To solve the "UV catastrophe" Planck assumed

1) Light comes in energy quanta
2) quantum of light of freq ω has size $\hbar\omega$

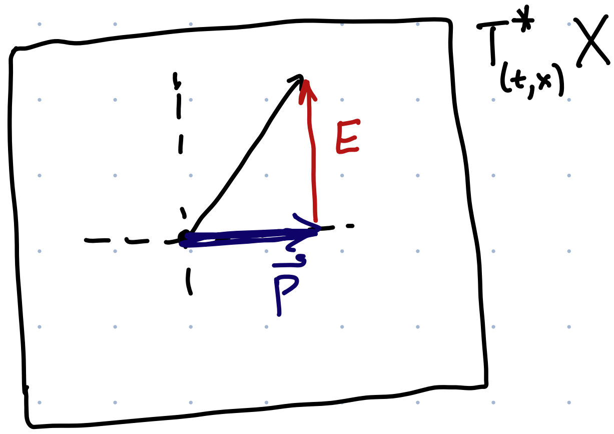
(size of quantum $\propto \omega$)

Special Relativity

$$X = (\mathbb{R}, t) \times (\mathbb{R}^3, x, y, z) \qquad \begin{pmatrix} \mathbb{R} \times M^3 \\ dt^2 - g_{Riem} \end{pmatrix}$$

$$g_{Lor} = dt^2 - (dx^2 + dy^2 + dz^2)$$

at $(t, x) \in X$ $\quad T^*_{(t,x)}X$ has coords $(\overset{\overset{\text{energy}}{\swarrow}}{E}, \underbrace{P_x, P_y, P_z}_{\text{spatial moment.}})$ $\quad P_{\parallel}$



Lorentz group
$\parallel$
$SO(1,3)$
$\parallel$
Symmetries of $(\mathbb{R}^4, \langle , \rangle_{Lor})$

do not preserve decomposition $\mathbb{R}^4 = \mathbb{R} \oplus \mathbb{R}^3$
$(E, \vec{P})$

Canonical quantity: $\|p\| = \sqrt{E^2 - |\vec{p}|^2}$ <u>rest mass</u>

for massless particle $\|p\| = 0$ i.e. $\boxed{E = |\vec{p}|}$

(moving at max speed : c speed of light).

Planck's assumption consistent with $E \propto \omega$
$|p| \propto \omega$

and so $\boxed{\exists \text{ of single type of photon particle.}}$

Massive particle

energy when $\vec{p} = 0$

$$E^2 - |\vec{p}|^2 = m^2$$

$$E^2 = \underset{\uparrow \text{ rest mass}}{m^2} + \underset{\uparrow \text{ spatial momentum}}{\overbrace{|\vec{p}|^2}^{\text{K.E.}}}$$

massless particle $E^2 = |\vec{p}|^2$

Lie groups in Classical + Quantum theory

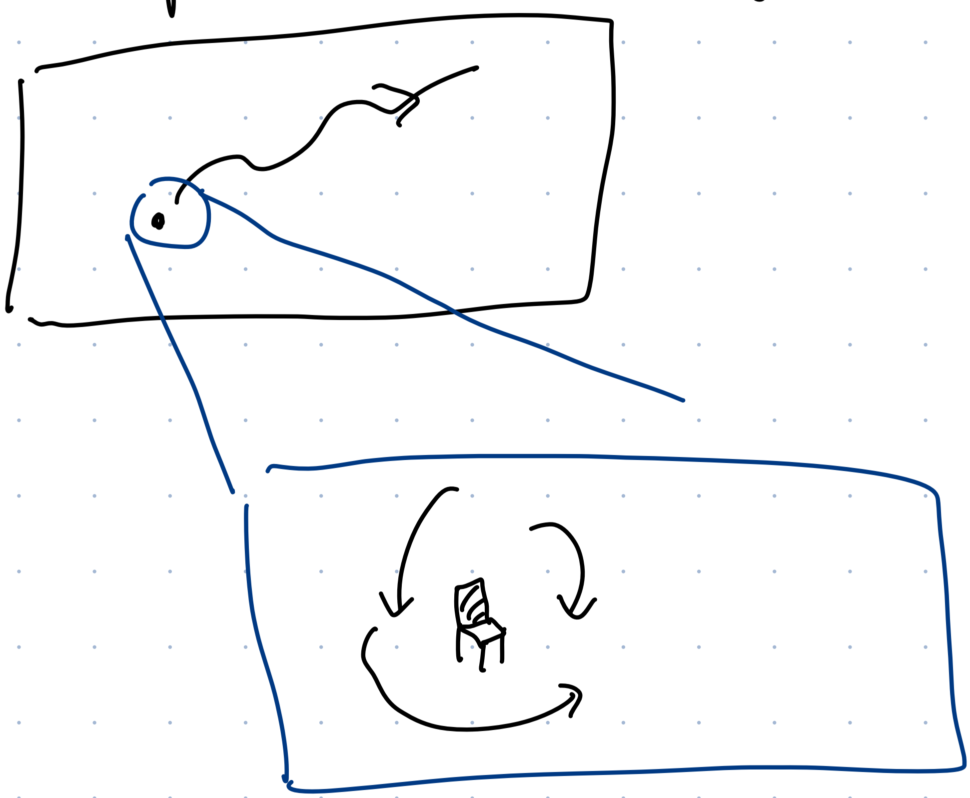
1. Angular momentum:

so far we considered (X, g) config sp. w/ Riem. metric
T^*X phase space

if X is space or spacetime

$T^*_x X$ = linear momenta

(modeling a particle moving in X).



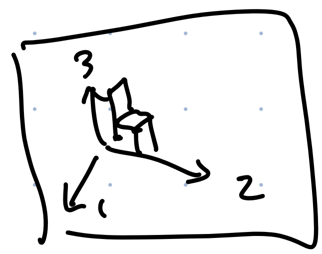
model a more complex object "rigid body"
with orientation in space.

Need to enlarge config. space: Assume X oriented

<u>config</u>: (x, F) x point in X (space)

F frame for $T_x X$

(a frame is (ordered) orthonormal basis comp. w/ orient.)

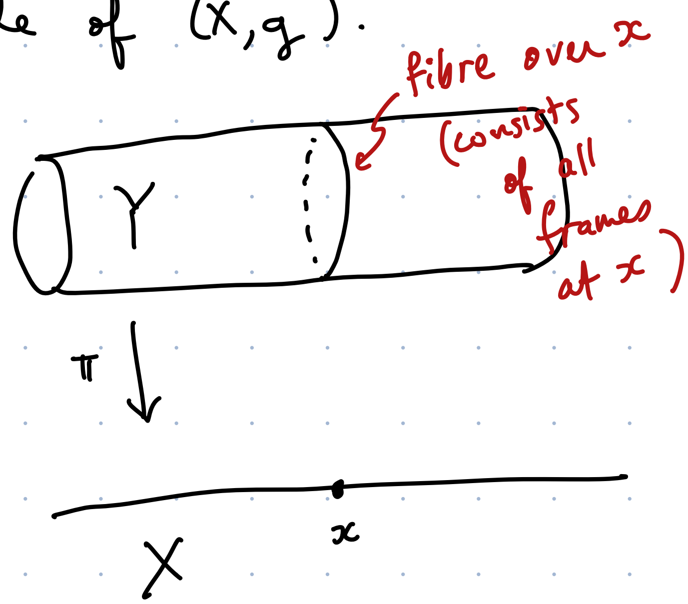


<u>new config. space</u>: The frame bundle of (X, g).

bundle $\quad Y \ni (x, F)$

bundle projection map $\rightarrow \quad \big\downarrow \pi$

base $\quad X$



This type of bundle is called a

Principal $SO(n)$ bundle (if $\dim X = n$)

i.e. $SO(n)$ = Rotation group in n dimensions acts on the bundle Y

Freely & Transitively on each fiber

i.e. given a pair of o.o.n frames F_1, F_2 @ x

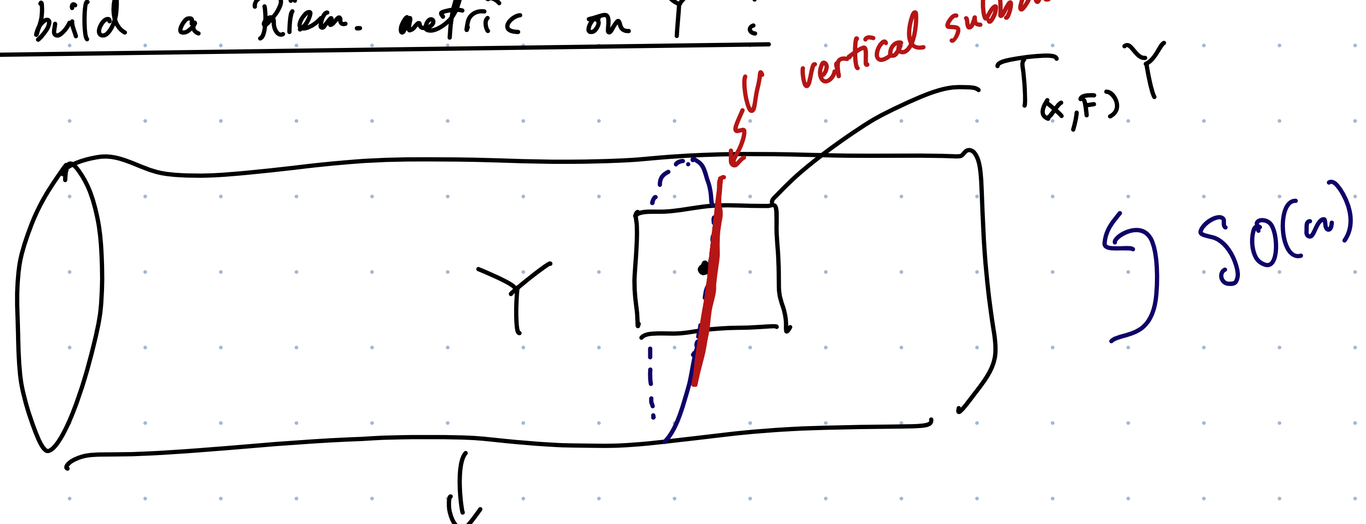
$\exists !$ $g \in SO(n)$ st. $F_1 g = F_2$

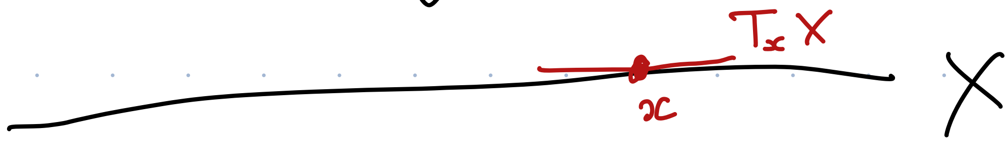
$\Rightarrow$ we can think of Y as a family of copies of $SO(n)$ param. by X: but these copies have no <u>preferred</u> identif. w/ $SO(n)$ (the fibres are "~~affine spaces~~ torsor" for $SO(n)$)

The rigid body is modeled by a point in Y.

<u>Problem:</u> we need a Hamiltonian f^n H on T^*Y
e.g. need a Riem. metric on Y instead of X.

<u>How to build a Riem. metric on Y?</u>





$$0 \to V \to TY \to \pi^* TX \to 0$$

all of these vector bundles carry an action of $SO(n)$

Action of $SO(n)$ on Y

$$\rho : SO(n) \times Y \longrightarrow Y$$

$$(\underset{T_1 SO(n)}{1}, y) \longmapsto y$$

$$d\rho : \underset{\overset{\shortparallel}{}}{\mathfrak{so}(n)} \times 0 \longrightarrow TY$$

for each $X \in \mathfrak{so}(n)$ skew-symm. matrices

$R \in SO(n)$ $\qquad \underbrace{RR^T = 1}_{\text{diff}} \quad \det R = 1$

$\mathfrak{so}(n) = \left(\begin{matrix} n\times n \text{ skew-symm.} \\ \text{matrices} \end{matrix} \right)$

$$\dot{R}R^T + R\dot{R}^T = 0 \Big|_{R=1}$$

$$\dot{R} + \dot{R}^T = 0$$

$\dot{R}$ is skew-symm

$d\rho(X)$ defines a vector field on Y

$\quad \| $

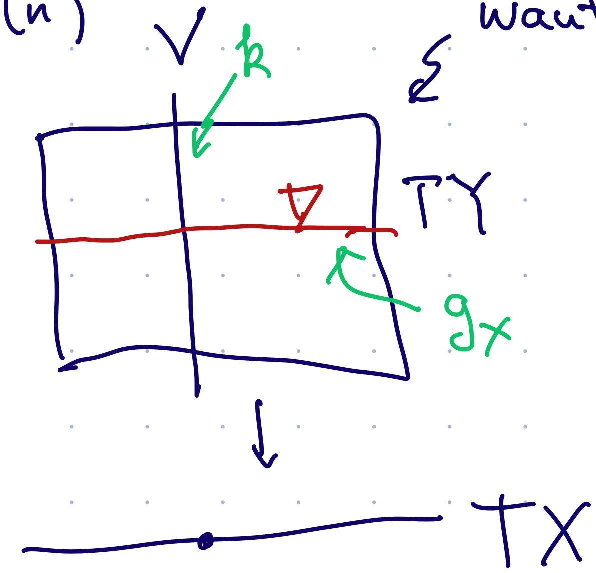
V_X infinitesimal action corresp. to $SO(n)$ action on Y.

$\{$ this vector field is vertical

generated by action vector fields V_X

$$0 \to V \to TY \underset{\nabla}{\rightleftarrows} \pi^*TX \to 0$$

$$V = Y \times so(n)$$



want Riem. metric on TY

have Riem. metric on X

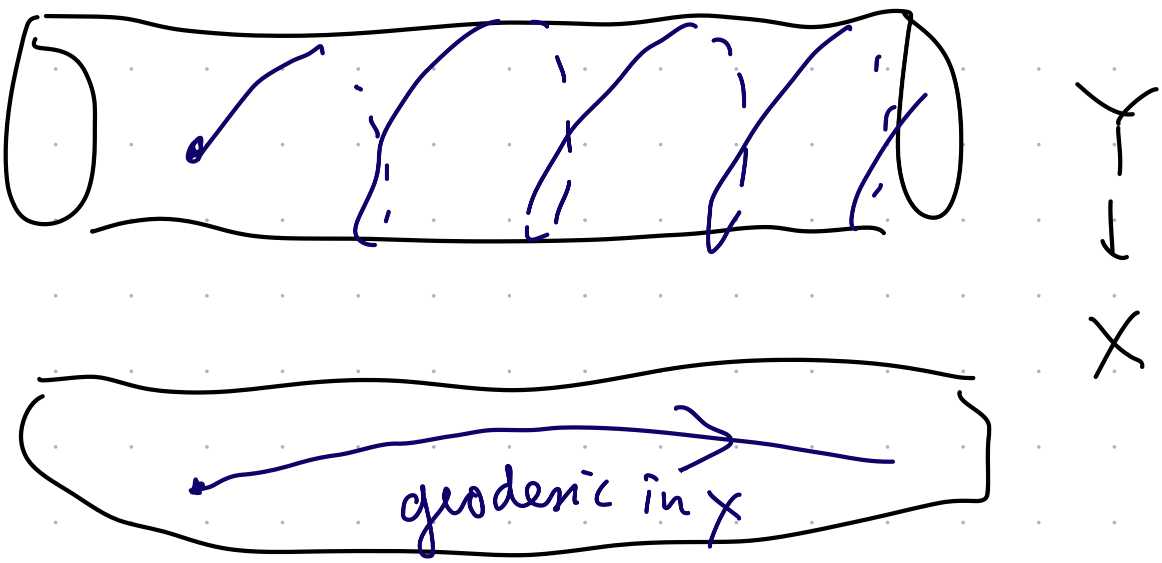
<u>strategy</u>: (1) choose a splitting

$$TY = TX \oplus V$$

provided by Levi-Civita conn. on X.

(2) want a pos. def inner prod. k on $\mathfrak{so}(n)$ Lie alg.

finally $g_X \oplus k$ gives a Riem. metric on Y.



when Lie algebra $\mathfrak{g}$ is "semi-simple" ($so(n)$ is!)

There is a canonical choice of inner product : Killing form

for $so(n)$ this represents the motion of a uniform Rigid body

In general, the choice of K is the Moment of inertia tensor. of the body.