Low-dimensional geometry—a variational approach

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Lecture 1: Open orbits and stable forms

Introduction

These lectures are about a variational approach to questions in geometry. In addition to providing a new viewpoint on known geometries, this approach motivates the definition of new kinds of geometrical structures. We are already familiar with the idea that certain geometries can be obtained from the variational principle; for example, the critical points of the Einstein-Hilbert functional are Einstein metrics.

We will focus mainly on 'low dimensions', which are, for us, dimensions 6,7 and 8 (unlike the topologists' 2, 3, and 4!) My personal starting point was a comment by Robert Bryant in his study of G_2.

Why G_2 is not exceptional

According to the classification of simple Lie groups, there are four infinite families and 5 exceptional groups:

$$A_k, \quad B_k, \quad C_k, \quad D_k,$$
$$F_4, \quad G_2, \quad E_6, \quad E_7, \quad E_8.$$

A standard description of G_2 is as follows: Consider the octonions $\mathbb{O}$, which is the largest normed division algebra over $\mathbb{R}$: it is 8-dimensional and non-associative. Then G_2 is the automorphism group of this algebra.

This algebraic description of G_2 gives the impression that it is indeed exceptional and depends on a highly specialized multiplication table for the octonions. However there is a more 'generic' way to obtain it, as we now describe. Since G_2 preserves the structure of $\mathbb{O}$, it fixes $V = Im\ \mathbb{O}$, which is the orthogonal complement to $1 \in \mathbb{O}$, and it preserves a cross product defined on this 7-dimensional space V by

$$x \times y = \tfrac{1}{2}[x, y].$$

Using the metric g on $\mathbb{O}$ (obtained from the norm), this cross product gives rise to a skew 3-form $\rho \in \wedge^3 V^*$ via

$$\rho(x, y, z) = g(x \times y, z). \tag{1}$$

Bryant's observation is that ρ lies in an *open* orbit of the action of $GL(7, \mathbb{R})$ on $\wedge^3 V^*$, and that the stabilizer of any 3-form in this orbit is exactly G_2. By a dimension count

$$\dim GL(7, \mathbb{R}) = 49 \qquad \dim \wedge^3 V^* = 35,$$

we see that $\dim Stab(\rho) = 14$, making the claim plausible. To see how G_2 can be obtained as the stabilizer of such a 3-form, we show how the cross product is recovered. View the 3-form as a map $\wedge^2 V \to V^*$. According to (1), the cross product is the composition of ρ with the inverse of the metric $g : V \to V^*$:

$$\wedge^2 V \xrightarrow{\ \rho\ } V^* \xrightarrow{\ g^{-1}\ } V, \qquad \wedge^2 V \xrightarrow{\ \times\ } V$$

However, the metric g must be constructed out of ρ itself. Let us see how this is done. For any $X, Y \in V$, define the symmetric bilinear map

$$(X, Y) \longmapsto i_X\rho \wedge i_Y\rho \wedge \rho \in \wedge^7 V^*.$$

This is not quite a metric, since it takes values in the determinant of V^*, rather than in $\mathbb{R}$. To correct this problem, view this bilinear map as defining a linear map

$$h : V \longrightarrow V^* \otimes \wedge^7 V^*,$$

and take its determinant

$$\wedge^7 h : \wedge^7 V \longrightarrow (\wedge^7 V^*)^8,$$

or in other words,

$$\wedge^7 h \in (\wedge^7 V^*)^9.$$

This has a unique 9^{th} root, which we denote by $(\wedge^7 h)^{1/9} \in \wedge^7 V^*$. Finally we define

$$g = \frac{h}{(\wedge^7 h)^{1/9}},$$

which for ρ in the open orbit, defines a positive-definite metric (depending non-linearly on ρ) on the 7-dimensional space V. Composing with ρ, we are able to reconstruct the full octonionic structure. In this way we see that G_2 may be thought of as the symmetry group of a 'generic' 3-form in 7 dimensions, and hence is not so exceptional after all.

More examples of open orbits

Let V be a real vector space of dimension n. There are very few examples of open orbits of $GL(V)$ acting on $\wedge^p V^*$, for obvious dimensional reasons. Indeed open orbits are only possible for low values of p. We provide the classification for $p > 1$ here:

- $(p = 2)$ If V is even-dimensional, there is an open orbit of non-degenerate 2-forms, each with stabilizer conjugate to $Sp(2m, \mathbb{R})$. If V is odd-dimensional there is the open orbit of maximal rank 2-forms.
- $(p = 3, n = 6)$ There is an open orbit of 3-forms in 6 dimensions with stabilizer $SL(3, \mathbb{C})$, acting in the usual representation on $\mathbb{C}^3 \cong \mathbb{R}^6$.
- $(p = 3, n = 7)$ This is the G_2 case studied above.
- $(p = 3, n = 8)$ There is an open orbit of 3-forms in 8 dimensions with stabilizer $PSU(3)$ acting in the adjoint representation.

Note that the symmetry $p \leftrightarrow n - p$ provides more open orbits, but it suffices to consider those mentioned above, and we shall in fact meet the complementary forms in a natural way. We must now investigate the geometrical consequences of this curious classification result.

Volume forms

An important observation is that each stabilizer group in the above classification preserves a volume form. The compact examples G_2 and $PSU(3)$ preserve a metric and hence a volume form, but $Sp(2n, \mathbb{R})$ preserves the Liouville volume $\omega^n/n!$, and $SL(3, \mathbb{C})$ preserves a complex 3-form Ω, and hence the real volume form $i\Omega \wedge \overline{\Omega}$. We will now investigate some elementary properties of this volume form.

Let $U \subset \wedge^p V^*$ be the open orbit of forms, and let

$$\phi : U \subset \wedge^p V^* \longrightarrow \wedge^n V^*$$

be the nonlinear map determining the invariant volume form. We easily determine the homogeneity of this map by applying scalar elements $\lambda \in GL(V)$: By invariance we know that $\phi(\lambda^p \rho) = \lambda^n \phi(\rho)$, and hence ϕ is homogeneous of degree $\frac{n}{p}$. The first derivative of ϕ at ρ is a linear map

$$D\phi : \wedge^p V^* \longrightarrow \wedge^n V^*,$$

and hence $D\phi(\dot\rho)$ must be given by wedge product of ρ by a form of complementary degree $n-p$:

$$D\phi(\dot\rho) = \hat\rho \wedge \dot\rho.$$

In this way we see that ρ determines not only a volume form $\phi(\rho)$ but also a form $\hat\rho$ of complementary degree, both in a nonlinear fashion. Let us work out two examples of this complementary form.

- In the symplectic case, $\phi(\omega) = \omega^m/m!$, and

$$D\phi(\dot\omega) = \frac{\omega^{m-1}\wedge\dot\omega}{(m-1)!},$$

  so that the complementary form $\hat\omega = \omega^{m-1}/(m-1)!$.

- In the G_2 case, the complementary form $\hat\rho$ must be a 4-form invariant under G_2. By the representation theory of G_2, it must be some multiple of $\star\rho$, where $\star$ is the Hodge star associated to the metric defined by ρ. After rescaling $\phi(\rho)$, we obtain the nonlinear relation

$$\hat\rho = \star_\rho \rho.$$

As a final comment on the volume form before moving on to geometry, we note that by the homogeneity of ϕ, if we apply the derivative of ϕ at ρ to the vector ρ, we obtain the relation

$$D\phi(\rho) = \hat\rho\wedge\rho = \tfrac{n}{p}\phi(\rho),$$

showing that the volume form can be recovered from the complementary form $\hat\rho$.

Stable forms on manifolds: the variational problem

Now suppose that M is a smooth n-dimensional manifold, and let $\rho \in \Omega^p(M)$ be, at each point $x \in M$, an element of the open orbit of $GL(n,\mathbb{R})$ acting on the tangent space T_xM.

$$\rho_x \in U_x \subset \wedge^p T_x^* M \quad \forall x \in M.$$

Such a differential form is called a *stable form.* The existence of such a stable form already places a global constraint on the manifold, i.e. that the structure group of the tangent bundle admits a topological reduction to the stabilizer group $Stab(\rho)$. For example, in order for an even-dimensional manifold to admit a nondegenerate 2-form, it must be almost complex, i.e. admit a topological reduction to $Sp(2n,\mathbb{R}) \simeq U(n)$.

Without any extra structure but the choice of this differential form, we may define a functional, which we call the *volume* associated to ρ:

$$V(\rho) = \int_M \phi(\rho).$$

In analogy with Hodge theory, we wish to find the critical points of this functional upon restriction to a de Rham cohomology class. In particular, we suppose that we can find *closed* stable forms ρ, and we attempt to determine critical points of V in the class $[\rho] \in H^p(M)$. Because the volume depends in a nonlinear way upon ρ, one might think of this as a kind of nonlinear version of Hodge theory.

A critical point occurs when the first variation of the volume functional vanishes, so let us calculate this first variation:

$$\delta V(\dot\rho) = \int_M D\phi(\dot\rho) = \int_M \hat\rho\wedge\dot\rho.$$

Since we are restricting the variations to a cohomology class, $\dot\rho$ is exact, i.e. $\dot\rho = d\sigma$. Hence we have

$$\delta V(\dot\rho) = \int_M \hat\rho \wedge d\sigma = \pm\int_M d\hat\rho\wedge\sigma.$$

This vanishes for all σ precisely when $d\hat\rho = 0$. That is, a closed stable form is a critical point of the volume functional if and only if *the complementary form $\hat\rho$ is closed.*

By considering such critical closed stable forms, we can recover several interesting geometries without any mention of the associated Riemannian invariants, such as holonomy:

- $Sp(2m, \mathbb{R})$ The condition $d\omega = 0$ on the stable form ω is precisely the definition of a symplectic structure. The complementary form $\hat{\omega}$ is automatically closed, and so the volume functional is simply a constant on each de Rham cohomology class.

- G_2 The conditions $d\rho = 0$ and $d \star \rho = 0$ are known, by a theorem of Fernandez and Gray, to be equivalent to the fact that the associated metric has holonomy G_2, in particular we obtain a Ricci-flat metric.

- $SL(3, \mathbb{C})$ The 3-form ρ together with its complement $\hat{\rho}$ determine the invariant complex 3-form $\Omega = \rho + i\hat{\rho}$ which determines an almost complex structure on the manifold. The condition $d\Omega = 0$ ensures that this almost complex structure is integrable. Then Ω is a nonvanishing holomorphic section of the canonical bundle, implying that M is a Calabi-Yau 3-fold (in terms of its complex geometry).

- $PSU(3)$ The nature of this geometry is a work in progress with Frederik Witt. The resulting metric is not Einstein but certain components of the Ricci tensor vanish. So far the group manifold $SU(3)$ is the only known compact example.

The variational approach gives information about moduli spaces, as we shall see, but no direct benefit to finding examples. A related problem however yields quite effectively to the variational approach.

The constrained variational problem: Weak holonomy G_2

In this section we show how to constrain the variational problem on the 4-forms of a 7-manifold in a simple way which gives rise to a geometry with weak holonomy G_2, as defined by Gray.

Given an exact p-form $d\beta \in \Omega^p_{ex}$ and a closed $(n-p)$-form $\gamma \in \Omega^{n-p}_{cl}$, their integral pairing vanishes, i.e.

$$\int_M d\beta \wedge \gamma = 0.$$

That is, we obtain a formal equivalence

$$(\Omega^p_{ex})^* = \Omega^{n-p}/\Omega^{n-p}_{cl},$$

and the exterior derivative maps the right hand side isomorphically to Ω^{n-p+1}_{ex}, i.e.

$$(\Omega^p_{ex})^* = \Omega^{n-p+1}_{ex}.$$

On a 7-dimensional manifold, we obtain in particular

$$(\Omega^4_{ex})^* = \Omega^4_{ex}.$$

In other words, the integral pairing defines a quadratic form

$$q(d\beta) = \int_M d\beta \wedge \beta$$

on the infinite-dimensional space of exact 4-forms of a 7-manifold.

Now that we have this quadratic form in hand, we might ask the question: *what are the critical points of the volume functional on exact 4-forms, given the constraint that $q(d\beta) = 1$?* Computing the variations at $\rho = d\beta$, we obtain

$$\delta V(\dot{\rho}) = \int_M \hat{\rho} \wedge d\dot{\beta}$$

$$\delta q(\dot{\rho}) = 2 \int_M \rho \wedge \dot{\beta},$$

And by introducing a Lagrange multiplier, we obtain the constrained variational equation:

$$d\hat{\rho} = \lambda \rho.$$

The complementary form $\hat{\rho} = \star\rho$ so

$$d \star \rho = \lambda\rho,$$

which is an equation defining a geometry called *weak holonomy* G_2; the metric is Einstein but with positive scalar curvature. It is a result of Bär that the metric cone on such a 7-manifold is an 8-manifold with holonomy $Spin(7)$. This is our first contact with $Spin(7)$, which does not fall directly within our classification: the 4-form which $Spin(7)$ preserves is not stable.

There do exist homogeneous examples of metrics with weak holonomy G_2, and it is possible to use the variational principle to obtain examples explicitly. This is done by searching for critical points among the invariant forms, a finite-dimensional problem.

For example, view S^7 as an $SU(2)$ bundle over S^4 acted on transitively by $Sp(2) \cdot Sp(1)$ with stabilizer $Sp(1) \cdot Sp(1)$. In this case, the space of invariant exact 4-forms form a 2-dimensional space. If $\alpha_1, \alpha_2, \alpha_3$ are the components of the connection form relative to the standard basis of $\mathfrak{su}(2)$ and $\omega_1 = d\alpha_1 + 2\alpha_2\alpha_3$ etc. are the components of the curvature, then the invariant exact 4-forms are spanned by

$$\begin{aligned} &d(\alpha_1\alpha_2\alpha_3), \\ &d(\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3). \end{aligned}$$

Finding the critical points of the volume functional, subject to the constraint $q(d\beta) = 1$, is an easy exercise with an interesting result: one obtains the *squashed* S^7, a well-known manifold with weak holonomy G_2.

Lecture 2: Geometrical structures on moduli spaces

Moduli space of critical points

In the previous lecture, we characterized geometries such as symplectic, G_2, and 3-dimensional Calabi-Yau geometry as critical points of the volume functional

$$V(\rho) = \int_M \phi(\rho),$$

restricted to a de Rham cohomology class, in analogy with Hodge theory. In Hodge theory, however, we know that there is a unique critical point (the harmonic representative) in each cohomology class. In our case, a critical point of the volume functional could never be unique, since $V(\rho)$ is diffeomorphism invariant, i.e. $V(f^*\rho) = V(\rho)$ for any diffeomorphism f. Nevertheless, we can hope that by quotienting by the action of $\mathrm{Diff}(M)$, we would obtain a unique critical class, and therefore a finite-dimensional *moduli space* parametrized by the cohomology group, formally

$$\mathcal{M} = \{\text{critical points of } V\} / \mathrm{Diff}(M).$$

Such a space is difficult to handle in practice due to the complicated nature of the action of $\mathrm{Diff}(M)$; a more tractable space would be a local moduli space; we produce such a space by showing that near a critical point there exists a smooth finite-dimensional slice of critical points in the closed p-forms, transverse to the action of $\mathrm{Diff}(M)$ and mapping via a local diffeomorphism to $H^p(M)$.

We will illustrate the procedure with a finite-dimensional version. Let F be a function, representing the functional V, of variables $x_1, \ldots, x_m$, representing the space of exact p-forms, and also of variables $t_1, \ldots, t_n$, representing coordinates for $H^p(M)$ (for example harmonic forms):

$$F(x_1, \ldots, x_m, t_1, \ldots, t_n).$$

Then its x-derivative is

$$DF_x : \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^m,$$

whose zeros are the critical points of F. If the second x-derivative, i.e. the Hessian, at a

$$D^2F_x(a) : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$

is an isomorphism, then the implicit function theorem says that there is a locally invertible smooth function $x(t)$ such that

$$DF_x(x(t),t) = 0.$$

In our infinite-dimensional case, we must of course utilize Sobolev Banach spaces of sufficiently high degree, but this procedure can in fact be implemented, as long as we can show that the Hessian D^2V is non-degenerate *transverse to the orbits of* $\mathrm{Diff}(M)$. This condition can be phrased in the following way:

Non-degeneracy condition If the Hessian at ρ, $D^2V(\rho)$, is degenerate in the $\dot\rho_1$ direction, i.e.

$$D^2V(\dot\rho_1,\dot\rho_2) = 0 \ \ \forall \dot\rho_2 \in \Omega^p_{cl},$$

then this implies that $\dot\rho_1$ must be in the infinitesimal orbit of $\mathrm{Diff}(M)$, i.e.

$$\dot\rho_1 = \mathcal{L}_X\rho \ \text{ for some vector field X.}$$

If this non-degeneracy condition holds, then we obtain a local moduli space $\mathcal{M}$ which is locally diffeomorphic to $H^p(M)$.

Example 1: Symplectic moduli space We have seen already that the volume functional is constant on the cohomology class of a symplectic structure, simply because

$$V(\omega) = \int_M \frac{\omega^m}{m!} = \frac{1}{m!}[\omega]^m([M]).$$

Calculating the Hessian at ω, we obtain

$$D^2V(\dot\omega_1,\dot\omega_2) = \text{const.}\int_M \omega^{m-2}\dot\omega_1\dot\omega_2,$$

which vanishes for all closed $\dot\omega_2$ only if $[\omega^{m-2}\dot\omega_1] = 0$, by Poincaré duality. If we now suppose that the strong Lefschetz property holds for ω at the level of $H^2(M)$, meaning that

$$H^2(M) \xrightarrow{[\omega]^{m-2}} H^{2m-2}(M)$$

is an isomorphism, then we can conclude that $[\dot\omega_1] = 0$, i.e. $\dot\omega_1 = d\theta$ for some 1-form θ. But then there must exist a vector field X with $\theta = i_X\omega$, and we conclude that

$$\dot\omega_1 = di_X\omega = \mathcal{L}_X\omega,$$

proving that the Hessian is non-degenerate transverse to the $\mathrm{Diff}(M)$ orbits. This gives a local moduli space as an open set in $H^2(M)$. Of course this is not too surprising since we have access to this information through Moser's theorem in symplectic geometry, but the example is illustrative nonetheless.

While we will not describe the other examples in detail, it is possible to show that the Hessian is non-degenerate in the G_2 case using harmonic theory, and in the Calabi-Yau case if the $\partial\bar\partial$-lemma holds.

Structure on the moduli space

Now that we have a concrete description of the local moduli space as an open subset $\mathcal{M} \subset H^p(M)$, it becomes clear that there are additional structures on this space.

- Since $\mathcal{M} \subset H^p(M)$, a vector space, there are flat coordinates on $\mathcal{M}$, which we denote by x_i.
- The critical value V of the volume functional determines a function on $\mathcal{M}$.
- The function V defines a pseudo-Riemannian metric on $\mathcal{M}$ of Hessian type:

$$g = \sum \frac{\partial^2 V}{\partial x_i \partial x_j} dx_i dx_j$$

There is an invariant way of describing such a metric, which we now explain. Suppose W is a vector space; then $W \times W^*$ is naturally a symplectic manifold using

$$\omega((w_1, \xi_1), (w_2, \xi_2)) = i_{w_1}\xi_2 - i_{w_2}\xi_1$$

and has a natural split-signature metric

$$g((w_1, \xi_1), (w_2, \xi_2)) = i_{w_1}\xi_2 + i_{w_2}\xi_1.$$

Now suppose that $L < W \times W^*$ is a Lagrangian submanifold transverse to W^*. Then the metric induced on L by g, and then projected to W, is a Hessian metric on W. For example, if L is the graph of df, for f a function on W, then we obtain the Hessian metric on W written above.

In our case, the vector space $W = H^p(M)$ has dual $W^* = H^{n-p}(M)$, and the subset

$$\{([\rho], [\hat{\rho}]) \in W \times W^*\}$$

is Lagrangian, generated by the volume function

$$V = \tfrac{p}{n} \int_M \rho \wedge \hat{\rho} = \tfrac{p}{n} [\rho] \cup [\hat{\rho}],$$

as we saw before. We will now investigate the properties of this metric induced on the moduli space, for our usual examples.

Metric on the symplectic moduli space

If our symplectic manifold is actually Kähler, then as a consequence of the Hodge-Riemann bilinear relations, the Hessian metric can be shown to be positive-definite on $\langle\omega\rangle \oplus H^{2,0} \oplus H^{0,2}$ and negative on the primitive $(1,1)$ forms $H_0^{1,1}$. Let us assume that $H^{2,0} = 0$. Then the Hessian has signature $(1, b_2 - 1)$, and by restricting our attention to forms such that

$$\int_M \omega^m = 1,$$

we obtain a Riemannian metric. This metric has been studied by P. Wilson [7] and his findings prompted him to conjecture the following:

Conjecture: For a Kähler manifold, the sectional curvatures of the Riemannian metric above are non-positive and bounded below by $-\frac{1}{2}m(m-1)$.

Now suppose that $H^{2,0} \neq 0$ and more particularly that M is actually a compact hyperkähler manifold of dimension $4k$. Then the signature of the Hessian metric is $(3, b_2 - 3)$, and the volume functional is actually the k^{th} power of the Beauville-Bogomolov form

$$\int_M \omega^{2k} = q(\omega)^k,$$

which is a topological invariant of a hyperkähler manifold. In this case the Hessian metric is a *homogeneous* pseudo-Riemannian metric of constant negative curvature.

Metric on the G_2 moduli space

In the G_2 case the 3-forms decompose as

$$\wedge^3 T^* = \mathbb{R}\rho \oplus i_X(\star\rho) \oplus \wedge^3_0,$$

where harmonic representatives in the second summand correspond to harmonic 1-forms, of which there are none if M is irreducible, due to the Ricci-flat condition. As a consequence, the Hessian metric has signature $(1, b_3 - 1)$. It would be possible to normalize as in the Kähler case and study the resulting Riemannian metric, but it has not yet been explored.

Special Kähler metrics on the Calabi-Yau moduli space

In this final example, we explore how the variational point of view provides a natural explanation for the Special Kähler metric on the moduli space of Calabi-Yau 3-folds. This moduli space is particularly important for the study of mirror symmetry, since it is conjectured to be isomorphic to the quantum cohomology of a different Calabi-Yau 3-fold, called the 'mirror'.

A special Kähler metric, as introduced by Freed, is defined as follows:

Definition: A special Kähler metric consists of

- A flat torsion-free symplectic connection (∇, ω)
- A complex structure J compatible with ω
- A locally defined vector field X such that $J = \nabla X \in C^\infty(T \otimes T^*)$.

The first component of the special Kähler structure is easily obtained: since $\mathcal{M} \subset H^3(M)$, we obtain immediately flat coordinates as well as a symplectic form given by the cup product:

$$\omega(a, b) = a \cup b \in H^6(M) = \mathbb{R},$$

Hence by taking $\nabla = D$, we obtain a flat symplectic connection.

The vector field X is obtained from the fact that there is a natural S^1 action on the moduli space, as follows: if $\rho + i\hat{\rho}$ is a holomorphic 3-form, so is $e^{i\theta}(\rho + i\hat{\rho})$. Therefore, there is an action on the open set of $H^3(M)$

$$[\rho] \longmapsto \cos\theta[\rho] + \sin\theta[\hat{\rho}],$$

and hence a vector field

$$\begin{aligned} X &= \tfrac{d}{d\theta}(\cos\theta[\rho] + \sin\theta[\hat{\rho}])|_{\theta=0} \\ &= [\hat{\rho}] \end{aligned}$$

In fact, since we have

$$(i_X\omega)([\dot{\rho}]) = [\hat{\rho} \wedge \dot{\rho}] = DV([\dot{\rho}]),$$

we see that V is a *moment map* for this Hamiltonian S^1-action on the moduli space.

To obtain the complex structure, we simply define $J = \nabla X$ and verify that it is an integrable almost complex structure compatible with ω. The fact that it is an almost complex structure follows from the fact that

$$\hat{\hat{\rho}} = -\rho.$$

Since X is Hamiltonian and ∇ preserves ω, it follows that J preserves ω; more explicitly, write

$$X = \sum \omega^{ij} \frac{\partial V}{\partial x_i} \frac{\partial}{\partial x_j}$$

and then

$$J = \sum \omega^{ij} \frac{\partial^2 V}{\partial x_i \partial x_j} dx_k \otimes \frac{\partial}{\partial x_j},$$

implying that

$$J_k^j = \sum \omega^{ij} g_{ik},$$

where g is the Hessian metric, as required.

To show that J is integrable, we explicitly compute some complex functions whose derivatives span $T^*_{1,0}$. In particular, define

$$z_j = x_j - i \sum \omega^{jk} \frac{\partial V}{\partial x_k},$$

so that

$$dz_j = dx_j - i \sum \omega^{jk} \frac{\partial^2 V}{\partial x_k \partial x_l} dx_l$$
$$= dx_j - i \sum J_j^l dx_l,$$

which are obviously forms of type $(1,0)$. The 1-forms $dz_1, \dots dz_{2n}$ span $E \subset T^* \otimes \mathbb{C}$ with $\dim E \leq \dim(T^*)^{1,0} = n$, but we see that $2dx_j = dz_j + d\bar{z}_j$, showing that $\dim(E + \bar{E}) = 2n$. Consequently $\dim E = n$, proving that J is integrable. Hence $\mathcal{M} \subset H^3(M)$ has a special Kähler metric.

Special Kähler structures appear on moduli spaces of superconformal field theories, and they can be used to create examples of (pseudo) hyperkähler manifolds. In particular, if we consider the cotangent bundle of $\mathcal{M}$, or more explicitly $\mathcal{M} \times \mathbb{R}^{2n}$, where we use y_k as cotangent coordinates, and if we define the following closed forms:

$$\omega_1 = \sum \frac{\partial^2 V}{\partial x_j \partial x_k} dx_j \wedge dy_k,$$

$$\omega_2 + i\omega_3 = -\tfrac{1}{2} \sum \omega_{jk} d(x_j + iy_j) \wedge d(x_k + iy_k),$$

then these satisfy the algebraic relations for a pseudo-hyperkähler structure (the associated metric g is not necessarily positive-definite). A natural question then arises as to whether this hyperkähler space parametrizes some extra structure on top of the Calabi-Yau space; the obvious choice are objects classified by a flat $H^3(M, \mathbb{R}/\mathbb{Z})$ bundle over the original moduli space $\mathcal{M}$.

Lecture 3: Generalized Calabi-Yau manifolds

So far we have seen that open orbits of the action of $GL(n, \mathbb{R})$ acting on the particular representation space $\wedge^p(\mathbb{R}^n)^*$ give rise to special geometries. Let us return to this idea in greater generality.

Let G be a Lie group with representation space V in which it has an open orbit: V is then called a *prehomogeneous vector space*. A classification of these structures was carried out by Kimura and Sato in 1977 [5]. Some of the groups they found are familiar to us from Berger's holonomy classification: the Lie groups which he identified as possible irreducible holonomy groups $G \subset SO(n)$ of a Riemannian (non-symmetric) manifold are as follows:

$$U(n), \quad SU(n), \quad Sp(n), \quad Sp(n) \cdot Sp(1), \quad G_2, \quad Spin(7), \quad Spin(9).$$

While $Spin(9)$ was later eliminated as a holonomy group, it remains true that these are the subgroups $G \subset SO(m)$ which act transitively on S^{m-1}. Because of this, the groups

$$\tilde{G} = \mathbb{R}^* \cdot G$$

have an open orbit in $\mathbb{R}^m$.

More examples of prehomogeneous vector spaces can be found in the work of Merkulov and Schwachhöfer [6], who extended Berger's work by classifying the possible holonomy groups of torsion-free affine connections in general; in particular they considered such connections which preserve a symplectic structure:

Symplectic holonomy group $G \subset Sp(V, \omega)$	V
$Sp(2m, \mathbb{R})$	$\mathbb{R}^{2m}$
$SL(2, \mathbb{R}) \cdot SO(n)$	$\mathbb{R}^2 \otimes \mathbb{R}^n$
$SL(2, \mathbb{R})$	$\mathrm{Sym}^3\mathbb{R}^2$
$SL(6, \mathbb{R})$	$\wedge^3(\mathbb{R}^6)^*$
$Sp(3, \mathbb{R})$	$\mathbb{R}^{14} \subset \wedge^3(\mathbb{R}^6)^*$
$Spin(6, 6)$	$\mathbb{R}^{32}$
E_7	$\mathbb{R}^{56}$

Representative list of holonomy groups for torsion-free symplectic connections.

These groups can also be characterized in terms of transitive actions on invariant hypersurfaces. Since they preserve a symplectic structure, there is an associated moment map

$$\mu : V \longrightarrow \mathfrak{g}^*,$$

and for semisimple groups the Killing inner product of μ with itself,

$$q = (\mu, \mu),$$

is an invariant quartic function on V. Except for $Sp(2m, \mathbb{R})$, where $q \equiv 0$, the symplectic holonomy groups act transitively on components of $q^{-1}(c)$, $c \neq 0$. For this reason we obtain, as before, the fact that the groups

$$\tilde{G} = \mathbb{R}^* \cdot G$$

have open orbits in V.

In the middle of this list is the case $SL(6, \mathbb{R})$ which led us to Calabi-Yau 3-folds. In the next section, we will focus on the group $Spin(6, 6)$, which acts in its 32-dimensional spin representation. The group $\mathbb{R}^* \cdot Spin(6, 6)$ has an open orbit with stabilizer $SU(3, 3)$, and we ask what geometrical structure underlies this group-theoretic fact.

$Spin(6, 6)$ acting on $T \oplus T^*$

Instead of viewing $Spin(6, 6)$ as the structure group of the tangent bundle of a pseudo-Riemannian 12-manifold, consider the sum $T \oplus T^*$ of the tangent and cotangent bundles of a six-dimensional manifold M. The bundle $T \oplus T^*$ has a natural indefinite metric of signature $(6, 6)$:

$$g(X + \xi, X + \xi) = -i_X \xi,$$

for which the tangent and cotangent bundles are maximally isotropic (null) sub-bundles. So, we think of $T \oplus T^*$ not as a $GL(6, \mathbb{R})$-bundle but as having structure group $SO(6, 6)$. This bundle always admits a spin structure, and since the metric has split signature, we can construct the spin representation as the exterior algebra on a maximal isotropic subspace. For example, choosing the maximal isotropic T^*,

$$S = \wedge^\bullet T^*$$

as the spinors for $T \oplus T^*$, where the Clifford action is given by

$$(X + \xi) \cdot \varphi = i_X \varphi + \xi \wedge \varphi.$$

We verify that

$$(X + \xi)^2 \cdot \varphi = -||X + \xi||^2 \varphi,$$

the defining relation of a Clifford module. The spin bundle decomposes into positive and negative spinors

$$S^+ = \wedge^{ev} T^*, \qquad S^- = \wedge^{od} T^*,$$

and in this way we obtain the 32-dimensional spaces containing the open orbits:

$$\boxed{\mathbb{R}^* \cdot Spin(6, 6) \text{ has an open orbit in } \wedge^{ev/od} T_x^*(M^6)}.$$

According to the symplectic holonomy classification, the spaces $\wedge^{ev/od} T^*$ are supposed to be endowed with $Spin(6, 6)$-invariant symplectic structures; these, givemn a choice of volume form, are nothing but the usual bilinear pairing of spinors, which can be defined as follows: let $\varphi_1, \varphi_2 \in \wedge^\bullet T^*$. Then

$$\langle \varphi_1, \varphi_2 \rangle = (\varphi_1 \wedge \sigma(\varphi_2))_{\text{top}} \in \wedge^{\text{top}} T^*,$$

where σ multiplies forms of degree p by $(-1)^{p(p-1)/2}$. In 6 dimensions, $\langle \ , \ \rangle$ is skew-symmetric and $Spin(6, 6)$-invariant on each of $S^\pm$.

Before we consider the geometry determined by a stable form of this type, it is worth explaining more concretely how $Spin(6,6)$ acts on differential forms. The Lie algebra $\mathfrak{so}(T \oplus T^*)$ consists of the skew-adjoint transformations

$$\mathfrak{so}(T \oplus T^*) = \left\{ \begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix} \; : \; A \in \mathrm{End}(T),\ \beta \in \wedge^2 T,\ B \in \wedge^2 T^* \right\}.$$

An important component of this Lie algebra for what follows is the space of 2-forms $B \in \wedge^2 T^*$, whose spinorial action on forms is simply wedge product $\rho \mapsto B \wedge \rho$. By exponentiation, therefore, we see that the group of 2-forms forms a subgroup $\Omega^2(M) \subset Spin(T \oplus T^*)$ and acts via the exponential map in the following way:

$$\rho \longmapsto e^B \rho = (1 + B + \tfrac{1}{2} B \wedge B + \cdots) \wedge \rho.$$

Stable forms of mixed degree and the volume functional

The underlying structure group $GL(n)$ embeds into $Spin(n,n)$ and as a $GL(n)$-bundle, the spin bundle should be written as

$$S = \wedge^\bullet T^* \otimes (\wedge^n T)^{1/2},$$

reflecting the way that $GL(n)$ acts through the spin representation. With this modification, we see that the quartic form $q = (\mu, \mu)$ on the positive spinors $S^+ = \wedge^{ev} T^* \otimes (\wedge^6 T)^{1/2}$ in 6 dimensions can be viewed as a $GL(6)$-equivariant map

$$\wedge^{ev} T^* \xrightarrow{\ q\ } (\wedge^6 T^*)^2 ,$$

so that we obtain a volume form

$$\phi(\rho) = \sqrt{|q(\rho)|} \in \wedge^6 T^*$$

for stable forms ρ, i.e. forms which lie pointwise in the open orbit of $\mathbb{R}^* \cdot Spin(6,6)$. We emphasize that unlike our previous situation, ρ is now a form of mixed degree and ϕ is invariant under the larger group of symmetries $Spin(T \oplus T^*)$.

As before, the first variation of the volume functional determines a complementary form, but in this case it is not the wedge product but the spinor pairing which must be used, i.e. there is a unique form $\hat\rho$ such that

$$D\phi(\dot\rho) = \langle \hat\rho, \dot\rho \rangle.$$

Following our earlier procedure, we restrict the functional to closed differential forms, obtaining the result that a critical point ρ is characterized by the condition that $\hat\rho$ is also closed, or in other words

$$d(\rho + i\hat\rho) = 0.$$

Geometrical interpretation of critical points: generalized Calabi-Yau structures

In this section we ask the question: *What sort of geometrical structure does a critical stable form for* $\mathbb{R}^* \cdot Spin(6,6)$ *represent?* The essential clue lies in the fact that while the real spinor ρ is generic, i.e. lies in an open orbit, the complex spinor $\varphi = \rho + i\hat\rho$ is not: it is a complex spinor of *pure type*, which means that the Clifford annihilator

$$E_\varphi = \{ X + \xi \in (T \oplus T^*) \otimes \mathbb{C} \; : (X + \xi) \cdot \varphi = 0 \}$$

is maximal isotropic in the natural indefinite metric. A few examples of pure spinors are:

- $1 \in \wedge^0 T^*$ is pure, since $E_1 = T \subset T \oplus T^*$, which is certainly maximal isotropic.
- Any decomposable form $\varphi = dx_1 \wedge \ldots \wedge dx_m$ is pure since $E_\varphi = \mathrm{Ker}\,\varphi \oplus \langle dx_1, \ldots, dx_m \rangle$, which is maximal isotropic.
- We can act by any 2-form B via the spin representation, obtaining the pure form $\varphi = e^B dx_1 \wedge \ldots \wedge dx_m$.

The correspondence between pure spinors and maximal isotropic subspaces is such that the pairing on spinors encodes the intersection theory of maximal isotropics; in particular,

$$\langle \varphi, \psi \rangle = 0 \Longleftrightarrow E_\varphi \cap E_\psi \neq \{0\}.$$

Now returning to our pure spinor $\varphi = \rho + i\hat{\rho}$, we observe that

$$\langle \rho + i\hat{\rho}, \rho - i\hat{\rho} \rangle = 2i\langle \hat{\rho}, \rho \rangle = 2i\lambda\phi(\rho),$$

for nonzero λ. Since on the open orbit $\phi(\rho) \neq 0$, this means that the maximal isotropic E_φ satisfies

$$E_\varphi \cap \overline{E_\varphi} = \{0\}.$$

This immediately implies that $(T \oplus T^*) \otimes \mathbb{C} = E_\varphi \oplus \overline{E_\varphi}$, implying that we have a complex structure on the bundle $T \oplus T^*$ with $\pm i$ eigenspaces $E_\varphi, \overline{E_\varphi}$, and which is compatible with the natural indefinite metric g on $T \oplus T^*$.

In this way, the stable form $\rho \in \wedge^{ev/od} T^*(M^6)$ gives rise to a complex structure $\mathcal{J}$ on $T \oplus T^*$ and hence a reduction of structure from $Spin(6,6)$ to $U(3,3)$; the fact that we have the form ρ instead of simply the line it generates reduces structure further to $SU(3,3)$.

This geometrical structure may now be generalized to higher dimensions:

Definition: A generalized Calabi-Yau manifold is a manifold M with a closed complex form φ such that φ is *pure* when considered as a spinor for $T \oplus T^*$ and $\langle \varphi, \overline{\varphi} \rangle$ is nowhere vanishing.

Examples:

- A Calabi-Yau manifold is generalized Calabi-Yau, with $\varphi = \Omega$, the holomorphic m-form. Note that Ω is pure since it is locally decomposable, and

$$\langle \varphi, \overline{\varphi} \rangle = \pm \Omega \wedge \overline{\Omega} \neq 0.$$

- A symplectic manifold is generalized Calabi-Yau, with $\varphi = e^{i\omega}$. This is clearly pure and satisfies

$$\langle e^{i\omega}, e^{-i\omega} \rangle = \langle e^{2i\omega}, 1 \rangle = c \cdot \omega^m \neq 0.$$

- We may transform any example by a real closed 2-form B via $\varphi \mapsto e^B \varphi$; this is called a *B-field* transformation.

This last example demonstrates that in addition to the diffeomorphism group, the additive group of real closed 2-forms acts as B-field transformations sending critical points to critical points. For this reason, it is clear that the moduli problem must be modified in some way, which we now address.

The moduli space of generalized Calabi-Yau structures in 6 dimensions

We wish to proceed exactly as before, by showing that the Hessian of the volume functional for even/odd stable forms on a 6-manifold is suitably nondegenerate, obtaining a finite-dimensional moduli space, and then investigate whether the map $\rho \mapsto \hat{\rho}$ defines a special Kähler structure on the moduli space. This would be straightforward if not for the fact that the volume function $\phi(\rho)$, in addition to being diffeomorphism invariant, is also invariant under the transformation

$$\rho \longmapsto e^B \rho,$$

for B a real closed 2-form. This means that we must quotient the locus of critical points by a larger symmetry group. But we mustn't quotient by the action of all such 2-forms, since the variational problem is occurring in a fixed cohomology class

$$[\rho] \in H^{ev/od}(M).$$

Hence we should consider the action of *exact* 2-forms, which would preserve the class, i.e.

$$e^{d\xi}\rho = \rho + d(\xi \wedge \rho + \tfrac{1}{2}\xi \wedge d\xi \wedge \rho + \cdots).$$

In conclusion, if one considers a local moduli space of nearby critical points modulo the action of the semidirect product of $\mathrm{Diff}_0(M)$ with the exact 2-forms Ω^2_{ex}, one obtains a finite-dimensional moduli space parametrized by an open set $\mathcal{M} \subset H^{ev/od}(M)$ in the de Rham cohomology. This moduli space inherits a special Kähler structure, as before.

Example: Symplectic structure. Let $\varphi = e^{i\omega}$ be the generalized Calabi-Yau structure associated to a 6-dimensional symplectic manifold, and suppose it satisfies strong Lefschetz so that the local moduli space is a well-defined open set in $H^{ev}(M)$. Then

$$Re[e^{i\omega}] = 1 - \tfrac{1}{2}[\omega]^2 \in H^0 \oplus H^4,$$

while by rescaling φ by a nonzero complex number

$$Re[(a+ib)e^{i\omega}] = a - b[\omega] - \tfrac{1}{2}a[\omega]^2 + \tfrac{1}{6}b[\omega]^3 \in H^{ev}.$$

Furthermore, applying a B-field transformation, we obtain a mapping

$$(\lambda, B + i\omega) \longmapsto \lambda Re[e^{B+i\omega}],$$

which defines a local coordinate chart

$$\mathbb{C} \times H^2(M, \mathbb{C}) \longrightarrow H^{ev}(M)$$

whose induced complex structure is actually that which appears in the special Kähler structure on the moduli space.

However, note that this 'exponential map' is not always surjective onto the moduli space. For example, let $M^6 = M_1^4 \times M_2^2$ be the product of a $K3$ surface with a symplectic surface, where we take the product symplectic structure $\omega_2 + \omega$ transformed by the B-field ω_1, where $\omega_1 + i\omega_2$ is a holomorphic 2-form on the K3 and ω is the symplectic structure on the surface. The spinor defining this product structure is

$$\varphi = e^{\omega_1}e^{i(\omega_2+\omega)},$$

and is clearly in the image of the holomorphic coordinate chart described above. However, introduce a parameter t via

$$\varphi_t = te^{(\omega_1+i\omega_2)/t}e^{i\omega}.$$

For $t \neq 0$, this is clearly in the image of the chart. However

$$\varphi_0 = (\omega_1 + i\omega_2)e^{i\omega},$$

the product of a complex with a symplectic structure, something which is definitely not in the image of the map.

In this lecture, we have seen how the concept of generalized Calabi-Yau manifold in arbitrary even dimension has arisen from interpreting geometrically the structure obtained as the critical point of a functional on stable forms in 6 dimensions. The starting point was a simple group-theoretic fact concerning open orbits, but following it up led to a different geometrical world. These lists contain the germs of a great deal of geometry.

Lecture 4: Generalized Riemannian structures

In the previous lecture, we investigated generalized Calabi-Yau geometry, which resulted from an open orbit of $\mathbb{R}^* \cdot Spin(6,6)$ in its 32-dimensional spin representation. Another open orbit from the list of prehomogeneous spaces is that of the group

$$\mathbb{R}^* \cdot Spin(7,7)$$

acting in its 64-dimensional spin representation with 28-dimensional stabilizer. To properly interpret the geometry which results from this fact, we will need a more complete picture of the geometry of $T \oplus T^*$.

The geometry of $T \oplus T^*$

Recall the basic scenario of generalized geometry: we replace the tangent bundle T by $T \oplus T^*$ as the basic object of consideration. This bundle is equipped with a natural inner product of signature (n, n), which we take to be

$$(X + \xi, X + \xi) = -i_X \xi,$$

and so it should be thought of as a bundle with natural structure group $SO(n, n)$. The lie algebra $\mathfrak{so}(n, n)$ consists of the skew adjoint transformations

$$\mathfrak{so}(n, n) = \wedge^2(T \oplus T^*) = \mathrm{End}(T) \oplus \wedge^2 T^* \oplus \wedge^2 T,$$

and so, in particular, 2-forms $B \in \wedge^2 T^*$ can be thought of as infinitesimal symmetries of the bundle $T \oplus T^*$.

As we have seen, spinors for the metric bundle $T \oplus T^*$ can be expressed as differential forms $S = \wedge^\bullet T^*$. This imposes variants of the usual tools. For example, we modify the usual Poincaré pairing of forms, in favour of the spinorial bilinear form

$$\langle \varphi, \psi \rangle = [\varphi \wedge \sigma(\psi)]_n.$$

and we use the Clifford action of $T \oplus T^*$ on forms

$$(X + \xi) \cdot \varphi = i_X \varphi + \xi \wedge \varphi.$$

instead of the interior product i_X. We have already seen this, in replacing the action of $\mathrm{Diff}(M)$ on closed forms, whose infinitesimal version is simply

$$\rho \longmapsto \mathcal{L}_X \rho = d i_X \rho,$$

by an action of the semidirect product of $\mathrm{Diff}(M)$ with $\Omega^2_{ex}(M)$, via

$$\rho \longmapsto \mathcal{L}_X \rho + d\xi \wedge \rho = d((X + \xi) \cdot \rho).$$

Our general philosophy is to express usual operations on forms in terms which are $Spin(n, n)$-invariant.

Another example of generalization from T to $T \oplus T^*$ is the definition of an analogue of the Lie bracket. We are familiar with the Cartan formula for the exterior derivative, which implies that

$$i_{[X,Y]}\alpha = d(i_X i_Y \alpha) + i_X d(i_Y \alpha) - i_Y d(i_X \alpha) + i_X i_Y d\alpha,$$

or equivalently,

$$i_{[X,Y]}\alpha = d(\tfrac{1}{2}[i_X, i_Y]\alpha) + i_X d(i_Y \alpha) - i_Y d(i_X \alpha) + \tfrac{1}{2}[i_X, i_Y] d\alpha.$$

Now let us replace our vector fields $X, Y \in T$ by sections $A = X + \xi, B = Y + \eta \in T \oplus T^*$, and replace all interior products by Clifford products, to define a bracket operation $[A, B]$ on sections of $T \oplus T^*$:

$$[A, B] \cdot \alpha = d(\tfrac{1}{2}(AB - BA) \cdot \alpha) + A \cdot d(B \cdot \alpha) - B \cdot d(A \cdot \alpha) + \tfrac{1}{2}(AB - BA) \cdot d\alpha.$$

This bracket is actually well-known in Poisson geometry as the *Courant bracket*, and can be written simply:

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \tfrac{1}{2} d(i_X \eta - i_Y \xi).$$

From our point of view, the bracket naturally arises from the interpretation of differential forms as spinors for $T \oplus T^*$.

Generalized Riemannian metrics

In this section we see how Riemannian geometry obtains an alternative interpretation in terms of the geometry of $T \oplus T^*$. A Riemmanian metric g determines an isomorphism

$$g : T \longrightarrow T^*$$

whose graph is a subbundle $V \subset T \oplus T^*$ which is positive definite in the natural indefinite metric. Therefore we obtain a decomposition

$$T \oplus T^* = V \oplus V^\perp,$$

where $V^\perp$ is the (negative definite) orthogonal complement to V. This reduces the structure group of $T \oplus T^*$ from $SO(n,n)$ to $SO(n) \times SO(n)$. The correspondence between Riemannian metrics and maximal positive-definite subspaces is not, however, 1-1 because of the fact that a positive-definite subspace may have a skew-symmetric component. A general maximal positive-definite subspace $V \subset T \oplus T^*$ is the graph of $g + B$, where g is a Riemannian metric and B is a 2-form. In short, a *generalized Riemannian metric* is a (non-closed) B-field transform of a usual Riemannian metric.

Equivalently, such a metric can be defined as a self-adjoint involution $R \in \mathrm{End}(T \oplus T^*)$, defined as $+1$ on V and -1 on $V^\perp$. One consequence of this point of view is that since $R \in O(n,n)$, we may lift it to $\tilde{R} \in Pin(n,n)$ after choosing an orientation, and this involutive operation on differential forms is a generalization of the Hodge star operator; for $B = 0$, $\tilde{R} = \sigma\star$, where $\star$ is the usual Hodge star.

Generalized G_2 structures

The stabilizer of a form in the open orbit of $\mathbb{R}^* \cdot Spin(7,7)$ is the compact group $G_2 \times G_2$, sitting inside $SO(7) \times SO(7)$. From the discussion in the previous section, we now know how this should be interpreted: we obtain a generalized metric $g + B$ with a reduction on both V and $V^\perp$ to G_2. Since $V, V^\perp$ are both *definite*, the natural projection to the null bundle T is an isomorphism, and so we obtain two almost-G_2 structures on the tangent bundle. As we know, $G_2 \subset Spin(7)$ is the stabilizer of a Riemannian spinor, so the data induced from the $G_2 \times G_2$ structure above consists of

- A Riemannian metric g,
- A 2-form B,
- Two unit Riemannian spinors ϕ^+, ϕ^-.

The precise relationship between this data and the stable form was determined by my student Frederik Witt:

Theorem [F. Witt] A stable form ρ in the open orbit of $\mathbb{R}^* \cdot Spin(7,7)$ which satisfies the variational equations

$$d\rho = 0 = d\hat{\rho}$$

is equivalent to the following data:

- A Riemannian metric g, 2-form B, and real function Φ,
- Unit spinors ϕ^+, ϕ^- which satisfy
$$\nabla^\pm \phi^\pm = 0,$$
where $\nabla^\pm$ are metric connections with skew torsion $\pm H (H = dB)$, and such that
- $(d\Phi \pm 2H) \cdot \phi^\pm = 0$.

The stable form can be expressed in terms of these data as follows:

$$\rho = e^\Phi e^B \phi^+ \otimes \phi^-.$$

It is comforting to note that the total number of degrees of freedom in the above data $(g, B, \Phi, \phi^+, \phi^-)$ are $(28, 21, 1, 7, 7)$ and therefore sum to 64, which is the dimension of the open orbit. Also, such a geometrical structure has actually arisen in the string theory literature, e.g. in the recent paper "Superstrings with Intrinsic Torsion" [3].

A search for examples of generalized G_2 which are not simply B-field transforms of usual G_2 structures is warranted, but unfortunately not in the compact case; the equations $(d\Phi + 2H) \cdot \phi^+ = 0$ and $\nabla^+ \phi^+ = 0$ imply that $\Delta(e^{-\Phi}) = e^{-\Phi}|H|^2$, which after integration over a compact manifold yields $H = 0$ and $\Phi =$ constant. In this case, both connections equal the Levi-Civita connection, $\phi^\pm$ are equal, and the resulting geometry is the B-field transform of a Riemannian metric with holonomy G_2.

Generalized Kähler structures

Another structure involving a generalized metric, studied by M. Gualtieri, is called a generalized Kähler metric. To define it, we first define a generalized complex structure:

Definition: A generalized complex structure is a complex structure J on $T \oplus T^*$ which preserves the natural inner product and whose $+i$-eigenbundle $E \subset (T \oplus T^*) \otimes \mathbb{C}$ is closed under the Courant bracket.

While generalized Calabi-Yau structures, defined earlier, are a special case of such a structure, it is clear that any complex manifold provides an example: if $I : T \to T$ is the complex structure tensor, then form

$$J = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}.$$

It is not difficult to verify that J is a generalized complex structure. Recall that any symplectic manifold is generalized Calabi-Yau; indeed forming

$$J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix},$$

we obtain the generalized complex structure associated with it.

From these two extreme examples we see that generalized complex geometry is a way of interpolating between complex and symplectic geometry. Recall that Kähler geometry occurs when we have a complex structure I and a compatible symplectic structure ω, in the sense that $-\omega I$ is a positive-definite Riemannian metric. By generalizing this situation we obtain the definition of a generalized Kähler structure.

Definition: A generalized Kähler structure on a manifold M consists of two generalized complex structures J_1, J_2 such that $J_1 J_2 = J_2 J_1$ and the inner product

$$(J_1 J_2(X + \xi), Y + \xi)$$

is positive-definite. In other words, the involution $R = J_1 J_2$ defines a generalized Riemannian metric on $T \oplus T^*$.

A generalized Kähler structure gives a reduction of structure for $T \oplus T^*$ to $U(n) \times U(n)$, and as in the $G_2 \times G_2$ case, this gives rise to two separate $U(n)$ structures on the tangent bundle. The precise conditions on these structures was determined by Gualtieri:

Theorem [M. Gualtieri [4]] A generalized Kähler structure is equivalent to:

- a Riemannian metric g and a 2-form B,
- metric connections ∇^+, ∇^- with torsion $\pm H$, $(H = dB)$,
- integrable complex structures I^+, I^-, both Hermitian with respect to g, and such that $\nabla^\pm I^\pm = 0$.

This kind of bi-Hermitian geometry first appeared in the physics literature in 1984, in the article [2]. In the following section we show how one can reduce this $U(n) \times U(n)$ structure further, to $SU(n) \times SU(n)$.

0.1 Generalized Calabi-Yau metrics

The Calabi-Yau condition on a Kähler manifold with holomorphically trivial canonical bundle is that the holomorphic m-form Ω and the Kähler form ω satisfy

$$\Omega \wedge \overline{\Omega} = c\omega^m,$$

where c is a constant. This equality of volume forms suggests a natural extension to the generalized framework:

Definition: A generalized Calabi-Yau metric consists of a pair of generalized Calabi-Yau structures defined by differential forms φ_1, φ_2 such that their associated generalized complex structures J_1, J_2 form a generalized Kähler structure and such that

$$\langle \varphi_1, \overline{\varphi_1} \rangle = \langle \varphi_2, \overline{\varphi_2} \rangle.$$

To gain some insight into the above constraint, we will investigate the 4-dimensional case in detail. A special feature of 4 dimensions is that our symmetry group $Spin(4,4)$ enjoys the triality isomorphism between its spin representations and its vector representation. A consequence of this fact is that the *pure* spinors are the same as the *null* spinors, i.e. φ is pure if and only if $\langle \varphi, \varphi \rangle = 0$. For even spinors $\alpha_0 + \alpha_2 + \alpha_4 \in \wedge^{ev} T^*$, this condition is that

$$2\alpha_0 \wedge \alpha_4 - \alpha_2 \wedge \alpha_2 = 0.$$

Suppose that $\varphi_1 = \rho_1 + i\rho_2$, $\varphi_2 = \rho_3 + i\rho_4$ define the generalized Calabi-Yau metric structure. Since these are pure, we deduce from $\langle \varphi_1, \varphi_1 \rangle = \langle \varphi_2, \varphi_2 \rangle = 0$ that

- $\langle \rho_1, \rho_1 \rangle = \langle \rho_2, \rho_2 \rangle$ and $\langle \rho_1, \rho_2 \rangle = 0$,
- $\langle \rho_3, \rho_3 \rangle = \langle \rho_4, \rho_4 \rangle$ and $\langle \rho_3, \rho_4 \rangle = 0$.

The fact that J_1, J_2 commute gives rise to the generalized Riemannian metric $R = J_1 J_2$, which determines a splitting $T \oplus T^* = V \oplus V^\perp$. The space $V \otimes \mathbb{C}$ then decomposes as

$$V \otimes \mathbb{C} = V_1 \oplus \overline{V_1},$$

the $\pm i$ eigenbundles of J_1. Correspondingly we have $V^\perp \otimes \mathbb{C} = V_1^\perp \oplus \overline{V_1^\perp}$. The Clifford annihilators E_1, E_2 of φ_1, φ_2 are simply the $+i$-eigenbundles of J_1, J_2, and so we have

$$E_1 = V_1 \oplus V_1^\perp,$$
$$E_2 = \overline{V_1} \oplus V_1^\perp.$$

Thus we obtain the conditions $\dim E_1 \cap E_2 = 2$, $\dim E_1 \cap \overline{E_2} = 2$, which yield $\langle \varphi_1, \varphi_2 \rangle = 0$ and $\langle \varphi_1, \overline{\varphi_2} \rangle = 0$, which happens if and only if

$$\langle \rho_1, \rho_3 \rangle = \langle \rho_2, \rho_3 \rangle = \langle \rho_1, \rho_4 \rangle = \langle \rho_2, \rho_4 \rangle = 0.$$

The above conditions, from parity considerations, are actually equivalent to the commuting of J_1, J_2. Finally, the Calabi-Yau metric condition, i.e. that $\langle \varphi_1, \overline{\varphi_1} \rangle = \langle \varphi_2, \overline{\varphi_2} \rangle$, implies that

$$\langle \rho_1, \rho_1 \rangle + \langle \rho_2, \rho_2 \rangle = \langle \rho_3, \rho_3 \rangle + \langle \rho_4, \rho_4 \rangle.$$

All the conditions obtained so far, which define a generalized Calabi-Yau metric in 4 dimensions, may be summarized in the following equations on four spinors $\rho_1, \rho_2, \rho_3, \rho_4$:

$$\langle \rho_i, \rho_j \rangle = \delta_{ij} \nu; \quad d\rho_i = 0,$$

where ν is a volume form on the 4-manifold. These are clearly expressed in $SO(4,4)$-invariant terms, but also note that phrased in this way, we encounter an $O(4)$ symmetry in the problem.

Let us solve these equations in the case where φ_i are *even* forms. Note that by the closure of φ_i, their zero degree components must be constants, and by a rotation in $O(4)$ we may set $\varphi_1^{(0)} = c$ for some real constant and $\varphi_2^{(0)} = \varphi_3^{(0)} = \varphi_4^{(0)} = 0$. Therefore

$$\begin{aligned}
\rho_1 &= c + \alpha_0 + \beta_0 \\
\rho_2 &= \alpha_1 + \beta_1 \\
\rho_3 &= \alpha_2 + \beta_2 \\
\rho_4 &= \alpha_3 + \beta_3.
\end{aligned}$$

The orthogonality of ρ_2, ρ_3, ρ_4 implies that

$$\alpha_1\alpha_2 = \alpha_2\alpha_3 = \alpha_3\alpha_1 = 0,$$

$$\alpha_2^2 = \alpha_3^2 = \alpha_1^2 = \nu.$$

These equations, together with $d\alpha_i = 0$, imply that $(\alpha_1, \alpha_2, \alpha_3) = (\omega_1, \omega_2, \omega_3)$ for a hyperkähler structure on M.

Now observe that $c \neq 0$ since if $c = 0$ we would have $\alpha_0 \wedge \omega_i = 0$, implying α_0^2 is a negative multiple of ν, a contradiction. Hence we may rescale ρ_1 so that $c = 1$, and letting $\alpha_0 = B$, we have

$$\rho_1 = 1 + B + \beta_0.$$

The condition $\langle\rho_1, \rho_2\rangle = 0$ implies that $\beta_1 = B\omega_1$, and the condition $\langle\rho_1, \rho_1\rangle = \langle\rho_2, \rho_2\rangle$ implies that $\beta_0 = (B^2 - \omega_1^2)/2$. Similar identities hold for ρ_3, ρ_4, and we obtain

$$\begin{aligned}
\rho_1 &= 1 + B + (B^2 - \nu)/2 = e^B(1 - \nu/2)\\
\rho_2 &= \omega_1 + B\omega_1 = e^B\omega_1\\
\rho_3 &= \omega_2 + B\omega_2 = e^B\omega_2\\
\rho_4 &= \omega_3 + B\omega_3 = e^B\omega_3.
\end{aligned}$$

Forming $\varphi_1 = \rho_1 + i\rho_2 = e^{B+i\omega_1}$ and $\varphi_2 = \rho_3 + i\rho_4 = e^B(\omega_2 + i\omega_3)$, we obtain the result: *an even generalized Calabi-Yau metric in 4 dimensions is nothing but the B-field transform of a hyperkähler metric.* We will investigate the odd case in the next lecture.

Lecture 5: Generalized *Spin*(7) geometry, and T-duality

We begin this lecture by addressing the question of how *Spin*(7) structures fit into the picture developed so far. Then we shall explain a procedure called T-duality for constructing examples of many of the special geometries described in these lectures.

Generalized *Spin*(7) structures

Like $SU(3)$, G_2 and $PSU(3)$, the group $Spin(7)$ viewed through its spin representation

$$Spin(7) \subset SO(8) \subset GL(8)$$

is the stabilizer of a differential form $\varphi \in \Omega^4(M)$. However, this 4-form does not lie in an open orbit; it satisfies certain pointwise algebraic conditions. Compensating for this algebraic complication is the relatively simple integrability condition: *a* $Spin(7)$ *4-form* φ *defines a metric of holonomy* $Spin(7)$ *if and only if* $d\varphi = 0$.

This situation can be generalized to a $Spin(7) \times Spin(7)$ structure on $T \oplus T^*$ of an 8-manifold M, in the following way. $Spin(7)$ is the stabilizer of a Riemannian spinor ϕ, and the tensor product

$$\phi \otimes \phi = 1 + \varphi + \nu \in \wedge^{ev}T^*,$$

where φ is the $Spin(7)$ 4-form and ν is the volume form of the associated metric. By looking at the $Spin(8,8)$ orbit of this differential form we obtain a class of even differential forms which reduce the structure of $T \oplus T^*$ to the group $Spin(7) \times Spin(7)$. One may then say that such a differential form defines a *generalized* $Spin(7)$ structure when it is closed. The question of what this integrability condition implies for the two induced $Spin(7)$ structures was worked out by F. Witt:

Theorem [F. Witt] A differential form ρ in the above-mentioned orbit which satisfies $d\rho = 0$ is equivalent to the following data:

- A Riemannian metric g and 2-form B, and real function Φ,
- Unit spinors ϕ^+, ϕ^- satisfying $\nabla^\pm \phi^\pm = 0$, where $\nabla^\pm$ are metric connections with torsion $\pm H$ ($H = dB$), and such that
- $(d\Phi \pm H) \cdot \phi^\pm = 0$.

Just as in the generalized G_2 case, we recover a geometry which appears in string theory [3] and which has no exotic compact examples.

Evolution equation for (generalized) $Spin(7)$ structures

We will now see how it is possible to use the variational approach we developed earlier to produce metrics with holonomy $Spin(7)$. The starting point is the observation that any hypersurface in a $Spin(7)$ 8-manifold acquires a reduction to G_2, since $G_2 \subset Spin(7)$ is the stabilizer of a vector (in the spin representation), which in this case is any normal vector field. Therefore, we will begin with a G_2 metric on a 7-manifold M^7 and attempt to extend it to a $Spin(7)$ structure on $M^7 \times \mathbb{R}$. Let t be the coordinate on $\mathbb{R}$ so that $\partial/\partial t$ is the vector stabilized by $G_2 \subset Spin(7)$. Then the 4-form

$$\varphi = \star\rho + dt \wedge \rho,$$

where ρ is an almost G_2 3-form on M^7 and $\star$ is the Hodge star on M^7, has the right algebraic type to define a reduction to $Spin(7)$ on $M^7 \times \mathbb{R}$. The integrability condition on ρ, namely $d\rho = d \star \rho = 0$, is equivalent to the closure of φ so the product is $Spin(7)$.

Now suppose that ρ depends on time. Then the closure of φ imposes conditions on how ρ must evolve in time, which we now derive:

$$0 = d\varphi = dt \wedge \frac{\partial(\star\rho)}{\partial t} + d(\star\rho) - dt \wedge d\rho.$$

So, putting $\sigma = \star\rho \in \Omega^4(M^7)$, we have the following evolution equations on a stable 4-form on a 7-dimensional manifold:

$$\boxed{d\sigma = 0; \quad \frac{\partial\sigma}{\partial t} = d \star \sigma}$$

We now show how the evolution of this closed 4-form may be viewed as a *gradient flow*. Recall that there is a quadratic form

$$q(d\beta) = \int_M d\beta \wedge \beta$$

on the space of exact 4-forms $\Omega^4_{ex}(M)$ in 7 dimensions. Define the affine space

$$\mathcal{A}_a = \left\{\sigma \in a \in H^4(M)\right\},$$

whose group of translations is Ω^4_{ex}. Hence the tangent space

$$T\mathcal{A}_a = \mathcal{A}_a \times \Omega^4_{ex},$$

and so, formally, q defines an indefinite metric g on the infinite-dimensional affine space A_a. Using this metric and the volume functional V, we can define *gradient flow* for stable forms $\sigma \in \mathcal{A}_a$.

Proposition: The gradient flow of $V(\sigma)$ for stable $\sigma \in \mathcal{A}_a$ defines a $Spin(7)$ metric on $M \times \mathbb{R}$.

Proof. The gradient flow equation is given by

$$g(\frac{\partial\sigma}{\partial t}, d\beta) = DV(d\beta) = \int_M \star\sigma \wedge d\beta.$$

By Stokes' formula and the definition of q, we obtain

$$\int_M \frac{\partial \sigma}{\partial t} \wedge \beta = \int_M d(\star\sigma) \wedge \beta.$$

Hence we see that the gradient flow on closed stable 4-forms σ is precisely the equation

$$\frac{\partial \sigma}{\partial t} = d \star \sigma,$$

as required. □

This evolution method is particularly useful in the cohomogeneity one case, when it can be used to derive ODEs whose solution yields metrics with holonomy $Spin(7)$. The procedure is as follows:

- Begin by writing down a basis for the invariant forms in a cohomology class; for example on $M = S^7$, as we saw before, the (necessarily exact) invariant 4-forms form a 2-dimensional space spanned by

$$d(\alpha_1 \wedge \alpha_2 \wedge \alpha_3), \quad d(\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3).$$

- One must then determine the volume as a nonlinear function on this vector space. This can often be simplified by using a normal form, identifying a change of basis to map to the normal form, and taking its determinant.
- Finally, determine the indefinite metric defined by q and write down the gradient flow equation, an ODE on a finite dimensional space.

In principle, the same procedure holds for evolving a generalized G_2 structure into a generalized $Spin(7)$ structure, since the variational formalism is similar.

T-duality

Another approach to producing examples of generalized geometrical structures is one which uses symmetry essentially and has its roots in the physics of string theory. A good reference for the subject of T-duality is the paper [1], and its application to the field of generalized geometry was studied by G. Cavalcanti and M. Gualtieri.

A basic feature of T-duality is that it deals with geometries which are "twisted by a closed 3-form H", which we now explain. Recall that the space of 2-forms acts by isometries on $T \oplus T^*$ and via exponentiation on differential forms via the spin representation. Conjugating the exterior derivative by this transformation, we obtain

$$e^{-B} d e^{B} = d + H, \quad H = dB,$$

known as the twisted de Rham differential. In general, H need not be exact; $d + H$ is a differential for any closed 3-form H. By replacing d by $d+H$, we may consider H-twisted versions of all the geometries we have considered in these lectures, by performing the variational procedure on twisted cohomology classes in

$$H_H^{ev/od}(M).$$

Locally, we are simply conjugating by e^B for some 2-form.

T-duality can be performed if one has the following data:

- $p : P \longrightarrow M$ a principal S^1-bundle,
- a connection $\theta \in \Omega^1(P)$ with curvature $d\theta = F$,
- an S^1-invariant closed 3-form $H \in \Omega^3(P)$ such that $[H] \in H^3(P, 2\pi\mathbb{Z})$.

Once this data is in place, then let X be the vector field generating the S^1 action on P. Then

$$i_X H = p^* F^T$$

for some closed 2-form F^T such that $[F^T] \in H^2(M, 2\pi\mathbb{Z})$. We may therefore interpret this 2-form as the curvature of a connection θ^T on a second principal bundle $p^T : P^T \longrightarrow M$, which we say is "T-dual" to P. Any S^1-invariant differential form ρ on P may be decomposed as

$$\rho = \rho_0 + \theta\rho_1,$$

where ρ_0, ρ_1 are pulled back from the base. In particular, we can write

$$H = H_0 + \theta F^T.$$

We may then equip P^T with an invariant 3-form as well, by defining

$$H^T = H_0 + \theta^T F.$$

Note that $dH^T = dH_0 + F^T F = dH = 0$.

The idea of T-duality is that we may transform S^1-invariant geometrical structures from P to P^T and back. The important observation along these lines is the following:

T-duality transform: The invariant from $\rho_0 + \theta\rho_1$ is d_H-closed on P if and only if the invariant form $\rho_1 - \theta^T \rho_0$ is d_{H^T}-closed on P^T.

Note that the parity of the form has been switched under this transformation. In fact, this transformation can be viewed in terms of the Clifford action of $T \oplus T^*$ on forms, since

$$(X - \theta) \cdot \rho_0 + \theta\rho_1 = \rho_1 - \theta\rho_0.$$

Because $(X - \theta)^2 = -i_X\theta = -1$, we see that $X - \theta$ is the Clifford algebra element representing reflection in the direction $X - \theta$. In particular, it determines an orthogonal, orientation-reversing map of $T \oplus T^*$. From this point of view, T-duality is interpreted as an orthogonal isomorphism

$$T \oplus T^*(P)/S^1 \cong T \oplus T^*(P^T)/S^1.$$

Because of this, any invariant structure on P defined by an $SO(n, n)$ orbit maps to a similar structure for P^T. For example, given a generalized G_2 structure ρ such that

$$d_H\rho = 0, \quad d_H\hat{\rho} = 0,$$

the transform above produces a generalized G_2 structure ρ^T on P^T satisfying

$$d_{H^T}\rho^T = 0, \quad d_{H^T}\hat{\rho}^T = 0.$$

Using this, one could start with a usual G_2 metric with S^1 symmetry, and produce a generalized G_2 metric on the T-dual, in a quite explicit and straightforward way.

To illustrate the ability of T-duality to produce interesting examples, let us work out an example of T-duality between 4-dimensional generalized Calabi-Yau metrics.

T-duality for 4-dimensional generalized Calabi-Yau metrics

We saw in the previous lecture that an even generalized Calabi-Yau structure in 4 dimensions is a hyperkähler manifold transformed by a B-field.

The case of an *odd* generalized Calabi-Yau is different, of course, although the same equations

$$\langle \rho_i, \rho_j \rangle = \delta_{ij}\nu$$

must be satisfied. Locally, we obtain the expression

$$\rho_i = e^B(dx_i + V \star dx_i),$$

where $\star$ is the Euclidean Hodge star. The ρ_i are closed if and only if

$$dx_i + V \star dx_i$$

is d_H-closed, for $H = dB$, yielding equations

$$d(V \star dx_i) + H \wedge dx_i = 0,$$

i.e. $H = - \star dV$. since H is closed, this means V must be harmonic:

$$\Delta V = 0.$$

Suppose one wishes to find an S^1-invariant hyperkähler metric. Then by T-duality, one could try to find an S^1-invariant odd generalized Calabi-Yau metric and then transform it back to obtain an even one, i.e. a hyperkähler metric, possibly with a B-field. We now implement this strategy.

Suppose that V depends only on the first three variables, and that we impose periodic boundary conditions on $x_4 = t$ so that we are now on a trivial S^1-bundle $P = \mathbb{R}^3 \times S^1$ with trivial connection 1-form $\theta = dt$ and hence $F = 0$. Then let $H = dt \wedge \star_3 dV$, so that $F^T = i_{\partial_t} H = \star_3 dV$ and $H^T = \theta^T F = 0$. The odd generalized Calabi-Yau is determined by forms

$$\rho_1 = dx_1 + V dt \wedge \star_3 dx_1 = dx_1 + V\theta \wedge dx_2 \wedge dx_3, \quad \text{etc.},$$

which are transformed by T-duality to forms

$$\rho_1^T = V dx_2 dx_3 - \theta dx_1, \quad \text{etc.}$$

on a nontrivial S^1-bundle P^T with curvature $F^T = \star_3 dV$ and connection form θ^T. These define a metric

$$g = V(dx_1^2 + dx_2^2 + dx_3^2) + V^{-1}(\theta^T)^2,$$

which is precisely the Gibbons-Hawking ansatz for S^1-invariant hyperkähler metrics. In this way, we see that T-duality can be used to produce quite interesting examples out of structures which at first glance seem rather trivial.

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