T-DUALITY AND GENERALIZED GEOMETRY WITH 3-FORM FLUX

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Abstract. We explain how T-duality, a relation discovered by physicists between circle bundles equipped with certain geometrical structures, can be understood as a Courant algebroid isomorphism between the spaces in question. This then allows us to transport generalized geometrical structures such as Dirac structures and generalized Riemannian metrics from one space to another of possibly different topology. In particular this includes the transport of twisted generalized complex and Kähler structures, and extends the usual *Buscher rules* well-known to physicists. We show how this applies to general affine torus bundles, give an interpretation of T-duality in terms of gerbes, and finally explain that T-duality between generalized complex manifolds may be viewed as a generalized complex submanifold (D-brane) of the product, in a way that establishes a direct analogy with the Fourier-Mukai transform in algebraic geometry.

Introduction

T-duality is an equivalence between quantum field theories with very different classical descriptions; for example type IIA and IIB string theory are T-dual when compactified on a circle. The precise relationship between T-dual Riemannian structures was first understood by Buscher in [7] and was developed further by Roček and Verlinde in [22]. It was realized that in order to phrase T-duality geometrically, one had to consider the interplay between the Neveu-Schwarz 3-form flux H, a closed 3-form with integral periods which entered the sigma model as the Wess-Zumino term, and the topology of the sigma model target. The precise relation between this 3-form flux and the topology of the T-dual spaces has recently been given a clear description by Bouwknegt, Evslin, and Mathai in [3] and it is their topological approach which we shall use as a basis to study the geometry of T-duality.

In this paper we explore and expand upon the realization in [14] that T-duality transformations can be understood in the framework of *generalized geometrical structures* introduced by Hitchin in [16]. In this formalism, one studies the geometry of the direct sum of the tangent and cotangent bundles of a manifold. This bundle is equipped with a natural orthogonal structure as well as the *Courant bracket*, an analog of the Lie bracket of vector fields, which depends upon the choice of a closed 3-form. In particular, an integrable orthogonal complex structure on this bundle, or generalized complex structure, is an object which encompasses complex and symplectic geometry as extremal special cases. As we shall see, T-duality can be viewed as an isomorphism between the underlying orthogonal and Courant structures of two possibly topologically distinct manifolds. It can therefore be used to transport a generalized complex structure

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from one manifold to the other, and in so doing, complex and symplectic structures on the two manifolds may be interchanged. This helps us to more fully understand the proposal of [23] that mirror symmetry between complex and symplectic structures on Calabi-Yau manifolds can be understood as an application of T-duality.

The action of T-duality on generalized complex structures was implicitly observed in [11], where both complex and symplectic structures in 6 dimensions were interpreted as spinors for $CL(6,6)$, a natural consideration from the point of view of supergravity. However, without the formalism of generalized complex structures, the intermediate geometrical structures were not recognized. Once the connection with generalized geometry was understood, several works appeared [13, 15, 17, 18, 20, 24, 25] which provide a physical motivation and justification for the use of generalized complex structures to understand mirror symmetry. From a mathematical point of view, Ben-Bassat [2] explored the action of T-duality on generalized complex structures on vector spaces and flat torus bundles, where one does not consider the 3-form flux H and therefore restricts the topological type of the bundles in question.

While we treat the most general case of T-duality of circle bundles with 3-form flux, it is important to clarify that for higher rank affine torus bundles, we only consider 3-forms H for which $i_X i_Y H = 0$ for X, Y tangent to the fibres. Mathai and Rosenberg [21] have shown that without this restriction, the T-dual manifold may be viewed as a noncommutative space. While this may also have an interesting interpretation in terms of generalized geometry, we do not explore it here.

In section 1, we review the definition of Bouwknegt *et al.* of T-duality as a relation between pairs (E, H), where E is a principal S^1-bundle over a fixed base B and $H \in H^3(E, \mathbb{Z})$. Choosing connections for the bundles and closed representatives for the cohomology classes, we restate one of their results as an isomorphism of H-twisted S^1-invariant de Rham complexes. We explain how this extends to affine torus bundles with the restriction on H stated above.

In section **??**, we show that if (E, H), $(\tilde{E}, \tilde{H})$ are T-dual, one can define an orthogonal isomorphism between the bundles $(TE \oplus T^*E)/S^1$ and $(T\tilde{E} \oplus T^*\tilde{E})/S^1$ which preserves the natural Courant bracket structure determined by the fluxes. This immediately allows the transport of any S^1-invariant generalized geometrical structures from E to $\tilde{E}$, and section 4 describes these transformation rules for generalized metrics, Dirac structures, and finally generalized complex and Kähler structures. In particular we describe how the type of a generalized complex structure changes under T-duality, and also how the Hodge diamond of a generalized Kähler manifold transforms under T-duality.

In section **??** we describe how, given a principal S^1-bundle equipped with a connection as well as a gerbe with connection on its total space (with curvature H), one canonically constructs a T-dual S^1-bundle as a moduli space, equipped with a connection and a gerbe with connection. It is in this sense that T-duality becomes a canonical construction, a fact tacitly assumed by physicists.

In section **??** we study the linear algebra of an orthogonal map between spaces of the form $V \oplus V^*$ in the context of the *orthogonal category*, an odd version of Weinstein's symplectic category. This allows us to view T-duality as a transform similar to a Fourier-Mukai transform. In the case of T-duality of generalized complex structures, we see that T-duality may be viewed as a generalized complex submanifold of the product. This may provide some insight into possible generalizations of the work of Donagi and Pantev [**?**] relating T-duality to Fourier-Mukai transforms for elliptic fibrations. In the final section we investigate several specific examples, including T-duality of generalized Kähler structures on Hopf surfaces and other Lie groups, on $\mathbb{C}P^1$, and on the Gibbons-Hawking hyperkähler manifold.

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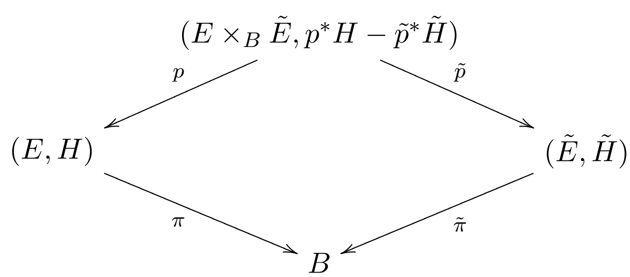
1. Topological T-duality

In this section we review the definition of topological T-duality as expressed by Bouwknegt, Evslin and Mathai [3]. They define T-duality as a relation between pairs (E, H) comprised of a principal circle bundle E and an integral cohomology class $H \in H^3(E, \mathbb{Z})$ on the total space of E. They show that bundles which are related by T-duality have isomorphic twisted cohomology and K-theory groups, up to a shift in degree.

Definition. Let $E \xrightarrow{\pi} B$ and $\tilde{E} \xrightarrow{\tilde{\pi}} B$ be two principal circle bundles equipped with 3-cohomology classes $H \in H^3(E, \mathbb{Z})$ and $\tilde{H} \in H^3(\tilde{E}, \mathbb{Z})$. The pairs (E, H) and $(\tilde{E}, \tilde{H})$ are defined to be *T-dual* if the following conditions hold:

i) $\pi_* \tilde{H} = c_1$ and $\pi_* H = \tilde{c_1}$, where $c_1 = c_1(E), \tilde{c_1} = c_1(\tilde{E})$ are the first Chern classes of the circle bundles;

ii) $p^* H - \tilde{p}^* \tilde{H} = 0$ in the cohomology of the fiber product $E \times_B \tilde{E}$, where $p : E \times_B \tilde{E} \to E$ and $\tilde{p} : E \times_B \tilde{E} \to \tilde{E}$ are the projections onto each circle bundle.



Although expressed as a relation, this definition in practice gives us a way to construct a T-dual for a given pair (E, H). Condition (i) implies that any T-dual to E must be a circle bundle $\tilde{E}$ with first Chern class $\tilde{c_1} = \pi_*(H) \in H^2(B,\mathbb{Z})$. To see that $\tilde{H}$ can be chosen to satisfy (ii) we use the Gysin sequence for E:

$$\cdots \longrightarrow H^1(B,\mathbb{Z}) \xrightarrow{c_1\cup} H^3(B,\mathbb{Z}) \xrightarrow{\pi^*} H^3(E,\mathbb{Z}) \xrightarrow{\pi_*} H^2(B,\mathbb{Z}) \xrightarrow{c_1\cup} H^4(B,Z) \longrightarrow \cdots$$

where c_1 is the first Chern class of the bundle.

Since $H \in H^3(E,\mathbb{Z})$, the above sequence gives that $\tilde{c_1} \cup c_1 = \pi_*(H) \cup c_1 = 0$. Hence, using the same sequence, but for $\tilde{E}$, we conclude that c_1 is in the image of $\tilde{\pi}_*$. Let $\tilde{H} \in H^3(\tilde{E},\mathbb{Z})$ be a cohomology class mapped to c_1 via $\tilde{\pi}$. Then $p_*(p^*H - \tilde{p}^*\tilde{H}) = -\pi^*c_1 = 0$ and similarly $\tilde{p}_*(p^*H - \tilde{p}^*\tilde{H}) = 0$. Hence the difference $p^*H - \tilde{p}^*\tilde{H}$ is actually the pull back of a 3-cohomology class from the base and changing $\tilde{H}$ by this pull back we can get $p^*H - \tilde{p}^*\tilde{H} = 0$, therefore obtaining one admissible $\tilde{H}$.

This approach also shows that $\tilde{H}$ is not unique, since above, when choosing $\tilde{H}$, we had the ambiguity coming from $c_1 \cup H^1(B)$ in the Gysin sequence.

Recall that the cohomology of the operator $d_A = d + A\cup$, for $A = H, \tilde{H}$ is the *A-twisted cohomology*, $H^\bullet_A$. The main theorem from [3] that concerns us relates the twisted cohomologies of E and $\tilde{E}$. If we choose representatives for the cohomology classes H and $\tilde{H}$ and let $F \in C^2(E \times_B \tilde{E})$ be a co-chain such that $dF = H - \tilde{H}$ then they establish:

Theorem 1.1. (Bouwknegt, Evslin and Mathai [3]): *The map* $\tau : H^\bullet_H(E,\mathbb{Z}) \to H^{\bullet+1}_{\tilde{H}}(\tilde{E},\mathbb{Z})$ *given by*

$$\tau(\rho) = \tilde{p}_* e^F p^* \rho, \tag{1.1}$$

is an isomorphism of twisted cohomologies.

Remark. Needless to say, τ does not preserve degrees. Nevertheless it is well behaved under the $\mathbb{Z}_2$-grading of cohomology as τ reverses the parity of its argument:

$$\tau(H^{ev/od}_H) \subset H^{od/ev}_{\tilde{H}}.$$

In this paper we are concerned with a more differential geometric version of the theorem above, so we remark that everything also holds rationally.

Definition. Using the notation above, (E,H) and $(\tilde{E},\tilde{H})$ are *rationally T-dual* to each other if conditions (i) and (ii) for T-duality hold modulo torsion.

In this paper we will only be concerned about rational T-duality, and will refer to it as just T-duality from now on.

Again this is a constructive definition. The bundle $\tilde{E}$ is determined by H and $\tilde{H}$ is well defined up to an element of $H^1(B,\mathbb{R}) \wedge c_1$. If we work with differential forms representing the cohomology class, then a 3-form $\tilde{H}$ will be defined up to an exact element.

With that ambiguity noticed, Bouwknegt *et al* present a standard construction of a T-dual. Given (E, H), H a closed 3-form representing an integral cohomology class, we choose a connection θ on E, so that $\theta(\partial/\partial\theta) = 1$, where $\partial/\partial\theta$ is the vector field generated by a fixed element in the Lie algebra of S^1 of period 1. A representative for the Chern class of this bundle is $d\theta = c_1$ and if we write $H = \tilde{c_1}\theta + h$, then $\pi_* H = \tilde{c_1}$. As H is integral, and $\int_{S^1}\theta = 1$, we get that $\tilde{c_1}$ is integral. Hence we can construct a circle bundle $\tilde{E}$ over B and choose a connection form $\tilde{\theta}$ such that $d\tilde{\theta} = \tilde{c_1}$. We associate the 3-form $\tilde{H} = c_1\tilde{\theta} + h$ to $\tilde{E}$ to find a T-dual pair to (E, H).

Observe that the ambiguity in the cohomology class $[\tilde{H}]$ can also be seen in this construction as the ambiguity in the choice of the connection $\tilde{\theta}$, which can be changed by a closed 1-form.

In this setting, the map τ from Theorem 1.1 can be expressed as a map between the complexes of invariant differential forms:

$$\tau : \Omega^\bullet_{S^1}(E) \to \Omega^\bullet_{S^1}(\tilde{E}) \qquad \tau(\rho) = \frac{1}{2\pi}\int_{S_1} e^{-\theta\wedge\tilde{\theta}}\rho, \tag{1.2}$$

where the S^1 where the integration takes place is the fiber of $E \times_M \tilde{E} \to \tilde{E}$, so the result is an invariant form in $\tilde{E}$. Any invariant form ρ in E can be written as $\rho = \theta\rho_1 + \rho_0$. In this case it is easy to check that

$$\tau(\theta\rho_1 + \rho_0) = \rho_1 - \tilde{\theta}\rho_0. \tag{1.3}$$

It is clear from (1.2) and that if we T-dualize twice and choose $\theta = \tilde{\tilde{\theta}}$ for the second T-duality, we get (E, H) back and $\tau^2 = -\mathrm{Id}$.

Now, $\Omega^\bullet_{S^1}(E)$ is naturally a $\mathbb{Z}_2$-graded differential complex — without the Leibniz rule — with differential $d_H = d + H$. Bouwknegt's main theorem [3] can be stated in the following way for forms:

Theorem 1.2. *The map $\tau : (\Omega^\bullet_{S^1}(E), d_H) \to (\Omega^\bullet_{S^1}(\tilde{E}), -d_{\tilde{H}})$ is an isomorphism of differential complexes.*

Proof. Given that τ has an inverse, obtained by T-dualizing again, we only have to check that τ preserves the differentials, i.e., $-d_{\tilde{H}} \circ \tau = \tau \circ d_H$. To obtain this relation we use equation (1.2):

$$\begin{aligned}
-d_{\tilde{H}}\tau(\rho) &= \frac{1}{2\pi}\int_{S^1} d_{\tilde{H}}(e^{-\theta\tilde{\theta}}\rho) \\
&= \frac{1}{2\pi}\int_{S^1} (H - \tilde{H})e^{-\theta\tilde{\theta}}\rho + e^{-\theta\tilde{\theta}}d\rho + \tilde{H}e^{\theta\tilde{\theta}}\rho \\
&= \frac{1}{2\pi}\int_{S^1} He^{\theta\tilde{\theta}}\rho + e^{\theta\tilde{\theta}}d\rho \\
&= \tau(d_H\rho)
\end{aligned}$$

□

Remark. If one considers τ as a map of the complexes of differential forms (no invariance required), it will not be invertible. Nonetheless, every d_H-cohomology class has an invariant representative, hence τ is a quasi-isomorphism.

Example 1.1. The Hopf fibration makes the 3-sphere, S^3, a principal S^1 bundle over S^2. The curvature of this bundle is a volume form of S^2, σ. So S^3 with zero twist is T-dual to $(S^2 \times S^1, \sigma \wedge \theta)$. On the other hand, still considering the Hopf fibration, the 3-sphere endowed with the 3-form $H = \theta \wedge \sigma$ is self T-dual.

Example 1.2. *(Lie groups)* Let (G, H) be a semi-simple Lie group with 3-form $H(X, Y, Z) = K([X, Y], Z)$, the Cartan form generating $H^3(G, \mathbb{Z})$, where K is the Killing form.

With a choice of an $S^1 < G$, we can think of G as a principal circle bundle. For $X = \partial/\partial\theta \in \mathfrak{g}$ tangent to S^1 and of length -1 according to the Killing form, a natural connection on G is given by $-K(X, \cdot)$. The curvature of this connection is given by

$$d(-K(X, \cdot))(Y, Z) = K(X, [Y, Z]) = H(X, Y, Z),$$

hence c_1 and $\tilde{c}_1$ are related by

$$c_1 = H(X, \cdot, \cdot) = X \lfloor H = \tilde{c}_1.$$

Which shows that semi-simple Lie groups with the Cartan 3-form are self T-dual. Of course, one can repeat this with any other circle making up the maximal torus.

1.1. **Principal Torus Bundles.** The construction of the T-dual described above can also be used to construct T-duals of principal torus bundles. What one has to do is just to split the torus into a product of circles and use the previous construction with a circle at a time (see [4]). However, this is only possible if

$$H(X, Y, \cdot) = 0 \qquad \text{if } X, Y \text{ are vertical.} \tag{1.4}$$

Mathai and Rosenberg studied the case when (1.4) fails in [21]. There they propose that the T-dual is a bundle of noncommutative tori.

Definition. Let (E, H) and $(\tilde{E}, \tilde{H})$ be a principal n-torus bundles over a base B. We say that E and $\tilde{E}$ are *T-dual* if there are bases for the torus $\{\partial_{\theta_i}\}$ and $\{\partial_{\tilde{\theta}_i}\}$, all of period 1, such that

- $\partial_{\theta_i} \cdot H$ is the Chern class of the S^1 bundle induced by $\partial_{\tilde{\theta}_i}$ and vice-versa;
- $H - \tilde{H}$ is exact in the correspondence space.

In this case, iterating Bouwknegt's theorem we get that the map

$$\tau(\rho) = \int_{T^k} e^{-(\theta_i)^t \cdot (\tilde{\theta}_i)} \rho$$

is an isomorphism of differenctial complexes, where $(\theta_i)^t$ denotes the line vector whose entries are the θ_i and $(\tilde{\theta}_i)$ a similar column vector.

A point to be clarified is that the final space T-dual to a principal torus bundle *is independent of the particular decomposition of the torus into circles.* This can be shown by a direct computation. Say $\{\partial/\partial\theta_i\}$ is a basis of the Lie algebra of the torus such that $\partial/\partial\theta_i$ integrates to a circle with period 1 and let $A \in SL_n(\mathbb{Z})$ be a matrix for change of basis: $\partial/\partial\overline{\theta_i} = A\partial/\partial\theta_i$. In this case one can check that $\overline{\theta_i} = A^{*-1}\theta_i$ furnish connections for the new basis and the T-dual connections are given by $\tilde{\overline{\theta_i}} = A\tilde{\theta}_i$. This shows that a change of basis by A causes a change of basis by A^{*-1} in the dual. If we denote by (v_i) the collumn vector whose components are v_i, then the new 3-form is given by

$$\tilde{\overline{H}} = (\overline{c_i})^t \cdot (\tilde{\overline{\theta_i}}) = (c_i)^t A^{-1} \cdot A(\tilde{\theta}_i) = (c_i)^t \cdot (\tilde{\theta}_i) = \tilde{H}.$$

Also, the map on forms, $\tilde{\tau}$, corresponding to the new basis is given by

$$\begin{aligned}\tilde{\tau}(\rho) &= \int exp(-(\overline{\theta_i})^t \cdot (\tilde{\overline{\theta_i}})) \wedge \rho = \int exp(-(\theta_i)^t A^{-1} \cdot A(\tilde{\theta}_i)) \wedge \rho \\ &= \int exp(-(\theta_i)^t \cdot (\tilde{\theta}_i)) \wedge \rho = \tau(\rho).\end{aligned}$$

Hence, the duals constructed from two different decompositions of the torus into circles are the same principal torus bundle, but with fibers decomposed as products of circles on two different ways. *The associated 3-form $\tilde{H}$ and the map τ do not depend on the particular decomposition of the torus.*

Example 1.3. *(Nilmanifolds)* Using the notation of [10], consider a 2-step nilmanifold E^{j+k} whose structure is given by $(0, \cdots, 0, c_1, \cdots, c_k)$, with $c_i \in \wedge^2\text{span}\{1, \cdots, j\}$. Take $H = 0$ to be the associated 3-form, so that (1.4) is trivially satisfied. From the structure constants, this nilmanifold is a principal k-torus bundle over a torus and there is a prefered way to decompose the fibers into circles so that Chern classes are the c_i. This choice of circle bundles gives us a way to T-dualize along the k circles making the torus bundle. After T-dualizing, we obtain a $k + j$-torus with 3-form $\sum c_i \wedge i$.

This shows that every 2-step nilmanifold with vanishing 3-form is T-dual to a torus with nonvanishing 3-form.

Example 1.4. *(Affine torus bundles)* Affine torus bundles can be described in the following way. Let E be a principal n-torus bundle over B, let G be a finite group of diffeomorphisms of B without fixed points and let $\alpha : G \hookrightarrow SL_n(\mathbb{Z})$ be a representation of G. Then G acts on E and the quotient is an *affine torus bundle.* Although there is no torus action on the fibers of $E/G \to B/G$, one can still define T^n*-invariant* forms on E/G as those which pull back to invariant form on E.

The action of G on the fibers of $E \to B$ gives rise to an action of G in the T-dual fibers by $\tilde{\alpha}(g) = \alpha(g)^{*-1}$, and hence G also acts on $\tilde{E}$. If E is endowed with a $G \times T^n$-invariant closed integral 3-form H, the T-dual will be endowed with a $G \times T^n$-invariant 3-form in which case we say that the quotients are T-dual to each other.

Observe that the computation we made for principal torus bundles shows that we have an isomorphism of the complexes of invariant forms $\Omega^\bullet_{T^n}(E/G) \cong \Omega^\bullet_{T^n}(\tilde{E}/G)$, as the map τ is invariant under the actions of G on E and $\tilde{E}$.

Although we will not delve into affine torus bundles, this example shows that all results we establish in the following sections also hold for affine torus bundles, and not just for principal circle bundles. This seems to be particularly relevant when one wants to study T-duality in the presence of singular fibers and when there is monodromy.

Remark. A word of warning. As shown by Bunke and Schick [6], differently of the case for principal circle bundles, the cohomology class of H does *not* determine the topology of the T-dual torus bundle. A simple example to illustrate this fact is given by a 2-torus bundle with nonvanishing Chern classes but with $[H] = 0$. Taking the 3-form $H = 0$ as a representative, a T-dual will be a flat torus bundle. Taking $H = d(\theta_1 \wedge \theta_2) = c_1\theta_2 - c_2\theta_1$ as a representative of the zero cohomology class, a T-dual will be the torus bundle with (nonzero) Chern classes $[c_1]$ and $[-c_2]$.

2. T-duality as a Map of Courant Algebroids

We have seen that T-duality comes with a map of differential algebras τ which is an isomorphism of the invariant differential exterior algebras. Now we introduce a map on invariant sections of generalized tangent spaces:

$$\varphi : T_{S^1}E \oplus T^*_{S^1}E \to T_{S^1}\tilde{E} \oplus T^*_{S^1}\tilde{E}.$$

Any invariant section of $TE \oplus TE^*$ can be writen as $X + f\partial/\partial\theta + \xi + g\theta$, where X is a horizontal vector and ξ is pull-back from the base. We define φ by:

$$\varphi(X + f\frac{\partial}{\partial\theta} + \xi + g\theta) = -X - g\frac{\partial}{\partial\tilde{\theta}} - \xi - f\tilde{\theta}. \tag{2.1}$$

The relevance of this map comes from the fact that there is a natural pairing on $TE \oplus T^*E$:

$$\langle X + \xi, Y + \eta\rangle = \frac{1}{2}(\xi(Y) + \eta(X))$$

and a bracket operation on $TE \oplus T^*E$, the Courant bracket, which, in the presence of a twisting 3-form, can be written as:

$$[X + \xi, Y + \eta]_H = [X, Y] + X\lfloor d\eta - Y\lfloor d\xi + \frac{1}{2}d(X\lfloor\eta - Y\lfloor\xi) - H(X, Y, \cdot). \tag{2.2}$$

Similarly to the Lie bracket, the Courant bracket can be defined by its action on forms:

$$2[v_1, v_2]_H \cdot \rho = v_1 \wedge v_2 \cdot d_H\varphi + d_H(v_1 \wedge v_2 \cdot \varphi)) + 2v_1 \cdot d_H(v_2 \cdot \varphi) - 2v_2 \cdot d_H(v_1 \cdot \varphi), \tag{2.3}$$

where $\cdot$ denotes the Clifford action of $v_i = X + \xi$ on φ:

$$(X + \xi) \cdot \varphi = X\lfloor\varphi + \xi \wedge \varphi.$$

Now we can state our main result.

Theorem 2.1. *The map* $\varphi : (T_{S^1}E \oplus T^*_{S^1}E, [\cdot,\cdot]_H) \to (T_{S^1}\tilde{E} \oplus T^*_{S^1}\tilde{E}, -[\cdot,\cdot]_{\tilde{H}})$ *is an orthogonal isomorphism of Courant algebroids and relates to* τ *acting on invariant forms via*

$$\tau(V \cdot \rho) = \varphi(V) \cdot \tau(\rho). \tag{2.4}$$

Proof. It is obvious from equation (2.1) that φ is orthogonal with respect to the natural pairing. To prove equation (2.4) we split an invariant form $\rho = \theta\rho_1 + \rho_0$ and $V = X + f\partial/\partial\theta + \xi + g\theta$. Then a direct computation using equation (1.3) gives:

$$\begin{aligned}\tau(V \cdot \rho) &= \tau(\theta(-X\lfloor\rho_1 - \xi\rho_1 + g\rho_0) + X\lfloor\rho_0 + f\rho_1 + \xi\rho_0) \\ &= -X\lfloor\rho_1 - \xi\rho_1 + g\rho_0 + \tilde{\theta}(-X\lfloor\rho_0 - f\rho_1 - \xi\rho_0).\end{aligned}$$

While

$$\begin{aligned}\varphi(V) \cdot \tau(\rho) &= (-X - g\partial/\partial\theta - \xi - f\theta)(\rho_1 - \tilde{\theta}\rho_0) \\ &= -X\lfloor\rho_1 - \xi\rho_1 + g\rho_0 + \tilde{\theta}(-X\lfloor\rho_0 - \xi\rho_0 - f\rho_1).\end{aligned}$$

Finally, we have established that under the isomorphisms φ of Clifford algebras and τ of Clifford modules, d_H corresponds to $-d_{\tilde{H}}$, hence the induced brackets (according to equation 2.3) are the same. □

Remark. As E is the total space of a circle bundle, its invariant tangent bundle sits in the Atiyah sequence:

$$0 \to 1 = T_1S^1 \to T_{S^1}E \to TB \to 0$$

or, taking duals,

$$0 \to T^*B \to T^*_{S^1}E \to T^*_1S^1 = 1^* \to 0.$$

The choice of a connection on E induces a splitting of the sequences above and an isomorphism

$$T_{S^1}E \oplus T^*_{S^1}E \cong TB \oplus T^*B \oplus 1 \oplus 1^*,$$

The argument also applies to $\tilde{E}$:

$$T_{S^1}\tilde{E} \oplus T^*_{S^1}\tilde{E} \cong TB \oplus T^*B \oplus 1 \oplus 1^*.$$

The map φ can be seen in this light as the permutation of the terms 1 and 1^*. This is Ben-Bassat's starting point for the study of mirror symmetry and generalized complex structures in [2].

Another piece of structure well behaved with respect to T-duality is the Mukai pairing. This is a pairing on spinors for $Spin(n,n)$ which is invariant under the action of $Spin(n,n)$. When one considers $Cl(TM \oplus T^*M)$, a natural choice of spinors is given by the exterior algebra $\wedge^\bullet T^*M$. The Mukai pairing on forms is given by

$$(\xi_1, \xi_2) = \sum_j (-1)^j (\xi_1^{2j} \wedge \xi_2^{n-2j} + \xi_1^{2j+1} \wedge \xi_2^{n-2j-1}), \tag{2.5}$$

where $\xi_i = \sum \xi_i^j$, with $\deg(\xi_i^j) = j$.

A map which will be important is $\psi : \wedge^n T^*_{S^1}E \to \wedge^n T^*_{S^1}\tilde{E}$, given by

$$\psi(\theta vol_B) = \tilde{\theta} vol_B. \tag{2.6}$$

We observe that this map does not depend on the particular choice of connection and relates to the Mukai pairing according to the following lemma.

Lemma 2.1. Let ξ_i, $i = 1, 2$ be two invariant forms on E of possibly mixed degree. Then

$$\psi(\xi_1, \xi_2) = -(\tau(\xi_1), \tau(\xi_2)).$$

Proof. From equation (1.3), a formal way to see the map τ is as Clifford action of $\partial/\partial\theta - \theta$ and then swap $\theta \mapsto \tilde{\theta}$. The result is obvious from this description and the fact that the Mukai pairing satisfies

$$(v \cdot \xi_1, v \cdot \xi_2) = \langle v, v \rangle (\xi_1, \xi_2).$$

$\square$

3. GENERALIZED STRUCTURES

In this section we introduce the structures we want to transport using T-duality. From Theorems 1.2 and 2.1, we get that any structure defined on E in terms of the natural pairing, Courant bracket and closed forms will correspond to one on $\tilde{E}$. The most immediate examples of such structures are Dirac structures and their complex counterpart, generalized complex structures. In what follows, H is a real closed 3-form.

Definition. An *H-twisted Dirac structure* on a manifold (M^n, H) is an n-dimensional distribution $L \leq TM \oplus T^*M$ which is closed under the H-twisted Courant bracket and isotropic with respect to the natural pairing.

Definition. An *H-twisted generalized complex structure* is an complex structure $\mathcal{J}$ of $TM \oplus T^*M$, i.e, $\mathcal{J}^2 = -\mathrm{Id}$, orthogonal with respect to the natural pairing, and for which the Nijenhuis operator vanishes:

$$[\mathcal{J}X, \mathcal{J}Y]_H - \mathcal{J}[\mathcal{J}X, Y]_H - \mathcal{J}[X, \mathcal{J}Y]_H - [X, Y]_H = 0, \qquad X, Y \in C^\infty(TM \oplus T^*M).$$

As with complex structures, a generalized complex structure can also be described in terms of its $+i$-eingespace $L < T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$.

Alternative definition. An *H-twisted generalized complex structure* on a manifold (M^{2n}, H) is a $4n$-dimensional distribution $L \leq T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ which is closed under the H-twisted Courant bracket, isotropic with respect to the natural pairing and satisfies $L \cap \overline{L} = \{0\}$.

According to [14], any twisted generalized complex structure can be described at a point as the Clifford annihilator of a line in $\wedge^\bullet T^*_{\mathbb{C}}M$. If ρ is a nonvanishing local section of this line bundle, each of the conditions imposed on the distribution L corresponds to one about ρ:

i) L is maximal if and only if $\rho = e^{B+i\omega} \wedge \Omega$, where Ω is a decomposable form and B and ω are 2-forms;

ii) $L \cap \overline{L} = \{0\}$ if and only if $(\rho, \overline{\rho}) \neq 0$, where $(\cdot, \cdot)$ is the Mukai pairing;

iii) L is closed under the twisted Courant bracket if and only if there is locally a section v of $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ such that $d_H\rho = v \cdot \rho$.

For a form $\rho = e^{B+i\omega} \wedge \Omega$, condition (*ii*) is equivalent to $\Omega \wedge \overline{\Omega} \wedge \omega^{n-k} \neq 0$, where k is the degree of Ω, also called the *type* of the generalized complex structure at that point.

Definition. The line subbundle of $\wedge^\bullet T\mathbb{C}^*M$ determining the generalized complex structure is the *canonical bundle*. A twisted generalized complex structure is a *twisted generalized Calabi–Yau* structure if the canonical bundle admits a nowhere vanishing d_H-closed section.

Example 3.1. Any complex structure J on M gives rise to a generalized complex structure $\mathcal{J}$. Using the natural decomposition of $TM \oplus T^*M$ we can express $\mathcal{J}$ in the matrix form as

$$\mathcal{J} = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}.$$

The $+i$-eigenspace for this structure is $L = T^{0,1}M \oplus T^{*1,0}M$ and the canonical bundle is $\wedge^{n,0}T^*M$. If H is a closed form of type $(2,1)+(1,2)$, then the structure is also an H-twisted generalized complex structure. This is a type n structure.

Example 3.2. Any symplectic structure ω on M also gives rise to a generalized complex structure:

$$\mathcal{J} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

The $+i$-eigenspace of this is

$$L = \{X - i\omega(X, \cdot) : X \in T_{\mathbb{C}}M\},$$

the canonical bundle is generated by $e^{i\omega}$, which is a nowhere vanishing closed section, hence this is a generalized Calabi–Yau structure. There is no nontrivial H for which this is also H-twisted, however, if $H = -db$, then $e^{b+i\omega}$ is an H-twisted generalized Calabi–Yau structure.

A twisted generalized complex structure $\mathcal{J}$ on M induces a splitting of the bundle $\wedge^\bullet T^*_{\mathbb{C}}M$ into subbundles U^k, similar to the (p,q)-decomposition induced by a complex structure. The bundle U^n is just the canonical bundle and we define

$$U^{n-k} = \wedge^k \overline{L} \cdot U^n.$$

In the case of a generalized complex structure induced by a complex structure these bundles are given by

$$U^k = \oplus_{p-q=k} \wedge^{p,q} T^*M,$$

while for generalized complex structure induced by a symplectic structure we have, from [9]:

$$U^k = e^{i\omega} e^{\frac{-\omega^{-1}}{2i}} \wedge^{n-k} T^*_{\mathbb{C}}(M).$$

In a Kähler manifold, we have the relation

$$e^{i\omega} e^{\frac{-\omega^{-1}}{2i}} \wedge^{p,q} T^*M = U_J^{p-q} \cap U_\omega^{n-p-q}. \tag{3.1}$$

For any twisted generalized complex structure, denoting by $\mathcal{U}^k$ the local sections of U^k, we have (see [14], Theorem 4.23):

$$d_H : \mathcal{U}^k \to \mathcal{U}^{k-1} + \mathcal{U}^{k+1}.$$

This gives a splitting of d_H into two operators:

$$\partial : \mathcal{U}^k \to \mathcal{U}^{k+1}, \qquad \overline{\partial} : \mathcal{U}^k \to \mathcal{U}^{k-1}.$$

In the complex case with no twist, these operators correspond to the complex operators with the same name.

Two final types of generalized complex structures that will interest us are the following.

Definition. An *H-twisted generalized Kähler structure* is a pair of twisted commuting generalized complex structures $\mathcal{J}_A$ and $\mathcal{J}_B$ such that $G = \mathcal{J}_A \mathcal{J}_B$ is a metric on $TM \oplus T^*M$:

$$\langle Gv, v \rangle > 0 \text{ if } v \neq 0.$$

An *H-twisted generalized Calabi-Yau metric* is a twisted generalized Kähler structure for which each of the generalized complex structures involved are generalized Calabi–Yau determined by d_H-closed forms ρ_A and ρ_B such that

$$(\rho_A, \overline{\rho_A}) = (\rho_B, \overline{\rho_B}).$$

The generalized Kähler condition can also be translated into conditions on the i-eigenspaces of $\mathcal{J}_A$ and $\mathcal{J}_B$. Namelly, a pair of subbundles determining generalized complex structures L_A ad L_B induce a generalized Kähler strucure if

i) $L_A \cap L_B$, $L_A \cap \overline{L_B}$ are $2n$-dimensional;
ii) The pairing

$$\langle \xi, \bar{\xi} \rangle$$

is positive definite in $L_A \cap \overline{L_B}$ and negative definite in $L_A \cap L_B$.

One last geometric structure that we will be able to transport via T-duality is the generalized metric.

Definition. A *generalized metric* on a vector space V is an orthogonal self adjoint map $G : V \oplus V^* \to V \oplus V^*$ for which $\langle Gv, v \rangle$ is positive definite.

If G is a generalized metric, then, being self adjoint and orthogonal, it must satisfy $G = G^* = G^{-1}$. Therefore $G^2 = 1$ and $V \oplus V^*$ splits as an orthogonal sum of ± 1-eigenspaces

$C_\pm < V \oplus V^*$. As G is positive definite, the natural pairing is $\pm$-definite in $C_\pm$ and the choice of a pair of such spaces clearly gives us a metric back:

$$\langle\langle V, W\rangle\rangle = \langle V_+, W_+\rangle - \langle V_-, W_-\rangle,$$

where $V_\pm$ and $W_\pm$ are the components of V and W in $C_\pm$. Therefore a metric is equivalent to a choice of orthogonal spaces $C_\pm$ where the natural pairing is definite.

Since V is maximal isotropic, any such C_+ can be written as the graph of an element in $\otimes^2 V^*$. More precisely, using the splitting $\otimes^2 V^* = Sym^2 V^* \oplus \wedge^2 V^*$ of a 2-tensor into its symmetric and skew-symmetric parts, we can write C_+ as the graph of $b + g$, where g is a symmetric 2-form and b is skew:

$$C_+ = \{X + b(X,\cdot) + g(X,\cdot) | X \in V\}.$$

The fact that the natural pairing is positive definite on C_+ places restrictions on g. Indeed,

$$g(X,X) = \langle X + b(X,\cdot) + g(X,\cdot), X + b(X,\cdot) + g(X,\cdot)\rangle > 0 \qquad \text{if } X \neq 0.$$

Hence g is a metric on V. Further, C_-, the orthogonal complement of C_+, is also a graph of $b_- + g_-$. But using orthogonality we can determine g_- and b_-:

$$\begin{aligned} 0 &= \langle X + b(X,\cdot) + g(X,\cdot), Y + b_-(Y,\cdot) + g_-(Y,\cdot)\rangle \\ &= b(X,Y) + b_-(Y,X) + g(X,Y) + g_-(Y,X), \end{aligned}$$

which holds for all $X, Y \in V$ if and only if $b_- = b$ and $g_- = -g$ and hence C_- is the graph of $b - g$.

This means that a metric on $V \oplus V^*$ compatible with the natural pairing is equivalent to a choice of metric g on V and 2-form b.

A particular example of generalized metric is given by the two generalized complex structures of generalized Kähler structure with $G = \mathcal{J}_A\mathcal{J}_B$ and $C_+ \otimes \mathbb{C} = L_A \cap \overline{L_B} \oplus \overline{L_A} \cap L_B$. One peculiarity of the generalized Kähler case is that $\mathcal{J}_A$ induces complex structures on both $C_\pm$ and, projecting to TM, we endow M with a bihermitian structure $J_\pm, g$. The data $(g, b, J_\pm)$ is actually enough to construct the generalized Kähler back:

Theorem 3.1. (Gualtieri [14], Theorem 6.37) *A bihermitian structure with 2-form $(g, b, J_\pm)$ on a manifold induces an H-twisted generalized Kähler as above if and only if*

$$d^c_+\omega_+ = -d^c_-\omega_- = H + db,$$

where $\omega_\pm = g(J_\pm\cdot,\cdot)$ and $d^c = iJ_\pm^{-1} dJ_\pm$.

We remark that a complex manifold with hermitian metric (M, J, g) is strong Kähler with torsion (SKT) if $dd^c\omega = 0$, but $d^c\omega \neq 0$, where ω is the Kähler 2-form associated with with the hermitian structure. By the above, any H-twisted generalized Kähler structure with nontrivial twist is a SKT structure.

4. T-duality and Generalized Structures

In this section we show that it is possible to transport all the structures introduced in the previous section using Theorems 1.2 an 2.1. We start with Dirac and generalized complex structures.

Theorem 4.1. *Any invariant twisted Dirac, generalized complex, generalized Calabi–Yau, generalized Kähler or generalized Calabi–Yau metric structure on E is transformed into a similar one via φ.*

Proof. If $L < TE \oplus T^*E$ ($L < T_{\mathbb{C}}E \oplus T^*_{\mathbb{C}}E$) is a twisted Dirac (generalized complex) structure on E, then, by Theorem 2.1 (3), $\varphi(L)$ is closed under the $\tilde{H}$-twisted Courant bracket. As φ is orthogonal, $\varphi(L)$ is still maximal isotropic, hence is a Dirac on $\tilde{E}$. For the generalized complex case, as φ is real we have $\varphi(L) \cap \overline{\varphi(L)} = \varphi(L) \cap \varphi(\overline{L}) = \varphi(L \cap \overline{L}) = \{0\}$.

If E has an H-twisted generalized Calabi–Yau structure defined by an invariant d_H-closed form ρ, with Clifford annihilator L, then the Clifford annihilator of $\tau(\rho)$ is $\varphi(L)$, showing that $\tau(\rho)$ is pure, i.e., its annihilator has maximal dimension. By Theorem 1.2, $\tau(\rho)$ is $d_{\tilde{H}}$-closed, hence it induces an $\tilde{H}$-twisted generalized Calabi–Yau structure on $\tilde{E}$.

If $\mathcal{J}_A$ and $\mathcal{J}_B$ are twisted structures furnishing a twisted generalized Kähler structure on E, then the T-dual generalized complex structures $\tilde{\mathcal{J}}_{A/B} = \varphi \mathcal{J}_{A/B} \varphi^{-1}$ will also commute. Since φ is orthogonal, $\tilde{G} = \varphi J_A J_B \varphi^{-1}$, is also a generalizaed metric, hence $\tilde{\mathcal{J}}_A, \tilde{\mathcal{J}}_B$ induce a generalized Kähler structure on the T-dual.

The claim about twisted generalized Calabi–Yau metric is a consequence of the generalized Kähler and generalized Calabi–Yau cases together with lemma 2.1. □

Corollary 1. No 2-step nilmanifold admits a left invariant generalized Kähler structure. In particular, no 6-nilmanifold admits such structure.

Proof. From Gualtieri's theorem (Theorem 3.1), any twisted generalized Kähler manifold admits an SKT structure. If a 2-step nilmanifold admits a generalized Kähler structure, according to Example 1.3, this nilmanifold can be T-dualized to a torus with nonzero 3-form, therefore furnishing the torus with an invariant SKT structure. But every invariant form in the torus is closed. In particular $d^c\omega = 0$ for the Kähler form induced by the metric and the complex structure, which can not happen in an SKT structure.

For the 6-dimensional case, we remark that Fino *et al* [12] have classified which 6-nilmanifolds admit invariant SKT structures (which would be the case for any admiting generalized Kähler structures) and those are all 2-step. □

Example 4.1. *(T-duality and the generalized Kähler structure of Lie groups I)* In his thesis, [14], Example 6.39, the second author shows that any compact semi-simple Lie group admits a twisted generalized Kähler structure, with twist given by the Cartan 3-form. These structures are obtained using the bihermitian point of view: any pair of left and right invariant complex structures on the Lie group J_l and J_r, orthogonal with respect to the Killing form satisfies the hypothesis of Theorem 3.1 with H the Cartan 3-form and $b = 0$. Any twisted generalized Kähler structure obtained this way will not be left nor right invariant since at any point it depends on J_l and J_r. However one can also show that J_l and J_r can be chosen to be biinvariant under the action of a maximal torus [9], and hence so will be the induced twisted generalized Kähler structure. In this case, according to Theorem 4.1 and Example 1.2, T-duality furnishes other twisted generalized Kähler structures on the Lie group.

The decomposition of $\wedge^\bullet T^*_{\mathbb{C}}M$ into subbundles U^k is also preserved from T-duality.

Corollary 2. If two generalized complex manifolds $(E, \mathcal{J}_1)$ and $(\tilde{E}, \mathcal{J}_2)$ correspond via T-duality, then $\tau(\mathcal{U}^k_E) = \mathcal{U}^k_{\tilde{E}}$ and also

$$\tau(\partial_E\psi) = -\partial_{\tilde{E}}\tau(\psi) \qquad \tau(\overline{\partial}_E\psi) = -\overline{\partial}_{\tilde{E}}\tau(\psi).$$

Proof. The T-dual generalized complex structure in $\tilde{E}$ is determined by $\tilde{L} = \varphi(L)$, where L is the $+i$-eigenspace of the generalized complex structure on E. Since φ is real, $\overline{\tilde{L}} = \varphi(\overline{L})$, and hence

$$\mathcal{U}^{n-k}_{\tilde{E}} = \Omega^k(\overline{\tilde{L}}) \cdot \tau(\rho) = \tau(\Omega^k(\overline{L}) \cdot \rho) = \tau(\mathcal{U}^k_E).$$

Finally, if $\alpha \in \mathcal{U}^k$, then

$$\partial_{\tilde{E}}\tau(\alpha) - \overline{\partial}_{\tilde{E}}\tau(\alpha) = d_{\tilde{H}}\tau(\alpha) = -\tau(d_H\alpha) = -\tau(\partial_E\alpha) + \tau(\overline{\partial}_E\alpha).$$

Since $\tau(\mathcal{U}^k) = \mathcal{U}^k_{\tilde{E}}$, we obtain the identities for the operators $\partial_{\tilde{E}}$ and $\overline{\partial}_{\tilde{E}}$. □

Example 4.2. *(Change of type of generalized complex structures)* As even and odd forms get swapped with T-duality along a circle, the type of a generalized complex structure is not preserved. However, it can only change, at a point, by ± 1. Indeed, if $\rho = e^{B+i\omega}\Omega$ is an invariant form determining a generalized complex structure there are two possibilities: If Ω is a pull back from the base, the type will increase by 1, otherwise will decrease by 1.

For a principal n-torus bundle, the rule is not so simple. If we let T^n be the fiber, $\rho = e^{B+i\omega}\Omega$ be a local trivialization of the canonical bundle and define

$$l = \max\{i : \wedge^i TT \cdot \Omega \neq 0\}$$

and

$$r = rank\omega|_V, \text{ where } V = \mathrm{Ann}(\Omega) \cap TT,$$

then the type, $\tilde{t}$ of the T-dual structure relates to the type, t, of the original structure by

$$\tilde{t} = t + n - 2l - r. \tag{4.1}$$

The following table sumarizes different ways the type changes for generalized complex structures in E^{2n} induced by complex and symplectic structures if the fibers are n-tori of some special types:

Struture on E	Fibers of E	l	r	Structure on $\tilde{E}$	Fibers of $\tilde{E}$
Complex	Complex	$n/2$	0	Complex	Complex
Complex	Real ($TT \cap J(TT) = \{0\}$)	n	0	Symplectic	Lagrangian
Symplectic	Symplectic	0	n	Symplectic	Symplectic
Symplectic	Lagrangian	0	0	Complex	Real

Table 1: Change of type of generalized complex structures under T-duality according to the type of fiber.

Example 4.3. *(Hopf surfaces)* Given two complex numbers a_1 and a_2, with $|a_1|, |a_2| > 1$, the quotient of $\mathbb{C}^2$ by the action $(z_1, z_2) \mapsto (a_1 z_1, a_2 z_2)$ is a primary Hopf surface (with the induced complex structure). Of all primary Hopf surfaces, these are the only ones admiting a T^2 action preserving the complex structure (see [1]). If $a_1 = a_2$, the orbits of the 2-torus action are elliptic surfaces and hence, according to Example 4.2, the T-dual will still be a complex manifold. If $a_1 \neq a_2$, then the orbits of the torus action are real except for the orbits passing through $(1, 0)$ and $(0, 1)$, which are elliptic. In this case, the T-dual will be generically symplectic except for the two special fibers corresponding to the elliptic curves, where there is type change. This example also shows that even if the initial structure on E has constant type, the same does not need to be true in the T-dual.

Example 4.4. *(Mirror symmetry of Betti numbers)* Consider the case of the mirror of a Calabi-Yau manifold along a special Lagrangian fibration. We have seen that the bundles $U^k_{\omega,J}$ induced by both the complex and symplectic structure are preserved by T-duality. Hence $U^{p,q} = U^p_\omega \cap U^q_J$ is also preserved, but, $U^{p,q}$ will be associated in the mirror to $U^p_{\tilde{J}} \cap U^q_{\tilde{\omega}}$, as complex and symplectic structure get swaped. Finally, as remarked previous section, equation (3.1), we have an isomorphism between $\Omega^{p,q}$ and $\mathcal{U}^{n-p-q,p-q}$. Making these identifications, we have

$$\Omega^{p,q}(E) \cong \mathcal{U}^{n-p-q,p-q}(E) \cong \tilde{\mathcal{U}}^{n-p-q,p-q}(\tilde{E}) \cong \Omega^{n-p,q}(\tilde{E}).$$

Which, in cohomology, gives the usual 'mirror symmetry' of the Hodge diamond.

4.1. **The Metric and the Buscher Rules.** Another geometric structure that can be transported via T-duality, in a less obvious way, is the generalized metric. Assume that a principal circle bundle E is endowed with an invariant generalized metric. The question we pose is what would be $\tilde{b}$ and $\tilde{g}$ in $\tilde{E}$ so that the map φ from (2.1) is an isometry?

Since φ is orthogonal with respect to the natural pairing, $\tilde{b}$ and $\tilde{g}$ will be the forms of which $\varphi(C_+)$ is the graph. Writing an invariant vector $V = X + a\partial/\partial\theta \in TE$ and g and b as

$$g = g_0 \theta \odot \theta + g_1 \odot \theta + g_2$$
$$b = b_1 \wedge \theta + b_2,$$

we have that the elements of C_+ are of the form:

$$X + a\frac{\partial}{\partial\theta} + (X\lfloor g_2 + ag_1 + X\lfloor b_2 - ab_1) + (g_1(X) + ag_0 + b_1(X))\theta.$$

Applying the map φ, we obtain the generic element of $\varphi(C_+) = \tilde{C_+}$. As this is a vector space, we can multiply the result by -1 to obtain that the generic element of $\tilde{C_+}$ is given by:

$$X + (g_1(X) + ag_0 + b_1(X))\frac{\partial}{\partial\tilde{\theta}} + (X\lfloor g_2 + ag_1 + X\lfloor b_2 - ab_1) + a\tilde{\theta}.$$

This is the graph of $\tilde{b} + \tilde{g}$:

$$\begin{aligned}\tilde{g} &= \frac{1}{g_0}\tilde{\theta}\odot\tilde{\theta} - \frac{b_1}{g_0}\odot\tilde{\theta} + g_2 + \frac{b_1\odot b_1 - g_1\odot g_1}{g_0}\\ \tilde{b} &= -\frac{g_1}{g_0}\wedge\tilde{\theta} + b_2 + \frac{g_1\wedge b_1}{g_0}\end{aligned} \tag{4.2}$$

Of course, in the generalized Kähler case, this is how the g and b induced by the structure transform. These equations, however, are not new. They had been encountered before by the physicists [7, 8], independently of generalized complex geometry and are called *Buscher rules!*

4.2. **The Bihermitian Structure.** The choice of a generalized metric (g, b) gives us two orthogonal spaces

$$C_\pm = \{X + b(X,\cdot) \pm g(X,\cdot) : X \in TM\},$$

and the projections $\pi_\pm : C_\pm \to TM$ are isomorphisms. Hence, any endomorphism $A \in End(TM)$ induces endomorphisms $A_\pm$ on $C_\pm$. Using the map φ we can transport this structure to a T-dual:

$$\begin{array}{ccc} A_+ \in End(C_+) & \xrightarrow{\quad\varphi\quad} & \tilde{A}_+ \in End(\tilde{C}_+) \\ \downarrow{\scriptstyle \pi_+} & & \downarrow{\scriptstyle \tilde{\pi}_+} \\ A \in End(TE) & & \tilde{A}_\pm \in End(T\tilde{E}) \\ \uparrow{\scriptstyle \pi_-} & & \uparrow{\scriptstyle \tilde{\pi}_-} \\ A_- \in End(C_-) & \xrightarrow{\quad\varphi\quad} & \tilde{A}_- \in End(\tilde{C}_-) \end{array}$$

As we are using the generalized metric to transport A and the maps $\pi_\pm$ and φ are orthogonal, the properties shared by A and $A_\pm$ will be metric related ones, e.g., self-adjointness, skew-adjointness and orthogonality. In the generalized Kähler case, it is clear that if we transport $J_\pm$ via $C_\pm$ we obtain the corresponding complex structures of the induced generalized Kähler structure in the dual:

$$\tilde{J}_\pm = \tilde{\pi}_\pm\varphi\pi_\pm^{-1}J_\pm(\tilde{\pi}_\pm\varphi\pi_\pm^{-1})^{-1}.$$

In the case of a metric connexion, $\theta = g(\frac{\partial}{\partial\theta},\cdot)/g(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta})$, we can give a very concrete description of $\tilde{J}_\pm$. We start describing the maps $\tilde{\pi}_\pm\varphi\pi_\pm^{-1}$. If V is orthogonal do $\partial/\partial\theta$, then

$g_1(V) = 0$ and

$$\tilde{\pi}_\pm\varphi\pi_\pm^{-1}(V) = \tilde{\pi}_\pm\varphi(V + b_1(V)\theta + b_2(V) \pm g_2(V,\cdot)) = \tilde{\pi}_\pm(V + b_1(V)\frac{\partial}{\partial\tilde{\theta}} + b_2(V) \pm g_2(V,\cdot))$$
$$= V + b_1(V)\frac{\partial}{\partial\tilde{\theta}}.$$

And for $\partial/\partial\theta$ we have

$$\tilde{\pi}_\pm\varphi\pi_\pm^{-1}(\partial/\partial\theta) = \tilde{\pi}_\pm\varphi(\partial/\partial\theta + b_1 \pm (\frac{1}{g_0}\theta + g_1)) = \tilde{\pi}_\pm(\frac{1}{g_0}\partial/\partial\tilde{\theta} + \tilde{\theta})) = \pm\frac{1}{g_0}\frac{\partial}{\partial\tilde{\theta}}.$$

Remark. The T-dual connection is *not* the metric connection for the T-dual metric. This is particularly clear in this case, as the vector $\tilde{\pi}_\pm\varphi\pi_\pm^{-1}(V) = V + b_1(V)\partial/\partial\tilde{\theta}$, although not horizontal for the T-dual connection, is perpendicular to $\partial/\partial\tilde{\theta}$ according to the dual metric. This means that if we use the metric connections of both sides, the map $\tilde{\pi}_\pm\varphi\pi_\pm^{-1}$ is the identity from the orthogonal complement of $\partial/\partial\theta$ to the orthogonal complement of $\partial/\partial\tilde{\theta}$.

Now, if we let $V_\pm$ be the orthogonal complement to $\mathrm{span}\{\partial/\partial\theta, J_\pm\partial/\partial\theta\}$ we can describe $\tilde{J}_\pm$ by

$$\tilde{J}_\pm w = \begin{cases} J_\pm w, & \text{if } w \in V_\pm \\ \pm\frac{1}{g_0}J_\pm\partial/\partial\theta & \text{if } w = \frac{\partial}{\partial\tilde{\theta}} \\ \mp g_0\frac{\partial}{\partial\tilde{\theta}} & \text{if } w = J_\pm\frac{\partial}{\partial\tilde{\theta}} \end{cases} \tag{4.3}$$

Therefore, if we identify $\partial/\partial\theta$ with $\partial/\partial\tilde{\theta}$ and their orthogonal complements with each other via TB, $\tilde{J}_+$ is essentially the same as J_+, but stretched in the directions of $\partial/\partial\theta$ and $J_+\partial/\partial\theta$ by g_0, while $\tilde{J}_-$ is J_- conjugated and stretched in those directions. In particular, J_+ and $\tilde{J}_+$ determine the same orientation while $\tilde{J}_-$ and J_- determine reverse orientations.

Example 4.5. *(T-duality and the generalized Kähler structure of Lie groups II)* As we mentioned in Example 4.1, the choice of a left and a right invariant complex structure J_l and J_r on a compact semi-simple Lie group furnishes a twisted generalized Kähler structure with twist given by the Cartan 3-form and $b = 0$. As the Lie group is self T-dual, if we chose $J_+ = J_r$ and $J_- = J_l$, the computation above shows that T-duality will furnish a new structure on the Lie group coming from J_r and $\tilde{J}_l$, where $\tilde{J}_l$ is still left invariant but induces the opposite orientation of J_l. Of course we can also swap the roles of $J_\pm$ to change the right invariant complex structure and keep the left invariant fixed.

5. FURTHER EXAMPLES

In this section we study some further instructive examples of T-duality.

Example 5.1. *(The symplectic 2-sphere)* Consider the standard circle action on the 2-sphere S^2 fixing north and south poles. If we remove the fixed points, we can see $S^2\backslash\{N,S\}$ as the trivial circle bundle over the interval $(-1,1)$. Adopting coordinate $(t,\theta) \in (-1,1)\times(0,2\pi)$, the round metric is given by

$$ds^2 = (1-t^2)d\theta^2 + \frac{1}{1-t^2}dt^2.$$

Using the Busher rules with $b=0$, the T-dual metric will be

$$ds^2 = \frac{1}{1-t^2}d\tilde{\theta}^2 + \frac{1}{1-t^2}dt^2.$$

Observe that the fixed points give rise to circles of infinite radius at a finite distance. This metric is not complete.

Any invariant symplectic structure on the sphere is given by $\omega = w(t)dt \wedge d\theta$ and we can still consider B-field transforms of that by any invariant 2-form $B = b(t)dt \wedge d\theta$: $\exp(b(t) + iw(t)dt \wedge d\theta)$. The dual structure is given by

$$\tau(\exp(b(t) + i\omega(t)dt \wedge d\theta)) = -d\theta - (b(t) + iw(t))dt.$$

If we let $z = \exp(\int_0^t \omega(t') + ib(t')dt' + i\theta)$, the complex structure on the T-dual is determined by $\frac{dz}{z}$. Therefore z is a holomorphic coordinate system in the T-dual which therefore is biholomorphic to an annulus with interior radius 1 and exterior radius $\int_{S^2}\omega$.

If we work with a pinched torus instead of the sphere, the coordinate z above gives a way to identify the inner circle with the outer circle: $z \mapsto \exp(\int_{S^2}\omega + iB)z$. Therefore the complex structure on the dual elliptic curve is determined by the cohomology classes of ω and B.

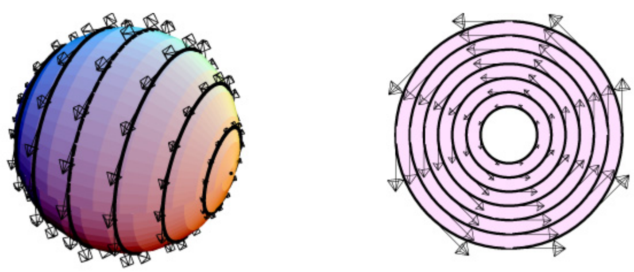


Figure 1. Symplectic sphere (S^2,ω) is T-dual to the complex annulus with radii 1 and $e^{\int_{S^2}\omega}$

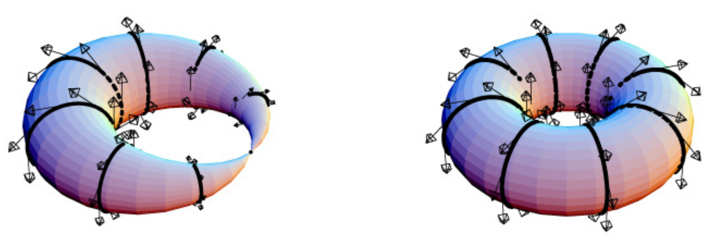


Figure 2. The symplectic pinched torus, i.e., the sphere (S^2,ω) with north and south pole indentified, is T-dual to the complex torus, i.e., the annulus with internal and external circles identified by $z \mapsto e^{\int_{S^2}\omega + iB}$.

Example 5.2. *(Odd 4-dimensional structures and the Gibbons–Hawking Ansatz)* The description of a generalized Calabi–Yau metric in 4 real dimensions can be divided in two cases, according to whether the induced complex structures $J_\pm$ determine the same orientation or not. If they determine different orientations, the differential forms defining the generalized Calabi–Yau structures are of odd degree (see [14], remark 6.14) and $J_\pm$ commute. The real distributions $S_\pm = \{v \in TM : J_+v = \pm J_-v\}$ are integrable, yielding a pair of transverse foliations for M. If we choose holomorphic coordinates (z_1, z_2) for J_+ respecting this decomposition, then $(z_1, \overline{z_2})$ furnish holomorphic coordinates for J_-.

As the metric g is of type (1,1) with respect to both $J_\pm$, it is of the form

$$g = g_{1\overline{1}}dz_1 d\overline{z_1} + g_{2\overline{2}}dz_2 d\overline{z_2}.$$

Recalling that graphs of the i-eigenspaces of $J_\pm$ via $b \pm g$ are the intersections $L_A \cap L_B$ and $L_A \cap \overline{L_B}$, we can recover L_A and L_B from $J_\pm, g$ and b. In this case, the differential forms annihilating $L_{A/B}$ are

$$\rho_A = e^{b+g_{2\overline{2}}dz_2\wedge d\overline{z_2}} \wedge f_A dz_1,$$
$$\rho_B = e^{b+g_{1\overline{1}}dz_1\wedge d\overline{z_1}} \wedge f_B dz_2.$$

The generalized Calabi–Yau condition $d\rho_{A/B} = 0$ implies that f_A is a holomorphic function on z_1 and f_B a holomorphic function on z_2, hence with a holomorphic change of coordinates, we have

$$\rho_A = e^{b+g_{2\overline{2}}dz_2\wedge d\overline{z_2}} \wedge dz_1,$$
$$\rho_B = e^{b+g_{1\overline{1}}dz_1\wedge d\overline{z_1}} \wedge dz_2.$$

After rescaling ρ_B, if necessary, the compatibility condition $(\rho_A, \overline{\rho_A}) = (\rho_B, \overline{\rho_B})$ becomes $g_{1\overline{1}} = g_{2\overline{2}}$ therefore showing that the metric is conformally flat. Call this conformal factor V. One can easily check that the other generalized Calabi–Yau conditions $(\rho_A, \rho_B) = (\rho_A, \overline{\rho_B}) = 0$ hold for these forms and hence give no further information.

Finally, the integrability conditions, $d\rho_{A/B} = 0$, give

$$db \wedge dz_i = dV \wedge *dz_i = (*dV) \wedge dz_i, \qquad i = 1, 2,$$

where $*$ is the Euclidean Hodge star. Therefore,

$$db = *dV, \tag{5.1}$$

showing that the conformal factor V is harmonic with respect to the flat metric.

Now, assume that the structure described above is realized in $S^1 \times \mathbb{R}^3$ in an invariant way, where some points of $\mathbb{R}^3$ may be removed so as to allow poles of V. The invariance of V implies it is a harmonic function on $\mathbb{R}^3$ and writing $b = b_1 \wedge \theta + b_2$, equation (5.1) implies that $db_1 = *_3 dV$ and $db_2 = 0$ (using the flat connection). According to the Busher rules, the T-dual metric will be given by

$$\tilde{g} = V(dx_1^2 + dx_2^2 + dx_3^2) + \frac{1}{V}(\tilde{\theta} - b_1)^2;$$
$$\tilde{b} = b_2,$$

with $db_1 = *_3 dV$ and b_2 closed, which is a B-field transform of the Hyperkähler metric given by the Gibbons–Hawking ansatz.

This example shows that T-duality can be used to produce interesting examples out of structures that at first glance may seem rather trivial.

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