{N-1}$ given by

$$\beta_1(p_1, p_2) = \frac{\Phi(p_2) - \Phi(p_1)}{||\Phi(p_2) - \Phi(p_1)||},$$

then Φ_v will fail to be injective. Note however that β_1 maps a $2n$-dimensional manifold to a $N-1$-manifold, and if $N > 2n+1$ then baby Sard's theorem implies the image has measure zero.

The immersion condition is a local one, which we may analyze in a chart (U, φ). Φ_v will fail to be an immersion in U precisely when v coincides with a vector in the normalized image of $D(\Phi \circ \varphi^{-1})$ where

$$\Phi \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N.$$

Hence we have a map (letting $N(w) = ||w||$)

$$\frac{D(\Phi \circ \varphi^{-1})}{N \circ D(\Phi \circ \varphi^{-1})} : U \times S^{n-1} \longrightarrow S^{N-1}.$$

The image has measure zero as long as $2n - 1 < N - 1$, which is certainly true since $2n < N - 1$. Taking union over countably many charts, we see that immersion fails on a set of measure zero in S^{N-1}.

Hence we see that Φ_v fails to be an embedding for a set of $v \in S^{N-1}$ of measure zero. Hence we may reduce N all the way to $N = 2n+1$. $\square$

Corollary 3.51. *We see from the proof that if we do not require injectivity but only that the manifold be* immersed *in $\mathbb{R}^N$, then we can take $N = 2n$ instead of $2n+1$.*

Theorem 3.52 (noncompact Whitney embedding in $\mathbb{R}^{2n+1}$)**.** *Any smooth n-manifold may be embedded in $\mathbb{R}^{2n+1}$ (or immersed in $\mathbb{R}^{2n}$).*

Proof. We saw that any manifold may be written as a countable union of increasing compact sets $M = \cup K_i$, and that a regular covering $\{(U_{i,k} \supset V_{i,k}, \varphi_{i,k})\}$ of M can be chosen so that for fixed i, $\{V_{i,k}\}_k$ is a finite cover of $K_{i+1}\backslash K_i^\circ$ and each $U_{i,k}$ is contained in $K_{i+2}\backslash K_{i-1}^\circ$.

This means that we can express M as the union of 3 open sets W_0, W_1, W_2, where

$$W_j = \bigcup_{i \equiv j (\mathrm{mod} 3)} (\cup_k U_{i,k}).$$

Each of the sets $R_i = \cup_k U_{i,k}$ may be injectively immersed in $\mathbb{R}^{2n+1}$ by the argument for compact manifolds, since they have a finite regular cover. Call these injective immersions $\Phi_i : R_i \longrightarrow \mathbb{R}^{2n+1}$. The image $\Phi_i(R_i)$ is bounded since all the charts are, by some radius r_i. The open sets $R_i,\ i \equiv j(\text{mod}3)$ for fixed j are disjoint, and by translating each $\Phi_i,\ i \equiv j(\text{mod}3)$ by an appropriate constant, we can ensure that their images in $\mathbb{R}^{2n+1}$ are disjoint as well.

Let $\Phi_i' = \Phi_i + (2(r_{i-1}+r_{i-2}+\cdots)+r_i)\overrightarrow{e}_1$. Then $\Psi_j = \cup_{i\equiv j(\text{mod}3)}\Phi_i' : W_j \longrightarrow \mathbb{R}^{2n+1}$ is an embedding.

Now that we have injective immersions Ψ_0, Ψ_1, Ψ_2 of W_0, W_1, W_2 in $\mathbb{R}^{2n+1}$, we may use the original argument for compact manifolds: Take the partition of unity subordinate to $U_{i,k}$ and resum it, obtaining a 3-element partition of unity $\{f_1, f_2, f_3\}$, with $f_j = \sum_{i\equiv j(\text{mod}3)} \sum_k f_{i,k}$. Then the map

$$\Psi = (f_1\Psi_1, f_2\Psi_2, f_3\Psi_3, f_1, f_2, f_3)$$

is an injective immersion of M into $\mathbb{R}^{6n+3}$. To see that it is in fact an embedding, note that any closed set $C \subset M$ may be written as a union of closed sets $C = C_1 \cup C_2 \cup C_3$, where $C_j = \cup_{i\equiv j(\text{mod}3)}(C \cap K_{i+1}\backslash K_i^\circ)$ is a disjoint union of compact sets. Ψ is injective, hence C_j is mapped to a disjoint union of compact sets, hence a closed set. Then $\Psi(C)$ is a union of 3 closed sets, hence closed, as required.

Using projection to hyperplanes we may again reduce to $\mathbb{R}^{2n+1}$, but if we exclude all hyperplanes perpendicular to $\text{Span}((e_1, 0, 0, 0, 0, 0), (0, e_1, 0, 0, 0, 0), (0, 0, e_1, 0, 0, 0))$, we obtain an injective immersion Ψ' which is *proper*, meaning that inverse images of compact sets are compact. This space of forbidden planes has measure zero as long as $N - 1 > 3$, so that we may reduce to $2n + 1$ for $n > 1$. We leave as an exercise the $n = 1$ case (or see Bredon for a slightly different proof).

The fact that the resulting injective immersion Ψ' is proper implies that it is an embedding, by the closed map lemma, as follows. $\square$

Lemma 3.53 (Closed map lemma for proper maps)**.** *Let $f : X \longrightarrow Y$ be a proper continuous map of topological manifolds. Then f is a closed map.*

Proof. Let $K \subset X$ be closed; we show that $f(K)$ contains all its limit points and hence is closed. Let $y \in Y$ be a limit point for $f(K)$. Choose a precompact neighbourhood U of y, so that y is also a limit point of $f(K) \cap \overline{U}$. Since f is proper, $f^{-1}(\overline{U})$ is compact, and hence $K \cap f^{-1}(\overline{U})$ is compact as well. But then by continuity, $f(K \cap f^{-1}(\overline{U})) = f(K) \cap \overline{U}$ is compact, implying it is closed. Hence $y \in f(K) \cap \overline{U} \subset f(K)$, as required. $\square$

We now use Whitney embedding to prove the existence of tubular neighbourhoods for submanifolds of $\mathbb{R}^N$, a key point in proving genericity of transversality. Tubular neighbourhoods also exist for submanifolds of any manifold, but we leave this corollary for the reader.

If $Y \subset \mathbb{R}^N$ is an embedded submanifold, the normal space at $y \in Y$ is defined by $N_yY = \{v \in \mathbb{R}^N\ :\ v\perp T_yY\}$. The collection of all normal spaces of all points in Y is called the normal bundle:

$$NY = \{(y,v) \in Y \times \mathbb{R}^N \ : \ v \in N_yY\}.$$

Proposition 3.54. *$NY \subset \mathbb{R}^N \times \mathbb{R}^N$ is an embedded submanifold of dimension N.*

Proof. Given $y \in Y$, choose coordinates $(u^1, \dots u^N)$ in a neighbourhood $U \subset \mathbb{R}^N$ of y so that $Y \cap U = \{u^{n+1} = \cdots = u^N = 0\}$. Define $\Phi : U \times \mathbb{R}^N \longrightarrow \mathbb{R}^{N-n} \times \mathbb{R}^n$ via

$$\Phi(x,v) = (u^{n+1}(x), \dots, u^N(x), \langle v, \tfrac{\partial}{\partial u^1}|_x\rangle, \dots, \langle v, \tfrac{\partial}{\partial u^n}|_x\rangle),$$

so that $\Phi^{-1}(0)$ is precisely $NY \cap (U \times \mathbb{R}^N)$. We then show that 0 is a regular value: observe that, writing v in terms of its components $v^j \frac{\partial}{\partial x^j}$ in the standard basis for $\mathbb{R}^N$,

$$\langle v, \tfrac{\partial}{\partial u^i}|_x\rangle = \langle v^j \tfrac{\partial}{\partial x^j}, \tfrac{\partial x^k}{\partial u^i}(u(x)) \tfrac{\partial}{\partial x^k}|_x\rangle = \sum_{j=1}^N v^j \tfrac{\partial x^j}{\partial u^i}(u(x))$$

Therefore the Jacobian of Φ is the $((N-n)+n) \times (N+N)$ matrix

$$D\Phi(x) = \begin{pmatrix} \frac{\partial u^j}{\partial x^i}(x) & 0 \\ * & \frac{\partial x^j}{\partial u^i}(u(x)) \end{pmatrix}$$

The N rows of this matrix are linearly independent, proving Φ is a submersion. $\square$

The normal bundle NY contains $Y \cong Y \times \{0\}$ as a regular submanifold, and is equipped with a smooth map $\pi : NY \longrightarrow Y$ sending $(y,v) \mapsto y$. The map π is a surjective submersion and is the bundle projection. The vector spaces $\pi^{-1}(y)$ for $y \in Y$ are called the fibers of the bundle and NY is an example of a vector bundle.

We may take advantage of the embedding in $\mathbb{R}^N$ to define a smooth map $E : NY \longrightarrow \mathbb{R}^N$ via

$$E(x,v) = x + v.$$

Definition 3.55. A tubular neighbourhood of the embedded submanifold $Y \subset \mathbb{R}^N$ is a neighbourhood U of Y in $\mathbb{R}^N$ that is the diffeomorphic image under E of an open subset $V \subset NY$ of the form

$$V = \{(y,v) \in NY \ : \ |v| < \delta(y)\},$$

for some positive continuous function $\delta : M \longrightarrow \mathbb{R}$.

If $U \subset \mathbb{R}^N$ is such a tubular neighbourhood of Y, then there does exist a positive continuous function $\epsilon : Y \longrightarrow \mathbb{R}$ such that $U_\epsilon = \{x \in \mathbb{R}^N \ : \ \exists y \in Y \text{ with } |x-y| < \epsilon(y)\}$ is contained in U. This is simply

$$\epsilon(y) = \sup\{r \ : \ B(y,r) \subset U\},$$

which is continuous since $\forall \epsilon > 0, \exists x \in U$ for which $\epsilon(y) \leq |x-y| + \epsilon$. For any other $y' \in Y$, this is $\leq |y-y'| + |x-y'| + \epsilon$. Since $|x-y'| \leq \epsilon(y')$, we have $|\epsilon(y) - \epsilon(y')| \leq |y-y'| + \epsilon$.

Theorem 3.56 (Tubular neighbourhood theorem)**.** *Every regular submanifold of $\mathbb{R}^N$ has a tubular neighbourhood.*

Proof. First we show that E is a local diffeomorphism near $y \in Y \subset NY$. if ι is the embedding of Y in $\mathbb{R}^N$, and $\iota' : Y \longrightarrow NY$ is the embedding in the normal bundle, then $E \circ \iota' = \iota$, hence we have $DE \circ D\iota' = D\iota$, showing that the image of $DE(y)$ contains T_yY. Now if ι is the embedding of N_yY in $\mathbb{R}^N$, and $\iota' : N_yY \longrightarrow NY$ is the embedding in the normal bundle, then $E \circ \iota' = \iota$. Hence we see that the image of $DE(y)$ contains N_yY, and hence the image is all of $T_y\mathbb{R}^N$. Hence E is a diffeomorphism on some neighbourhood

$$V_\delta(y) = \{(y', v') \in NY \ : \ |y' - y| < \delta, \ |v'| < \delta\}, \quad \delta > 0.$$

Now for $y \in Y$ let $r(y) = \sup\{\delta \ : \ E|_{V_\delta(y)} \text{ is a diffeomorphism}\}$ if this is ≤ 1 and let $r(y) = 1$ otherwise. The function $r(y)$ is continuous, since if $|y - y'| < r(y)$, then $V_\delta(y') \subset V_{r(y)}(y)$ for $\delta = r(y) - |y - y'|$. This means that $r(y') \geq \delta$, i.e. $r(y) - r(y') \leq |y - y'|$. Switching y and y', this remains true, hence $|r(y) - r(y')| \leq |y - y'|$, yielding continuity.

Finally, let $V = \{(y, v) \in NY \ : \ |v| < \frac{1}{2}r(y)\}$. We show that E is injective on V. Suppose $(y, v), (y', v') \in V$ are such that $E(y, v) = E(y', v')$, and suppose wlog $r(y') \leq r(y)$. Then since $y + v = y' + v'$, we have

$$|y - y'| = |v - v'| \leq |v| + |v'| \leq \tfrac{1}{2}r(y) + \tfrac{1}{2}r(y') \leq r(y).$$

Hence y, y' are in $V_{r(y)}(y)$, on which E is a diffeomorphism. The required tubular neighbourhood is then $U = E(V)$. □

4 Vector fields

4.1 Derivations

The space $C^\infty(M, \mathbb{R})$ of smooth functions on M is not only a vector space but also a ring, with multiplication $(fg)(p) := f(p)g(p)$. That this defines a smooth function is clear from the fact that it is a composition of the form

$$M \xrightarrow{\Delta} M \times M \xrightarrow{f \times g} \mathbb{R} \times \mathbb{R} \xrightarrow{m} \mathbb{R}\ .$$

Given a smooth map $\varphi : M \longrightarrow N$ of manifolds, we obtain a natural operation $\varphi^* : C^\infty(N, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$, given by $f \mapsto f \circ \varphi$. This is called the pullback of functions, and defines a homomorphism of rings since $\Delta \circ \varphi = (\varphi \times \varphi) \circ \Delta$.

The association $M \mapsto C^\infty(M, \mathbb{R})$ and $\varphi \mapsto \varphi^*$ is therefore a *contravariant* functor from the category of manifolds to the category of rings, and is the basis for algebraic geometry, the algebraic representation of geometrical objects.

It is easy to see from this that any diffeomorphism $\varphi : M \longrightarrow M$ defines an automorphism φ^* of $C^\infty(M, \mathbb{R})$, but actually all automorphisms are of this form (Exercise!).

The concept of derivation of an algebra A is the infinitesimal version of an automorphism of A. That is, if $\phi_t : A \longrightarrow A$ is a family of automorphisms of A starting at Id, so that $\phi_t(ab) = \phi_t(a)\phi_t(b)$, then the map $a \mapsto \frac{d}{dt}|_{t=0}\phi_t(a)$ is a derivation.

Definition 4.1. A derivation of the $\mathbb{R}$-algebra A is a $\mathbb{R}$-linear map $D : A \longrightarrow A$ such that $D(ab) = (Da)b + a(Db)$. The space of all derivations is denoted $\mathrm{Der}(A)$.

If automorphisms of $C^\infty(M, \mathbb{R})$ correspond to diffeomorphisms, then it is natural to ask what derivations correspond to. We now show that they correspond to vector fields.

The vector fields $\Gamma^\infty(M, TM)$ form a vector space over $\mathbb{R}$ of infinite dimension (unless M is a finite set). They also form a module over the ring of smooth functions $C^\infty(M, \mathbb{R})$ via pointwise multiplication: for $f \in C^\infty(M, \mathbb{R})$ and $X \in \Gamma^\infty(M, TM)$, $fX : x \mapsto f(x)X(x)$ is a smooth vector field (why?)

The important property of vector fields which we are interested in is that they act as $\mathbb{R}$-derivations of the algebra of smooth functions. Locally, it is clear that a vector field $X = \sum_i a^i \frac{\partial}{\partial x^i}$ gives a derivation of the algebra of smooth functions, via the formula $X(f) = \sum_i a^i \frac{\partial f}{\partial x^i}$, since

$$X(fg) = \sum_i a^i \left(\tfrac{\partial f}{\partial x^i} g + f \tfrac{\partial g}{\partial x^i}\right) = X(f)g + fX(g).$$

We wish to verify that this local action extends to a well-defined global derivation on $C^\infty(M, \mathbb{R})$.

Definition 4.2. The differential of a function $f \in C^\infty(M, \mathbb{R})$ is the function on TM given by composing $Tf : TM \to T\mathbb{R}$ with the second projection $p_2 : T\mathbb{R} = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

$$df = p_2 \circ Tf \tag{81}$$

Recall that if (U, φ) is a chart for M, then $(TU, D\varphi)$ is a chart for TM. More explicitly, if $(x_1, \ldots, x_n)$ is the coordinate system on U given by φ, then the induced coordinate system on TU is $(x_1 \circ \pi, \ldots, x_n \circ \pi, dx_1, \ldots, dx_n)$. Often, we omit the bundle projection π and we write ξ_i for the differential dx_i, and so the induced coordinates are $(x_1, \ldots, x_n, \xi_1, \ldots, x_n)$.

Definition 4.3. Let $X \in \Gamma(M, TM)$ be a vector field. Then we define

$$X(f) = df \circ X.$$

This is called the directional (or Lie) derivative of f along X.

In coordinates, if $X = \sum a_i \partial/\partial x_i$, then $X(f) = \sum a_i \partial f/\partial x_i$, coinciding with the usual directional derivative mentioned above. This shows that $f \mapsto X(f)$ has the derivation property (since it satisfies it locally), but we can alternatively see that it is a derivation by using the property

$$d(fg) = fdg + gdf$$

of the differential of a product (here fdg is really $\pi^* fdg$).

Theorem 4.4. *The map $X \mapsto (f \mapsto X(f))$ is an isomorphism*

$$\Gamma(M, TM) \to Der(C^\infty(M, \mathbb{R})).$$

Proof. First we prove the result for an open set $U \subset \mathbb{R}^n$. Let D be a derivation of $C^\infty(U, \mathbb{R})$ and define the smooth functions $a^i = D(x^i)$. Then we claim $D = \sum_i a^i \frac{\partial}{\partial x^i}$. We prove this by testing against smooth functions. Any smooth function f on $\mathbb{R}^n$ may be written

$$f(x) = f(0) + \sum_i x^i g_i(x),$$

with $g_i(0) = \frac{\partial f}{\partial x^i}(0)$ (simply take $g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(tx)dt$). Translating the origin to $y \in U$, we obtain for any $z \in U$

$$f(z) = f(y) + \sum_i (x^i(z) - x^i(y))g_i(z), \qquad g_i(y) = \tfrac{\partial f}{\partial x^i}(y).$$

Applying D, we obtain

$$Df(z) = \sum_i (Dx^i)g_i(z) - \sum_i (x^i(z) - x^i(y))Dg_i(z).$$

Letting z approach y, we obtain

$$Df(y) = \sum_i a^i \tfrac{\partial f}{\partial x^i}(y) = X(f)(y),$$

as required.

To prove the global result, let $(V_i \subset U_i, \varphi_i)$ be a regular covering and θ_i an associated partition of unity. Then for each i, $\theta_i D : f \mapsto \theta_i D(f)$ is also a derivation of $C^\infty(M, \mathbb{R})$. This derivation defines a unique derivation D_i of $C^\infty(U_i, \mathbb{R})$ such that $D_i(f|_{U_i}) = (\theta_i Df)|_{U_i}$, since for any point $p \in U_i$, a given function $g \in C^\infty(U_i, \mathbb{R})$ may be replaced with a function $\tilde{g} \in C^\infty(M, \mathbb{R})$ which agrees with g on a small neighbourhood of p, and we define $(D_i g)(p) = \theta_i(p)D\tilde{g}(p)$. This definition is independent of $\tilde{g}$, since if $h_1 = h_2$ on an open set W, $Dh_1 = Dh_2$ on that open set (let $\psi = 1$ in a neighbourhood of p and vanish outside W; then $h_1 - h_2 = (h_1 - h_2)(1 - \psi)$ and applying D we obtain zero in W).

The derivation D_i is then represented by a vector field X_i, which must vanish outside the support of θ_i. Hence it may be extended by zero to a global vector field which we also call X_i. Finally we observe that for $X = \sum_i X_i$, we have

$$X(f) = \sum_i X_i(f) = \sum_i D_i(f) = D(f),$$

as required. ∎

4.2 Flows

Since vector fields are derivations, we have a natural source of examples, coming from infinitesimal automorphisms of M:

Example 4.5. Let φ_t : be a smooth family of maps $M \to M$ with $\varphi_0 = \mathrm{Id}$. That is, let $\varphi : (-\epsilon, \epsilon) \times M \longrightarrow M$ be smooth with $\varphi \circ j_0 = \mathrm{id}$, for $j_t(x) = (t, x)$. Then $X(f)(p) = \frac{d}{dt}|_{t=0}(\varphi_t^* f)(p)$ defines a smooth vector field. A better way of seeing it is to rewrite it as follows: Let $\frac{\partial}{\partial t}$ be the coordinate vector field on $(-\epsilon, \epsilon)$ and observe

$$X(f) = \tfrac{\partial}{\partial t}(\varphi^* f) \circ j_0.$$

Remark 4.6. A better way of describing the vector field from Example 4.5 is to note that the pullback of $T\varphi$ by j_0 is a bundle map from $j_0^* TU$ to TM over the identity map $M \to M$, and then X is simply the image of $j_0^* \frac{\partial}{\partial t}$ under $j_0^* T\varphi$, or informally

$$X = \varphi_*|_{t=0}(\tfrac{\partial}{\partial t}) \tag{82}$$

Essentially, a smooth vector field may always be expressed in this way, i.e. as the derivative of a family of automorphisms of M. The only caveat is that ϵ must be allowed to vary along the manifold M if it is noncompact. This gives rise to the notion of a “local 1-parameter group of diffeomorphisms”, as follows:

Definition 4.7. A local 1-parameter group of diffeomorphisms is an open set $U \subset \mathbb{R} \times M$ containing $\{0\} \times M$ and a smooth map

$$\begin{aligned} \Phi : & U \longrightarrow M \\ & (t, x) \mapsto \varphi_t(x) \end{aligned}$$

such that $\mathbb{R} \times \{x\} \cap U$ is connected, $\varphi_0(x) = x$ for all x and if $(t, x), (t + t', x), (t', \varphi_t(x))$ are all in U then $\varphi_{t'}(\varphi_t(x)) = \varphi_{t+t'}(x)$.

The derivative (82) of this family of diffeomorphisms is a vector field X, and we say that Φ is the *flow* of X.

Then the local existence and uniqueness of solutions to systems of ODE implies that every smooth vector field $X \in \Gamma(M, TM)$ gives rise to a local 1-parameter group of diffeomorphisms (U, Φ) such that the curve $\gamma_x : t \mapsto \varphi_t(x)$ satisfies $(\gamma_x)_*(\frac{d}{dt}) = X(\gamma_x(t))$ (this means that γ_x is an integral curve or “trajectory” of the “dynamical system” defined by X). Furthermore, if (U', Φ') are another such data, then $\Phi = \Phi'$ on $U \cap U'$.

Remark 4.8. We can rephrase the system of ODEs as the initial value problem

$$\begin{aligned} \Phi_* \tfrac{\partial}{\partial t} &= X, \\ \Phi \circ j_0 &= \mathrm{id}_M. \end{aligned}$$

This makes it very clear that 82 holds. In fact, the existence and uniqueness theorem is slightly more general, in that it allows the vector field to depend on time, so that X may be defined on $\mathbb{R} \times X$ with vanishing first projection $Tp_1(X) = 0$, and Φ may then be extended to a map $\tilde{\Phi} : U \to \mathbb{R} \times M$ with the property $\tilde{\Phi}_* \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + X$ and $\tilde{\Phi} \circ j_0 = j_0$. Uniqueness is then the statement that if $X = 0$ then $\tilde{\Phi}$ must be the identity.

Definition 4.9. A vector field $X \in \Gamma(M, TM)$ is called *complete* when it generates a global 1-parameter group of diffeomorphisms, i.e. $U = \mathbb{R} \times M$ in the above discussion.

We omit the proof of the following theorem, though it is not difficult to show that if $[0, \omega)$ is the maximal interval on which a trajectory γ is defined for non-negative times, and if the image $\gamma([0, \omega))$ has compact closure in M, then ω must be infinity.

Theorem 4.10. *If M is compact, then every smooth vector field is complete.*

Example 4.11. The vector field $X = x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$ is not complete. For initial condition x_0, have integral curve $\gamma(t) = x_0(1 - tx_0)^{-1}$, which gives $\Phi(t, x_0) = x_0(1 - tx_0)^{-1}$, which is well-defined on $\{1 - tx > 0\}$.

4.3 Commuting flows

Given two derivations D_1, D_2 of an algebra, the commutator $[D_1, D_2]$ is another derivation. In fact, if D_1 and D_2 arise from families of automorphisms φ_t, ψ_t respectively (with $\varphi_0 = \psi_0 = \mathrm{id}$), then the family of automorphisms $\varphi_t \psi_t \varphi_t^{-1} \psi_t^{-1}$ has zero first derivative but has second derivative given by $[D_1, D_2]$. This explains why derivations, or infinitesimal symmetries, always have the structure of a Lie algebra.

Using the correspondence between $\Gamma(M, TM)$ and $\mathrm{Der}(C^\infty(M, \mathbb{R}))$, we see that vector fields are endowed with a Lie bracket, given simply by their commutator when viewed as derivations.

Example 4.12. Let $X = \sum \alpha_i \partial_i$ and $Y = \sum \beta_i \partial_i$ be vector fields in coordinates. Then the Lie bracket $[X, Y] = \sum \gamma_i \partial_i$, where

$$\begin{aligned} \gamma_i &= X(Y(x_i)) - Y(X(x_i)) \\ &= X(\beta_i) - Y(\alpha_i) \\ &= \sum (\alpha_k \partial_k \beta_i - \beta_k \partial_k \alpha_i). \end{aligned} \tag{83}$$

The usefulness of the Lie bracket is clear from the fact that if X, Y are vector fields generating flows φ_t, ψ_s respectively, then it follows that $[X, Y]$ coincides with the time derivative of the family of vector fields $(T\varphi_{-t}) \circ Y \circ \varphi_t)$ at $t = 0$, and if $[X, Y] = 0$, then this guarantees $(\varphi_t)_* Y = Y$, and therefore that φ_t commutes with ψ_s.

Lemma 4.13. *If Φ is the flow of X, then $\Phi_* X = X$, i.e. $(\varphi_t)_* X = X$ over the appropriate domain.*

Proof. Let $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the sum, and let p_A denote the projection $A \times B \to A$. Then the 1-parameter group property may be written as the following identity[4] of maps $\mathbb{R}_1 \times \mathbb{R}_2 \times M \to M$ (we label the factors of $\mathbb{R}$ to keep track of order):

$$\Phi \circ (a \times \mathrm{id}_M) = \Phi \circ (p_{\mathbb{R}_1}, \Phi \circ p_{\mathbb{R}_2 \times M})$$

[4] We also use the notation (f, g) for maps $f : A \to B$ and $g : A \to C$ to mean $\Delta \circ (f \times g)$, where $\Delta : A \to A \times A$ is the diagonal embedding.

Then we simply differentiate and apply both sides to the vector field $\partial/\partial t_2$:

$$\Phi_*(\partial/\partial t) = \Phi_*(\Phi_*(\partial/\partial t)),$$

yielding $X = \Phi_* X$, as required. □

The fact that the diffeomorphisms φ_t preserve X automatically imply that they commute with the flows generated by X; this is no surprise, as we have $\varphi_t\varphi_{t'} = \varphi_{t+t'}$. Now we compute the way the flow of X acts on a different vector field Y.

Lemma 4.14. *Let Φ be the flow of X. If $[X,Y]=0$ for a vector field Y, then $\Phi_* Y = Y$, i.e. $(\varphi_t)_* Y = Y$ for all t.*

Proof. Extend Φ to $\tilde{\Phi} = (p_{\mathbb{R}}, \Phi) : U \to \mathbb{R} \times M$. Then we have

$$\tilde{\Phi}_* \partial_t = \partial_t + X,$$

and also $\varphi_t = p_M \circ \tilde{\Phi} \circ j_t$. At $t = 0$, it is clear that $\tilde{\Phi}_* Y = Y$, since $\varphi_0 = \mathrm{id}_M$. So, we would like to compute $[\partial_t, \tilde{\Phi}_* Y]$, which measures the time derivative of $(\varphi_t)_* Y$:

$$[\partial_t, \tilde{\Phi}_* Y] = [\tilde{\Phi}_* \partial_t - X, \tilde{\Phi}_* Y],$$

but we may use the fact that $\tilde{\Phi}_* X = X$, from the previous Lemma. Hence we have

$$[\partial_t, \tilde{\Phi}_* Y] = \Phi_*[\partial_t, Y] - \Phi_*[X,Y], \tag{84}$$

where we have used the fact that diffeomorphisms preserve Lie brackets. Since Y is time-independent, the first term vanishes, and we obtain the result. □

Remark 4.15. Equation 84 has independent interest, as it expresses the Lie bracket of vector fields as the derivative of the action of the flow of one vector field on the other. To be precise, restricting Equation 84 to the $t = 0$ slice, we obtain

$$\tfrac{d}{dt}|_{t=0}(\varphi_t)_* Y = -[X,Y].$$

Finally, since φ_t preserves Y, it will commute with any flow generated by Y, yielding the following result.

Theorem 4.16. *If X, Y are vector fields generating flows φ_t, ψ_s, then $[X,Y] = 0$ if and only if $\varphi_t\psi_s = \psi_s\varphi_t$ for all s, t.*

5 Vector bundles

Definition 5.1. A smooth real vector bundle of rank k over the base manifold M is a manifold E (called the total space), together with a smooth surjection $\pi : E \longrightarrow M$ (called the bundle projection), such that

- $\forall p \in M$, $\pi^{-1}(p) = E_p$ has the structure of k-dimensional vector space,

- Each $p \in M$ has a neighbourhood U and a diffeomorphism $\Phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ (called a local trivialization of E over U) such that $\pi_1(\Phi(\pi^{-1}(x))) = x$, where $\pi_1 : U \times \mathbb{R}^k \longrightarrow U$ is the first projection, and also that $\Phi : \pi^{-1}(x) \longrightarrow \{x\} \times \mathbb{R}^k$ is a linear map, for all $x \in M$.

Given two local trivializations $\Phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^k$ and $\Phi_j : \pi^{-1}(U_j) \longrightarrow U_j \times \mathbb{R}^k$, we obtain a smooth gluing map $\Phi_j \circ \Phi_i^{-1} : U_{ij} \times \mathbb{R}^k \longrightarrow U_{ij} \times \mathbb{R}^k$, where $U_{ij} = U_i \cap U_j$. This map preserves images to M, and hence it sends (x, v) to $(x, g_{ji}(v))$, where g_{ji} is an invertible $k \times k$ matrix smoothly depending on x. That is, the gluing map is uniquely specified by a smooth map

$$g_{ji} : U_{ij} \longrightarrow GL(k, \mathbb{R}).$$

These are called transition functions of the bundle, and since they come from $\Phi_j \circ \Phi_i^{-1}$, they clearly satisfy $g_{ij} = g_{ji}^{-1}$ as well as the "cocycle condition"

$$g_{ij} g_{jk} g_{ki} = \mathrm{Id}|_{U_i \cap U_j \cap U_k}.$$

Example 5.2. To build a vector bundle, choose an open cover $\{U_i\}$ and form the pieces $\{U_i \times \mathbb{R}^k\}$ Then glue these together on double overlaps $\{U_{ij}\}$ via functions $g_{ij} : U_{ij} \longrightarrow GL(k, \mathbb{R})$. As long as g_{ij} satisfy $g_{ij} = g_{ji}^{-1}$ as well as the cocycle condition, the resulting space has a vector bundle structure.

Example 5.3. Let $S^2 = U_0 \sqcup U_1$ for $U_i = \mathbb{R}^2$, as before. Then on $U_{01} = \mathbb{R}^2 \backslash \{0\} = \mathbb{C} \backslash \{0\}$, define

$$g_{01}(z) = [z^k], \quad k \in \mathbb{Z}.$$

In real coordinates $z = re^{i\theta}$, $g_{01}(r, \theta) = r^k \begin{pmatrix} cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \end{pmatrix}$. This defines a vector bundle $E_k \longrightarrow S^2$ of rank 2 for each $k \in \mathbb{Z}$ (or a complex vector bundle of rank 1, since $g_{01} : U_{01} \longrightarrow GL(1, \mathbb{C})$). Actually, since the map g_{01} is actually holomorphic as a function of z, we have defined *holomorphic* vector bundles on $\mathbb{C}P^1$.

Example 5.4 (The tangent bundle)**.** The tangent bundle TM is indeed a vector bundle, of rank $\dim M$. For any chart (U, φ) of M, there is an associated local trivialization $(\pi^{-1}(U), \Phi)$ of TM, and the transition function $g_{ji} : U_{ij} \longrightarrow GL(n, \mathbb{R})$ between two trivializations obtained from $(U_i, \varphi_i), (U_j, \varphi_j)$ is simply the Jacobian matrix

$$g_{ji} : p \mapsto D(\varphi_j \circ \varphi_i^{-1})(p).$$

Just as for the tangent bundle, we can define the analog of a vector-valued function, where the function has values in a vector bundle:

Definition 5.5. A smooth section of the vector bundle $E \xrightarrow{\pi} M$ is a smooth map $s : M \longrightarrow E$ such that $\pi \circ s = \mathrm{Id}_M$. The set of all smooth sections, denoted $\Gamma^\infty(M, E)$, is an infinite-dimensional real vector space, and is also a module over the ring $C^\infty(M, \mathbb{R})$.

Having introduced vector bundles, we must define the notion of *morphism* between vector bundles, so as to form a category.

Definition 5.6. A smooth bundle map between the bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ is a pair (f, F) of smooth maps $f : M \longrightarrow M'$ and $F : E \longrightarrow E'$ such that $\pi' \circ F = f \circ \pi$ and such that $F : E_p \longrightarrow E'_{f(p)}$ is a linear map for all p.

Example 5.7. I claim that the bundles $E_k \xrightarrow{\pi} S^2$ are all non-isomorphic, except that E_k is isomorphic to E_{-k} over the antipodal map $S^2 \longrightarrow S^2$.

Example 5.8. Suppose $f : M \longrightarrow N$ is a smooth map. Then $f_* : TM \longrightarrow TN$ is a bundle map covering f, i.e. (f_*, f) defines a bundle map.

Example 5.9 (Pullback bundle)**.** if $f : M \longrightarrow N$ is a smooth map and $E \xrightarrow{\pi} N$ is a vector bundle over N, then we may form the fiber product $M_f \times_\pi E$, which then is a bundle over M with local trivializations $(f^{-1}(U_i), f^* g_{ij})$, where (U_i, g_{ij}) is the local transition data for E over N. This bundle is called the pullback bundle and is denoted by f^*E. The natural projection to E defines a vector bundle map back to E:

$$\begin{tikzcd} f^*E \arrow[r, "p_2"] \arrow[d, "p_1"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

There is also a natural pullback map on sections: given a section $s \in \Gamma^\infty(N, E)$, the composition $s \circ f$ gives a map $M \longrightarrow E$. This then determines a smooth map $f^*s : M \longrightarrow f^*E$ by the universal property of the fiber product. We therefore obtain a pullback map

$$f^* : \Gamma^\infty(N, E) \longrightarrow \Gamma^\infty(M, f^*E).$$

Example 5.10. If $f : M \longrightarrow N$ is an embedding, then so is the bundle map $f_* : TM \longrightarrow TN$. By the universal property of the fiber product we obtain a bundle map, also denoted f_*, from TM to f^*TN. This is a vector bundle inclusion and $f^*TN/f_*TM = NM$ is a vector bundle over M called the normal bundle of M. *Note: we haven't covered subbundles and quotient bundles in detail. I'll leave this as an exercise.*

5.1 Associated bundles

We now describe a functorial construction of vector bundles, using functors from vector spaces. Consider the category $\mathbf{Vect}_\mathbb{R}$ of finite-dimensional real vector spaces and linear maps. We will describe several functors from $\mathbf{Vect}_\mathbb{R}$ to itself.

Example 5.11. If $V \in \mathbf{Vect}_\mathbb{R}$, then $V^* \in \mathbf{Vect}_\mathbb{R}$, and if $f : V \longrightarrow W$ then $f^* : W^* \longrightarrow V^*$. Since the composition of duals is the dual of the composition, duality defines a contravariant functor $* : \mathbf{Vect}_\mathbb{R} \longrightarrow \mathbf{Vect}_\mathbb{R}$.

Example 5.12. If $V, W \in \mathbf{Vect}_\mathbb{R}$, then $V \oplus W \in \mathbf{Vect}_\mathbb{R}$, and this defines a covariant functor $\mathbf{Vect}_\mathbb{R} \times \mathbf{Vect}_\mathbb{R} \longrightarrow \mathbf{Vect}_\mathbb{R}$.

Example 5.13. If $V, W \in \mathbf{Vect}_\mathbb{R}$, then $V \otimes W \in \mathbf{Vect}_\mathbb{R}$ and this again defines a covariant functor $\mathbf{Vect}_\mathbb{R} \times \mathbf{Vect}_\mathbb{R} \longrightarrow \mathbf{Vect}_\mathbb{R}$.

Example 5.14. If $V \in \mathbf{Vect}_{\mathbb{R}}$, then

$$\otimes^\bullet V = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \cdots \oplus (\otimes^k V) \oplus \cdots$$

is an infinite-dimensional vector space, with a product $a \otimes b$. Quotienting by the double-sided ideal $I = \langle v \otimes v \ : \ v \in V \rangle$, we obtain the exterior algebra

$$\wedge^\bullet V = \mathbb{R} \oplus V \oplus \wedge^2 V \oplus \cdots \oplus \wedge^n V,$$

with $n = \dim V$. The product is customarily denoted $(a, b) \mapsto a \wedge b$. The direct sum decompositions above, where $\wedge^k V$ or $\otimes^k V$ is labeled by the integer k, are called $\mathbb{Z}$-gradings, and since the product takes $\wedge^k \times \wedge^l \longrightarrow \wedge^{k+l}$, these algebras are called Z-graded algebras.

If $(v_1, \ldots v_n)$ is a basis for V, then $v_{i_1} \wedge \cdots \wedge v_{i_k}$ for $i_1 < \cdots < i_k$ form a basis for $\wedge^k V$. This space then has dimension $\binom{n}{k}$, hence the algebra $\wedge^\bullet V$ has dimension 2^n.

Note in particular that $\wedge^n V$ has dimension 1, is also called the determinant line $\det V$, and a choice of nonzero element in $\det V$ is called an "orientation" on the vector space V.

Recall that if $f : V \longrightarrow W$ is a linear map, then $\wedge^k f : \wedge^k V \longrightarrow \wedge^k W$ is defined on monomials via

$$\wedge^k f(a_1 \wedge \cdots \wedge a_k) = f(a_1) \wedge \cdots \wedge f(a_k).$$

In particular, if $A : V \longrightarrow V$ is a linear map, then for $n = \dim V$, the top exterior power $\wedge^n A : \wedge^n V \longrightarrow \wedge^n V$ is a linear map of a 1-dimensional space onto itself, and is hence given by a number, called $\det A$, the determinant of A.

We may now apply any of these functors to vector *bundles*. The main observation is that if F is a vector space functor as above, we may apply it to any vector bundle $E \xrightarrow{\pi} M$ to obtain a new vector bundle

$$F(E) = \sqcup_{p \in M} F(E_p).$$

If (U_i) is an atlas for M and E has local trivializations $(U_i \times \mathbb{R}^k)$, glued together via $g_{ji} : U_{ij} \longrightarrow GL(k, \mathbb{R})$, then $F(E)$ may be given the local trivialization $(U_i \times F(\mathbb{R}^k))$, glued together via $F(g_{ji})$. This new vector bundle $F(E)$ is called the "associated" vector bundle to E, given by the functor F.

Example 5.15. If $E \longrightarrow M$ is a vector bundle, then $E^* \longrightarrow M$ is the dual vector bundle. If E, F are vector bundles then $E \oplus F$ is called the direct or "Whitney" sum, and has rank $\mathrm{rk}\ E + \mathrm{rk}\ F$. $E \otimes F$ is the tensor product bundle, which has rank $\mathrm{rk}\ E \cdot \mathrm{rk}\ F$.

Example 5.16. If $E \longrightarrow M$ is a vector bundle of rank n, then $\otimes^k E$ and $\wedge^k E$ are its tensor power bundles, of rank n^k and $\binom{n}{k}$, respectively. The top exterior power $\wedge^n E$ has rank 1, and is hence a line bundle. If this line bundle is trivial (i.e. isomorphic to $M \times \mathbb{R}$) then E is said to be an orientable bundle.

Example 5.17. Starting with the tangent bundle $TM \longrightarrow M$, we may form the cotangent bundle T^*M, the bundle of tensors of type (r, s), $\otimes^r TM \otimes \otimes^s T^*M$.

We may also form the bundle of multivectors $\wedge^k TM$, which has sections $\Gamma^\infty(M, \wedge^k TM)$ called multivector fields.

Finally, we may form the bundle of k-forms, $\wedge^k T^*M$, whose sections $\Gamma^\infty(M, \wedge^k T^*M) = \Omega^k(M)$ are called differential k-forms, and will occupy us for some time.

We have now produced several vector bundles by applying functors to the tangent bundle. We are familiar with vector fields, which are sections of TM, and we know that a vector field is written locally in coordinates $(x^1, \dots, x^n)$ as

$$X = \sum_i a^i \tfrac{\partial}{\partial x^i},$$

with coefficients a^i smooth functions.

There is an easy way to produce examples of 1-forms in $\Omega^1(M)$, using smooth functions f. We note that the action $X \mapsto X(f)$ defines a dual vector at each point of M, since $(X(f))_p$ depends only on the vector X_p and not the behaviour of X away from p. Recall that $X(f) = Df_2(X)$.

Definition 5.18. The exterior derivative of a function f, denoted df, is the section of T^*M given by the fiber projection Df_2.

Since $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, we see that $(dx^1, \dots, dx^n)$ is the dual basis to $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$. Therefore, a section of T^*M has local expression

$$\xi = \sum_i \xi_i dx^i,$$

for ξ_i smooth functions, given by $\xi_i = \xi(\frac{\partial}{\partial x^i})$. In particular, the exterior derivative of a function df can be written

$$df = \sum_i \tfrac{\partial f}{\partial x^i} dx^i.$$

A section of the tensor bundle $\otimes^r TM \otimes \otimes^s T^*M$ can be written as

$$\Theta = \sum_{\substack{i_1, \cdots, i_r \\ j_1, \cdots, j_s}} a^{i_1 \cdots i_r}_{j_1 \cdots j_s} \tfrac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \tfrac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s},$$

where $a^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ are n^{r+s} smooth functions.

A general differential form $\rho \in \Omega^k(M)$ can be written

$$\rho = \sum_{i_1 < \cdots < i_k} \rho_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

6 Differential forms

There are several properties of differential forms which make them indispensible: first, the k-forms are intended to give a notion of k-dimensional volume (this is why they are multilinear and skew-symmetric, like the determinant) and in a way compatible with the boundary map (this leads to the exterior derivative, which we define below). Second, they behave well functorially, as we see now.

Given a smooth map $f : M \longrightarrow N$, we obtain bundle maps $f_* : TM \longrightarrow TN$ and hence $f^* := \wedge^k (f_*)^* : \wedge^k T^*N \longrightarrow \wedge^k T^*M$. Hence we have the diagram

$$\begin{tikzcd} \wedge^k T^*M \arrow[d, "\pi_M"'] & \wedge^k T^*N \arrow[l, "f^*"] \arrow[d, "\pi_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

The interesting thing is that if $\rho \in \Omega^k(N)$ is a differential form on N, then it is a section of π_N. Composing with f, f^*, we obtain a section $f^*\rho := f^* \circ \rho \circ f$ of π_M. Hence we obtain a natural map

$$\Omega^k(N) \xrightarrow{f^*} \Omega^k(M).$$

Such a natural map does not exist (in either direction) for multivector fields, for instance.

Suppose that $\rho \in \Omega^k(N)$ is given in a coordinate chart by

$$\rho = \sum \rho_{i_1 \cdots i_k} dy^{i_1} \wedge \cdots \wedge dy^{i_k}.$$

Now choose a coordinate chart for M with coordinates $x^1, \ldots x^m$. What is the local expression for $f^*\rho$? We need only compute $f^* dy_i$. We use a notation where f^k denotes the k^{th} component of f in the coordinates $(y^1, \ldots y^n)$, i.e. $f^k = y^k \circ f$.

$$\begin{aligned} f^* dy_i(\tfrac{\partial}{\partial x^j}) &= dy_i(f_* \tfrac{\partial}{\partial x^j}) && (85) \\ &= dy_i(\sum_k \tfrac{\partial f^k}{\partial x^j} \tfrac{\partial}{\partial y_k}) && (86) \\ &= \tfrac{\partial f^i}{\partial x^j}. && (87) \end{aligned}$$

Hence we conclude that

$$f^* dy_i = \sum_j \tfrac{\partial f^i}{\partial x^j} dx^j.$$

Finally we compute

$$\begin{aligned} f^*\rho &= \sum_{i_1 < \cdots < i_k} f^*\rho_{i_1 \cdots i_k} f^*(dy^{i_1}) \wedge \cdots \wedge f^*(dy^{i_k}) && (88) \\ &= \sum_{i_1 < \cdots < i_k} (\rho_{i_1 \cdots i_k} \circ f) \sum_{j_1} \cdots \sum_{j_k} \tfrac{\partial f^{i_1}}{\partial x^{j_1}} \cdots \tfrac{\partial f^{i_k}}{\partial x^{j_k}} dx^{j_1} \wedge \cdots dx^{j_k}. && (89) \end{aligned}$$

6.1 The exterior derivative

Differential forms are equipped with a natural differential operator, which extends the exterior derivative of functions to all forms: $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$. The exterior derivative is uniquely specified by the following requirements: first, it satisfies $d(df) = 0$ for all functions f. Second, it is a graded derivation of the algebra of exterior differential forms of degree 1, i.e.

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta.$$

This allows us to compute its action on any 1-form $d(\xi_i dx^i) = d\xi_i \wedge dx^i$, and hence, in coordinates, we have

$$d(\rho dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \sum_k \tfrac{\partial \rho}{\partial x^k} dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Extending by linearity, this gives a local definition of d on all forms. Does it actually satisfy the requirements? this is a simple calculation: let $\tau_p = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ and $\tau_q = dx^{j_1} \wedge \cdots \wedge dx^{j_q}$. Then

$$d((f\tau_p)\wedge(g\tau_q)) = d(fg\tau_p\wedge\tau_q) = (gdf+fdg)\wedge\tau_p\wedge\tau_q = d(f\tau_p)\wedge g\tau_q+(-1)^p f\tau_p\wedge d(g\tau_q),$$

as required.

Therefore we have defined d, and since the definition is coordinate-independent, we can be satisfied that d is well-defined.

Definition 6.1. d is the unique degree $+1$ graded derivation of $\Omega^\bullet(M)$ such that $df(X) = X(f)$ and $d(df) = 0$ for all functions f.

Example 6.2. Consider $M = \mathbb{R}^3$. For $f \in \Omega^0(M)$, we have

$$df = \tfrac{\partial f}{\partial x^1} dx^1 + \tfrac{\partial f}{\partial x^2} dx^2 + \tfrac{\partial f}{\partial x^3} dx^3.$$

Similarly, for $A = A_1 dx^1 + A_2 dx^2 + A_3 dx^3$, we have

$$dA = (\tfrac{\partial A_2}{\partial x^1} - \tfrac{\partial A_1}{\partial x^2})dx^1 \wedge dx^2 + (\tfrac{\partial A_3}{\partial x^1} - \tfrac{\partial A_1}{\partial x^3})dx^1 \wedge dx^3 + (\tfrac{\partial A_3}{\partial x^2} - \tfrac{\partial A_2}{\partial x^3})dx^2 \wedge dx^3$$

Finally, for $B = B_{12} dx^1 \wedge dx^2 + B_{13} dx^1 \wedge dx^3 + B_{23} dx^2 \wedge dx^3$, we have

$$dB = (\tfrac{\partial B_{12}}{\partial x^3} - \tfrac{\partial B_{13}}{\partial x^2} + \tfrac{\partial B_{23}}{\partial x^1})dx^1 \wedge dx^2 \wedge dx^3.$$

Definition 6.3. The form $\rho \in \Omega^\bullet(M)$ is called *closed* when $d\rho = 0$ and *exact* when $\rho = d\tau$ for some τ.

Example 6.4. A function $f \in \Omega^0(M)$ is closed if and only if it is constant on each connected component of M: This is because, in coordinates, we have

$$df = \tfrac{\partial f}{\partial x^1} dx^1 + \cdots + \tfrac{\partial f}{\partial x^n} dx^n,$$

and if this vanishes, then all partial derivatives of f must vanish, and hence f must be constant.

Theorem 6.5. *The exterior derivative of an exact form is zero, i.e.* $d \circ d = 0$. *Usually written* $d^2 = 0$.

Proof. The graded commutator $[d_1, d_2] = d_1 \circ d_2 - (-1)^{|d_1||d_2|} d_2 \circ d_1$ of derivations of degree $|d_1|, |d_2|$ is always (why?) a derivation of degree $|d_1| + |d_2|$. Hence we see $[d, d] = d \circ d - (-1)^{1\cdot 1} d \circ d = 2d^2$ is a derivation of degree 2 (and so is d^2). Hence to show it vanishes we must test on functions and exact 1-forms, which locally generate forms since every form is of the form $f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

But $d(df) = 0$ by definition and this certainly implies $d^2(df) = 0$, showing that $d^2 = 0$. ∎

6.2 de Rham Cohomology

The fact that $d^2 = 0$ is dual to the fact that $\partial(\partial M) = \emptyset$ for a manifold with boundary M. We will see later that Stokes' theorem explains this duality. Because of the fact $d^2 = 0$, we have a very special algebraic structure: we have a sequence of vector spaces $\Omega^k(M)$, and maps $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ which are such that any successive composition is zero. This means that the image of d is contained in the kernel of the next d in the sequence. This arrangement of vector spaces and operators is called a *cochain complex* of vector spaces [5]. We often simply refer to this as a "complex" and omit the term "cochain". The reason for the "co" is that the differential increases the degree k, which is opposite to the usual boundary map on manifolds, which decreases k. We will see chain complexes when we study homology.

A complex of vector spaces is usually drawn as a linear sequence of symbols and arrows as follows: if $f : U \longrightarrow V$ is a linear map and $g : V \longrightarrow W$ is a linear map such that $g \circ f = 0$, then we write

$$U \xrightarrow{f} V \xrightarrow{g} W$$

In general, this simply means that $\mathrm{im} f \subset \ker g$, and to measure the difference between them we look at the quotient $\ker g / \mathrm{im} f$, which is called the **cohomology** of the complex at the position V (or homology, if d decreases degree). If we are lucky, and the complex has no cohomology at V, meaning that $\ker g$ is precisely equal to $\mathrm{im} f$, then we say that the complex is **exact** at V. If the complex is exact everywhere, we call it an exact sequence (and it has no cohomology!) In our case, we have a longer cochain complex:

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M) \longrightarrow 0$$

There is a bit of terminology to learn: we have seen that if $d\rho = 0$ then ρ is called *closed*. But these are also called **cocycles** and denoted $Z^k(M)$. Similarly the exact forms $d\alpha$ are also called **coboundaries**, and are denoted $B^k(M)$. Hence the cohomology groups may be written $H^k_{dR}(M) = Z^k_{dR}(M)/B^k_{dR}(M)$.

Definition 6.6. The de Rham complex is the complex $(\Omega^\bullet(M), d)$, and its cohomology at $\Omega^k(M)$ is called $H^k_{dR}(M)$, the de Rham cohomology.

Exercise: Check that the graded vector space $H^\bullet_{dR}(M) = \bigoplus_{k\in\mathbb{Z}} H^k(M)$ inherits a product from the wedge product of forms, making it into a $\mathbb{Z}$-graded ring. This is called the de Rham cohomology ring of M, and the product is called the *cup product*.

It is clear from the definition of d that it commutes with pullback via diffeomorphisms, in the sense $f^* \circ d = d \circ f^*$. But this is only a special case of a more fundamental property of d:

Theorem 6.7. *Exterior differentiation commutes with pullback: for $f : M \longrightarrow N$ a smooth map, $f^* \circ d_N = d_M \circ f^*$.*

[5] since this complex appears for $\Omega^\bullet(U)$ for any open set $U \subset M$, this is actually a cochain complex of *sheaves* of vector spaces, but this won't concern us right away.

Proof. We need only check this on functions g and exact 1-forms dg: let X be a vector field on M and $g \in C^\infty(N, \mathbb{R})$.

$$f^*(dg)(X) = dg(f_*X) = \pi_2 g_* f_* X = \pi_2(g \circ f)_* X = d(f^*g)(X),$$

giving $f^*dg = df^*g$, as required. For exact 1-forms we have $f^*d(dg) = 0$ and $d(f^*dg) = d(df^*g) = 0$ by the result for functions. $\square$

This theorem may be interpreted as follows: The differential forms give us a $\mathbb{Z}$-graded ring, $\Omega^\bullet(M)$, which is equipped with a differential $d : \Omega^k \longrightarrow \Omega^{k+1}$. This sequence of vector spaces and maps which compose to zero is called a *cochain complex*. Beyond it being a cochain complex, it is equipped with a wedge product.

Cochain complexes $(C^\bullet, d_C)$ may be considered as objects of a new category, whose morphisms consist of a sum of linear maps $\psi_k : C^k \longrightarrow D^k$ commuting with the differentials, i.e. $d_D \circ \psi_k = \psi_{k+1} \circ d_C$. The previous theorem shows that pullback f^* defines a morphism of cochain complexes $\Omega^\bullet(N) \longrightarrow \Omega^\bullet(M)$; indeed it even preserves the wedge product, hence it is a morphism of differential graded algebras.

Corollary 6.8. *We may interpret the previous result as showing that $\Omega^\bullet$ is a functor from manifolds to differential graded algebras (or, if we forget the wedge product, to the category of cochain complexes). As a result, we see that the de Rham cohomology $H^\bullet_{dR}$ may be viewed as a functor, from smooth manifolds to $\mathbb{Z}$-graded commutative rings.*

Example 6.9. S^1 is connected, and hence $H^0_{dR}(S^1) = \mathbb{R}$. So it remains to compute $H^1_{dR}(S^1)$.

Let $\frac{\partial}{\partial\theta}$ be the rotational vector field on S^1 of unit Euclidean norm, and let $d\theta$ be its dual 1-form, i.e. $d\theta(\frac{\partial}{\partial\theta}) = 1$. Note that θ is not a well-defined function on S^1, so the notation $d\theta$ may be misleading at first.

Of course, $d(d\theta) = 0$, since $\Omega^2(S^1) = 0$. We might ask, is there a function $f(\theta)$ such that $df = d\theta$? This would mean $\frac{\partial f}{\partial\theta} = 1$, and hence $f = \theta + c_2$. But since f is a function on S^1, we must have $f(\theta+2\pi) = f(\theta)$, which is a contradiction. Hence $d\theta$ is not exact, and $[d\theta] \neq 0$ in $H^1_{dR}(S^1)$.

Any other 1-form will be closed, and can be represented as $gd\theta$ for $g \in C^\infty(S^1, \mathbb{R})$. Let $\overline{g} = \frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi} g(\theta)d\theta$ be the average value of g, and consider $g_0 = g - \overline{g}$. Then define

$$f(\theta) = \int_{t=0}^{t=\theta} g_0(t)dt.$$

Clearly we have $\frac{\partial f}{\partial\theta} = g_0(\theta)$, and furthermore f is a well-defined function on S^1, since $f(\theta + 2\pi) = f(\theta)$. Hence we have that $g_0 = df$, and hence $g = \overline{g} + df$, showing that $[gd\theta] = \overline{g}[d\theta]$.

Hence $H^1_{dR}(S^1) = \mathbb{R}$, and as a ring, $H^0_{dR} + H^1_{dR}$ is simply $\mathbb{R}[x]/(x^2)$.

Note that technically we have proven that $H^1_{dR}(S^1) \cong \mathbb{R}$, but we will see from the definition of integration later that this isomorphism is canonical.

The de Rham cohomology is an important invariant of a smooth manifold (in fact it doesn't even depend on the smooth structure, only the topological structure). To compute it, there are many tools available.

There are three particularly important tools: first, there is Poincaré's lemma, telling us the cohomology of $\mathbb{R}^n$. Second, there is integration, which allows us to prove that certain cohomology classes are non-trivial. Third, there is the Mayer-Vietoris sequence, which allows us to compute the cohomology of a union of open sets, given knowledge about the cohomology of each set in the union.

Lemma 6.10. *Consider the embeddings $J_i : M \longrightarrow M \times [0,1]$ given by $x \mapsto (x,i)$ for $i = 0,1$. The induced morphisms of de Rham complexes J_0^* and J_1^* are chain homotopic morphisms, meaning that there is a linear map $K : \Omega^k(M \times [0,1]) \longrightarrow \Omega^{k-1}(M)$ such that*

$$J_1^* - J_0^* = dK + Kd$$

This shows that on closed forms, J_i^* may differ, but only by an exact form.

Proof. Let t be the coordinate on $[0,1]$. Define $Kf = 0$ for $f \in \Omega^0(M \times [0,1])$, and $K\alpha = 0$ if $\alpha = f\rho$ for $\rho \in \Omega^k(M)$. But for $\beta = fdt \wedge \rho$ we define

$$K\beta = (\int_0^1 f dt)\rho.$$

Then we verify that

$$dKf + Kdf = 0 + \int_0^1 \tfrac{\partial f}{\partial t} dt = (J_1^* - J_0^*)f,$$

$$dK\alpha + Kd\alpha = 0 + (\int_0^1 \tfrac{\partial f}{\partial t} dt)\rho = (J_1^* - J_0^*)\alpha,$$

$$dK\beta + Kd\beta = (\int_0^1 d_M f dt) \wedge \rho + (\int_0^1 f dt) \wedge d\rho + K(df \wedge dt \wedge \rho - f dt \wedge d\rho) = 0,$$

which agrees with $(J_1^* - J_0^*)\beta = 0 - 0 = 0$. Note that we have used $K(df \wedge dt \wedge \rho) = K(-dt \wedge d_M f \wedge \rho) = -(\int_0^1 d_M f) \wedge \rho$, and the notation $d_M f$ is a time-dependent 1-form whose value at time t is the exterior derivative on M of the function $f(-,t) \in \Omega^0(M)$. □

The previous theorem can be used in a clever way to prove that homotopic maps $M \longrightarrow N$ induce the same map on cohomology:

Theorem 6.11. *Let $f : M \longrightarrow N$ and $g : M \longrightarrow N$ be smooth maps which are (smoothly) homotopic. Then $f^* = g^*$ as maps $H^\bullet(N) \longrightarrow H^\bullet(M)$.*

Proof. Let $H : M \times [0,1] \longrightarrow N$ be a (smooth) homotopy between f, g, and let J_0, J_1 be the embeddings $M \longrightarrow M \times [0,1]$ from the previous result, so that $H \circ J_0 = f$ and $H \circ J_1 = g$. Recall that $J_1^* - J_0^* = dK + Kd$, so we have

$$g^* - f^* = (J_1^* - J_0^*)H^* = (dK + Kd)H^* = dKH^* + KH^*d$$

This shows that f^*, g^* differ, on closed forms, only by exact terms, and hence are equal on cohomology. □

Corollary 6.12. *If M, N are (smoothly) homotopic, then $H^\bullet_{dR}(M) \cong H^\bullet_{dR}(N)$.*

Proof. M, N are homotopic iff we have maps $f : M \longrightarrow N$, $g : N \longrightarrow M$ with $fg \sim 1$ and $gf \sim 1$. This shows that $f^*g^* = 1$ and $g^*f^* = 1$, hence f^*, g^* are inverses of each other on cohomology, and hence isomorphisms. □

Corollary 6.13 (Poincaré lemma)**.** *Since $\mathbb{R}^n$ is homotopic to the 1-point space ($\mathbb{R}^0$), we have*

$$H^k_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} \text{ for } k = 0 \\ 0 \text{ for } k > 0 \end{cases}$$

As a note, we should mention that the homotopy in the previous theorem need not be smooth, since any homotopy may be deformed (using a continuous homotopy) to a smooth homotopy, by smooth approximation. Hence we finally obtain that the de Rham cohomology is a homotopy invariant of smooth manifolds.

6.3 Integration

Since we are accustomed to the idea that a function may be integrated over a subset of $\mathbb{R}^n$, we might think that if we have a function on a manifold, we can compute its local integrals and take a sum. This, however, makes no sense, because the answer will depend on the particular coordinate system you choose in each open set: for example, if $f : U \longrightarrow \mathbb{R}$ is a smooth function on $U \subset \mathbb{R}^n$ and $\varphi : V \longrightarrow U$ is a diffeomorphism onto $V \subset \mathbb{R}^n$, then we have the usual change of variables formula for the (Lebesgue or Riemann) integral:

$$\int_U f dx^1 dx^2 \cdots dx^n = \int_V \varphi^* f \left|\det[\tfrac{\partial \varphi_i}{\partial x^j}]\right| dx^1 \cdots dx^n.$$

The extra factor of the absolute value of the Jacobian determinant shows that the integral of f is coordinate-dependant. For this reason, it makes more sense to view the left hand side not as the integral of f but rather as the integral of $\nu = f dx^1 \wedge \cdots \wedge dx^n$. Then, the right hand side is indeed the integral of $\varphi^*\nu$ (which includes the Jacobian determinant in its expression automatically) , as long as φ^* has positive Jacobian determinant.

Therefore, the integral of a differential n-form will be well-defined on an n-manifold M, as long as we can choose an atlas where the Jacobian determinants of the gluing maps are all positive: This is precisely the choice of an *orientation* on M, as we now show.

Definition 6.14. A n-manifold M is called *orientable* when $\det T^*M := \wedge^n T^*M$ is isomorphic to the trivial line bundle. An orientation is the choice of an equivalence class of nonvanishing sections v, where $v \sim v'$ iff $fv = v'$ for $f \in C^\infty(M, \mathbb{R})$. M is called *oriented* when an orientation is chosen, and if M is connected and orientable, there are two possible orientations.

$\mathbb{R}^n$ has a natural orientation by $dx^1 \wedge \cdots \wedge dx^n$; if M is orientable, we may choose charts which preserve orientation, as we now show.

Proposition 6.15. *If the n-manifold M is oriented by $[v]$, it is possible to choose an orientation-preserving atlas (U_i, φ_i) in the sense that $\varphi_i^* dx^1 \wedge \cdots \wedge dx^n \sim v$ for all i. In particular, the Jacobian determinants for this atlas are all positive.*

Proof. Choose any atlas (U_i, φ_i). For each i, either $\varphi_i^* dx^1 \wedge \cdots \wedge dx^n \sim v$, and if not, replace φ_i with $q \circ \varphi$, where $q : (x^1, \ldots, x^n) \mapsto (-x^1, \ldots, x^n)$. This completes the proof. $\square$

Now we can define the integral on an oriented n-manifold M, by defining the integral on chart images and asking it to be compatible with these charts:

Theorem 6.16. *Let M be an oriented n-manifold. Then there is a unique linear map $\int_M : \Omega_c^n(M) \longrightarrow \mathbb{R}$ on compactly supported n-forms which has the following property: if h is an orientation-preserving diffeomorphism from $V \subset \mathbb{R}^n$ to $U \subset M$, and if $\alpha \in \Omega_c^n(M)$ has support contained in U, then*

$$\int_M \alpha = \int_V h^* \alpha.$$

Proof. Let $\alpha \in \Omega_c^n(M)$ and choose an orientation-preserving, locally finite atlas (U_i, φ_i) with subordinate partition of unity (θ_i). Then using the required properties (and noting that α is nonzero in only finitely many U_i), we have

$$\int_M \alpha = \sum_i \int_M \theta_i \alpha = \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* \theta_i \alpha.$$

This proves the uniqueness of the integral. To show existence, we must prove that the above expression actually satisfies the defining condition, and hence can be used as an explicit definition of the integral.

Let $h : V \longrightarrow U$ be an orientation-preserving diffeomorphism from $V \subset \mathbb{R}^n$ to $U \subset M$, and suppose α has support in U. Then $\varphi_i \circ h$ are orientation-preserving, and

$$\begin{aligned}
\int_M \alpha &= \sum_i \int_{\varphi_i(U_i) \cap \varphi_i(U)} (\varphi_i^{-1})^* \theta_i \alpha \\
&= \sum_i \int_{V \cap h^{-1}(U_i)} (\varphi_i \circ h)^* (\varphi_i^{-1})^* \theta_i \alpha \\
&= \sum_i \int_{V \cap h^{-1}(U_i)} h^* (\theta_i \alpha) \\
&= \int_V h^* \alpha,
\end{aligned}$$

as required. $\square$

6.4 Stokes' Theorem

Having defined the integral, we wish to explain the duality between d and ∂: A $n-1$-form α on a n-manifold may be pulled back to the boundary ∂M and integrated. On the other hand, it can be differentiated and integrated over M. The fact that these are equal is Stokes' theorem, and is a generalization of the fundamental theorem of calculus.

First we must some simple observations concerning the behaviour of forms in a neighbourhood of the boundary.

Recall the operation of contraction with a vector field X, which maps $\rho \in \Omega^k(M)$ to $i_X\rho \in \Omega^{k-1}(M)$, defined by the condition of being a graded derivation $i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge i_X\beta$ such that $i_X f = 0$ and $i_X df = X(f)$ for all $f \in C^\infty(M, \mathbb{R})$.

Proposition 6.17. *Let M be a manifold with boundary. If M is orientable, then so is ∂M. Furthermore, an orientation on M induces one on ∂M.*

Proof. Given a locally finite atlas (U_i) of ∂M, in each U_i we can pick a nonvanishing outward-pointing vector field X_i in $\Gamma^\infty(U_i, j^*TM)$, for $j : \partial M \longrightarrow M$ the inclusion. Let (θ_i) be a subordinate partition of unity, and form $X = \sum_i \theta_i X_i$. This is a vector field on ∂M, tangent to M and pointing outward everywhere along the boundary.

Given an orientation $[v]$ of M, we can form $[i_X v]$, which is then an orientation of ∂M. This depends only on $[v]$ and X being a nonvanishing outward vector field. $\square$

We now verify a local computation leading to Stokes' theorem. If

$$\alpha = \sum_i a_i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^m$$

is a degree $m-1$ form with compact support in $U \subset H^m$, and if U does not intersect the boundary ∂H^m, then by the fundamental theorem of calculus,

$$\int_U d\alpha = \sum_i (-1)^{i-1} \int_U \tfrac{\partial a_i}{\partial x^i} dx^1 \cdots dx^m = 0.$$

Now suppose that $V = U \cap \partial H^m \neq \emptyset$. Then

$$\begin{aligned}
\int_U d\alpha &= \sum_i (-1)^{i-1} \int_U \tfrac{\partial a_i}{\partial x^i} dx^1 \cdots dx^m \\
&= -(-1)^{m-1} \int_V a_m(x_1, \ldots, x_{m-1}, 0) dx^1 \cdots dx^{m-1} \\
&= \int_V a_m(x_1, \ldots, x_{m-1}, 0) i_{-\frac{\partial}{\partial x^m}} (dx^1 \wedge \cdots dx^m) \\
&= \int_V j^*\alpha,
\end{aligned}$$

where the last integral is with respect to the orientation induced by the outward vector field.

Theorem 6.18 (Stokes' theorem)**.** *Let M be an oriented manifold with boundary, and let the boundary be oriented with respect to an outward pointing vector field. Then for $\alpha \in \Omega_c^{m-1}(M)$ and $j : \partial M \longrightarrow M$ the inclusion of the boundary, we have*

$$\int_M d\alpha = \int_{\partial M} j^*\alpha.$$

Proof. For a locally finite atlas (U_i, φ_i), we have

$$\int_M d\alpha = \int_M d(\sum_i \theta_i\alpha) = \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* d(\theta_i\alpha)$$

By the local calculation above, if $\varphi_i(U_i) \cap \partial H^m = \emptyset$, the summand on the right hand side vanishes. On the other hand, if $\varphi_i(U_i) \cap \partial H^m \neq \emptyset$, we obtain (letting $\psi_i = \varphi_i|_{U_i \cap \partial M}$ and $j' : \partial H^m \longrightarrow \mathbb{R}^n$), using the local result,

$$\begin{aligned}\int_{\varphi_i(U_i)} (\varphi_i^{-1})^* d(\theta_i\alpha) &= \int_{\varphi_i(U_i)\cap\partial H^m} j'^*(\varphi_i^{-1})^*(\theta_i\alpha)\\ &= \int_{\varphi_i(U_i)\cap\partial H^m} (\psi_i^{-1})^*(j^*(\theta_i\alpha)).\end{aligned}$$

This then shows that $\int_M d\alpha = \int_{\partial M} j^*\alpha$, as desired. □

Corollary 6.19. *If $\partial M = \emptyset$, then for all $\alpha \in \Omega_c^{n-1}(M)$, we have $\int_M d\alpha = 0$.*

Corollary 6.20. *Let M be orientable and compact, and let $v \in \Omega^n(M)$ be nonvanishing. Then $\int_M v > 0$, when M is oriented by $[v]$. Hence, v cannot be exact, by the previous corollary. This tells us that the class $[v] \in H^n_{dR}(M)$ cannot be zero. In this way, integration of a closed form may often be used to show that it is nontrivial in de Rham cohomology.*

6.5 The Mayer-Vietoris sequence

Decompose a manifold M into a union of open sets $M = U \cup V$. We wish to relate the de Rham cohomology of M to that of U and V separately, and also that of $U \cap V$. These 4 manifolds are related by obvious inclusion maps as follows:

$$\begin{tikzcd} U \cup V & U \sqcup V \arrow[l] & U \cap V \arrow[l, shift right, "\partial_U"'] \arrow[l, shift left, "\partial_V"] \end{tikzcd}$$

Applying the functor $\Omega^\bullet$, we obtain morphisms of complexes in the other direction, given by simple restriction (pullback under inclusion):

$$\begin{tikzcd} \Omega^\bullet(U \cup V) \arrow[r] & \Omega^\bullet(U) \oplus \Omega^\bullet(V) \arrow[r, shift left, "\partial_V^*"] \arrow[r, shift right, "\partial_U^*"'] & \Omega^\bullet(U \cap V) \end{tikzcd}$$

Now we notice the following: if forms $\omega \in \Omega^\bullet(U)$ and $\tau \in \Omega^\bullet(V)$ come from a single global form on $U \cup V$, then they are killed by $\partial_V^* - \partial_U^*$. Hence we obtain a complex of (morphisms of cochain complexes):

$$0 \longrightarrow \Omega^\bullet(U \cup V) \longrightarrow \Omega^\bullet(U) \oplus \Omega^\bullet(V) \xrightarrow{\partial_V^* - \partial_U^*} \Omega^\bullet(U \cap V) \longrightarrow 0 \tag{90}$$

It is clear that this complex is exact at the first position, since a form must vanish if it vanishes on U and V. Similarly, if forms on U, V agree on $U \cap V$, they must glue to a form on $U \cup V$. Hence the complex is exact at the middle position. We now show that the complex is exact at the last position.

Theorem 6.21. *The above complex (of de Rham complexes) is exact. It may be called a "short exact sequence" of cochain complexes.*

Proof. Let $\alpha \in \Omega^q(U \cap V)$. We wish to write α as a difference $\tau - \omega$ with $\tau \in \Omega^q(U)$ and $\omega \in \Omega^q(V)$. Let (ρ_U, ρ_V) be a partition of unity subordinate to (U, V). Then we have $\alpha = \rho_U\alpha - (-\rho_V\alpha)$ in $U \cap V$. Now observe that $\rho_U\alpha$ may be extended by zero in V (call the result τ), while $-\rho_V\alpha$ may be extended by zero in U (call the result ω). Then we have $\alpha = (\partial_V^* - \partial_U^*)(\tau, \omega)$, as required. □

It is not surprising that given an exact sequence of morphisms of complexes

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0,$$

we obtain maps between the cohomology groups of the complexes

$$H^k(A^\bullet) \xrightarrow{f_*} H^k(B^\bullet) \xrightarrow{g_*} H^k(C^\bullet).$$

And it is not difficult to see that this sequence is exact at the middle term: Let $[\rho] \in H^k(B^\bullet)$, for $\rho \in B^k$ such that $d_B\rho = 0$. Suppose that $g(\rho) = 0$ in C^k, so that there exists $\tau \in A^k$ with $f(\tau) = \rho$. Then since f is a morphism of complexes, it follows that $f(d_A\tau) = d_B f(\tau) = d_B\rho = 0$. Since $f : A^{k+1} \longrightarrow B^{k+1}$ is injective, this implies that $d_A\tau = 0$, so we have $f_*[\tau] = [\rho]$, as required.

The interesting thing is that the maps g_* are not necessarily surjective, nor are f_* necessarily injective. In fact, there is a natural map $\delta : H^k(C^\bullet) \longrightarrow H^{k+1}(A^\bullet)$ (called the connecting homomorphism) which extends the 3-term sequence to a full complex involving all cohomology groups of arbitrary degree:

If $[\alpha] \in H^k(C^\bullet)$, where $d_C\alpha = 0$, then there must exist $\xi \in B^k$ with $g(\xi) = \alpha$, and $g(d_B\xi) = d_C(g(\xi)) = d_C\alpha = 0$, so that there must exist $\beta \in A^{k+1}$ with $f(\beta) = d_B\xi$, and $f(d_A\beta) = d_B(f(\beta) = 0$. Hence this determines a class $[\beta] \in H^{k+1}(A^\bullet)$, and one can check that this does not depend on the choices made. We then define $\delta([\alpha]) = [\beta]$.

Exercise: with this definition of δ, we obtain a "long exact sequence" of vector spaces as follows:

$$\begin{tikzcd}
H^\bullet(A) \arrow[rr, "f_*"] & & H^\bullet(B) \arrow[dl, "g_*"] \\
& H^\bullet(C) \arrow[ul, "\delta^{+1}"] &
\end{tikzcd}$$

In the diagram above, f_* and g_* preserve degree but δ has degree $+1$. Therefore, from the complex of complexes (90), we immediately obtain a long exact sequence of vector spaces, called the Mayer-Vietoris sequence:

$$\cdots \longrightarrow H^k(U \cup V) \longrightarrow H^k(U) \oplus H^k(V) \longrightarrow H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(U \cup V) \longrightarrow \cdots,$$

where the first map is simply a restriction map, the second map is the difference of the restrictions $\delta_V^* - \delta_U^*$, and the third map is the connecting homomorphism δ, which can be written explicitly as follows:

$$\delta[\alpha] = [\beta], \quad \beta = -d(\rho_V \alpha) = d(\rho_U \alpha).$$

(notice that β has support contained in $U \cap V$.)

6.6 Examples of cohomology computations

Example 6.22 (Circle)**.** Here we present another computation of $H^\bullet_{dR}(S^1)$, by the Mayer-Vietoris sequence. Express $S_1 = U_0 \cup U_1$ as before, with $U_i \cong \mathbb{R}$, so that $H^0(U_i) = \mathbb{R}$, $H^1_{dR}(U_i) = 0$ by the Poincaré lemma. Since $U_0 \cap U_1 \cong \mathbb{R} \sqcup \mathbb{R}$, we have $H^0(U_0 \cap U_1) = \mathbb{R} \oplus \mathbb{R}$ and $H^1(U_0 \cap U_1) = 0$. Since we know that $H^2_{dR}(S^1) = 0$, the Mayer-Vietoris sequence only has 4 a priori nonzero terms:

$$0 \longrightarrow H^0(S^1) \longrightarrow \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta_1^* - \delta_0^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta} H^1(S^1) \longrightarrow 0.$$

The middle map takes $(c_1, c_0) \mapsto c_1 - c_0$ and hence has 1-dimensional kernel. Hence $H^0(S^1) = \mathbb{R}$. Furthermore the kernel of δ must only be 1-dimensional, hence $H^1(S^1) = \mathbb{R}$ as well. Exercise: Using a partition of unity, determine an explicit representative for the class in $H^1_{dR}(S^1)$, starting with the function on $U_0 \cap U_1$ which takes values 0,1 on each respective connected component.

Example 6.23 (Spheres)**.** To determine the cohomology of S^2, decompose into the usual coordinate charts U_0, U_1, so that $U_i \cong \mathbb{R}^2$, while $U_0 \cap U_1 \sim S^1$. The first line of the Mayer-Vietoris sequence is

$$0 \longrightarrow H^0(S^2) \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R}.$$

The third map is nontrivial, since it is just the subtraction. Hence this first line must be exact, and $H^0(S^2) = \mathbb{R}$ (not surprising since S^2 is connected). The second line then reads (we can start it with zero since the first line was exact)

$$0 \longrightarrow H^1(S^2) \longrightarrow 0 \longrightarrow H^1(S^1) = \mathbb{R},$$

where the second zero comes from the fact that $H^1(\mathbb{R}^2) = 0$. This then shows us that $H^1(S^2) = 0$. The last term, together with the third line now give

$$0 \longrightarrow H^1(S^1) = \mathbb{R} \longrightarrow H^2(S^2) \longrightarrow 0,$$

showing that $H^2(S^2) = \mathbb{R}$.

Continuing this process, we obtain the de Rham cohomology of all spheres:

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R}, \text{ for } k = 0 \text{ or } n, \\ 0 \text{ otherwise.} \end{cases}$$