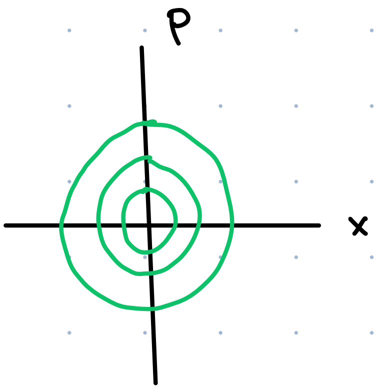
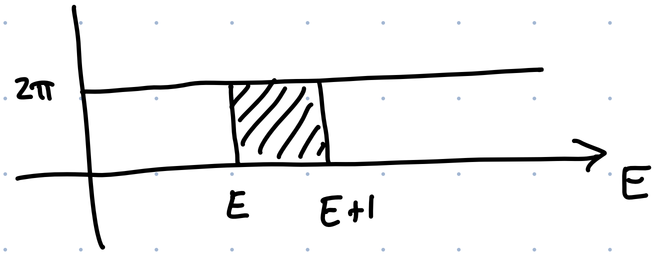
<u>Last time:</u> density of states by energy for Harmonic oscill.



$$H = \frac{1}{2}(x^2 + p^2)$$

$$H_*(dp \wedge dx) = H_*(dH \wedge d\theta) = 2\pi \, dH$$

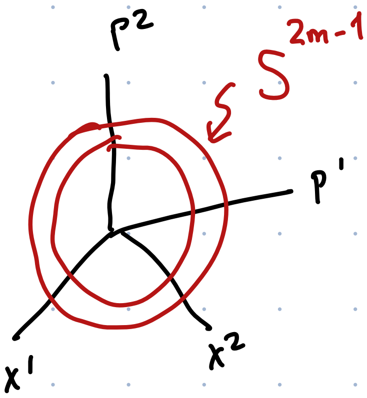


Combining multiple copies of Harmonic osc :

$$X(m) = \mathbb{R}^m \qquad M(m) = (T^*\mathbb{R})^m = \mathbb{R}^m \times \mathbb{R}^{m*}$$

m copies of HO

$$H = \sum_{i=1}^{m} H_i = \frac{1}{2}\left(\sum_{i=1}^{m} x_i^2 + \sum_{i=1}^{m} p_i^2\right)$$



energy level sets are spheres of dim $2m-1$

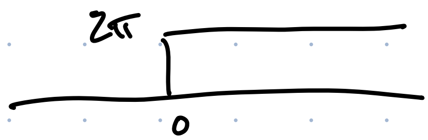
$\Rightarrow$ much faster growth of volume of state space as a f^n of energy

Compute $H_*(v_{\text{Liouville}} = \frac{\omega^m}{m!} = dp_1 \wedge \cdots \wedge dp_m \wedge dx^1 \wedge \cdots \wedge dx^n)$

$$\begin{tikzcd}
\mathbb{R}^{2m} \arrow[r, "\vec{H} = (H_1, \cdots, H_m)"] \arrow[dr, "H"'] & \mathbb{R}^m \arrow[d, "a \text{ addition.}"] \\
& \mathbb{R}
\end{tikzcd}$$

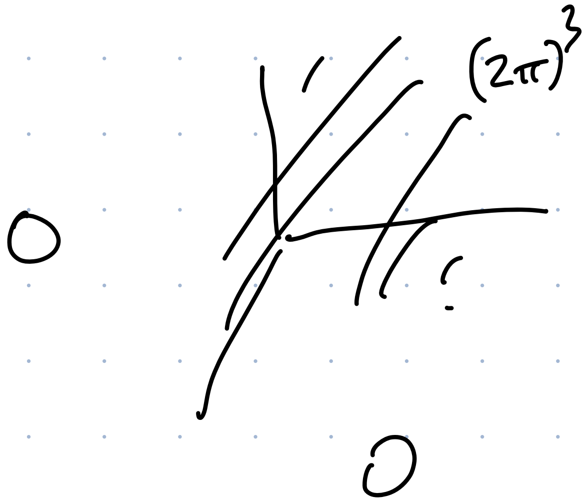
$$\vec{H}_*(v_{\text{Liouville}}) = \begin{cases} (2\pi)^m \cdot 1 & \text{in positive corner } E_i \geq 0 \\ 0 & \text{else} \end{cases}$$

$m = 1$

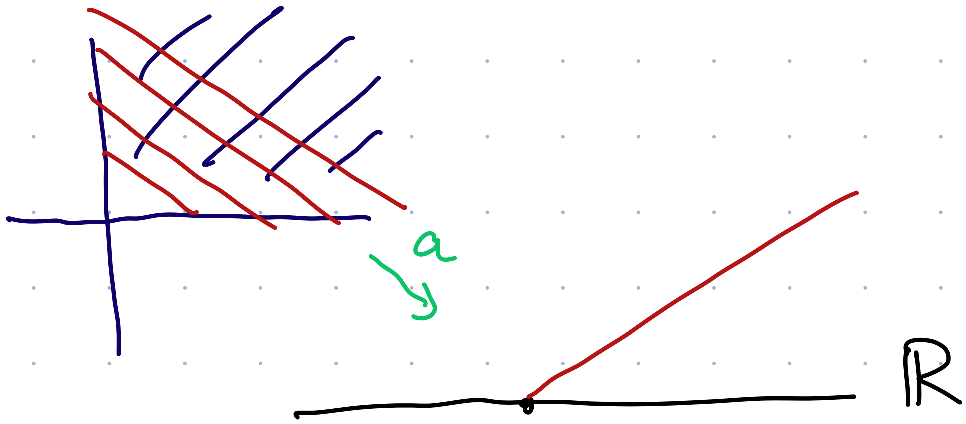


$m = 2$

$m = 3$



applying sum: $a_*\left(\vec{H}_* \, vol_{Liouville}\right)$



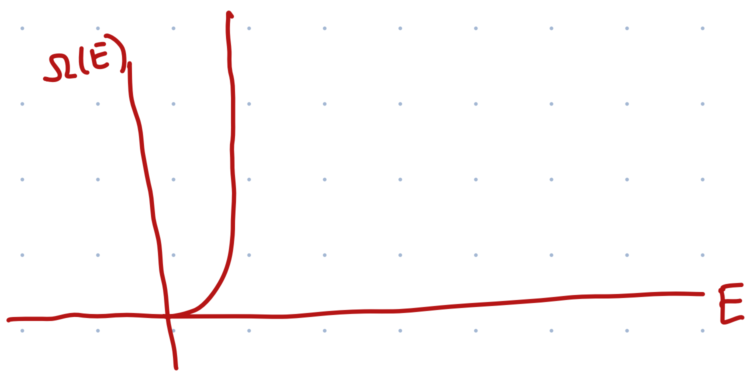
$m=2$

$\Omega(E)\,dE = C_2\, E\, dE$

linear

$m=3 \quad \Omega(E)\,dE = C_3\, E^2\, dE$

in general for m oscillators $\Omega(E)\,dE = C_m\, E^{m-1}\, dE$



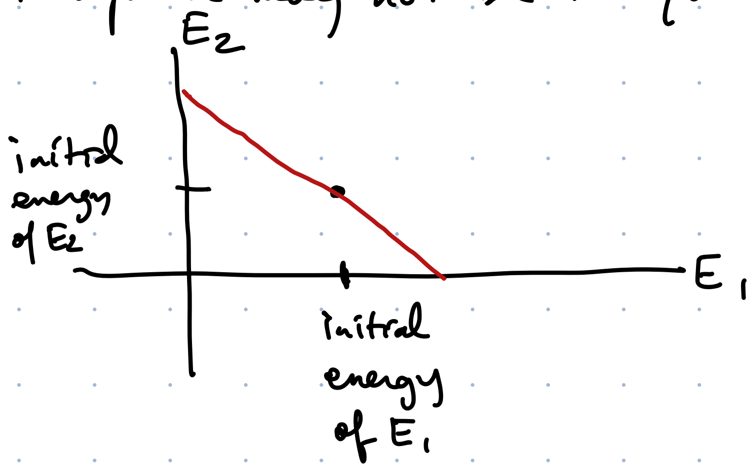
<u>Def:</u> Entropy of system, as a f^n of total energy:

$$S(E) = \log \Omega(E)$$

Temperature & Boltzmann distribution

Idea: i) Equipartition of energy
when system is at equilibrium,
states of equal energy are equally likely.

ii) If two systems (M_1, ω_1, H_1), (M_2, ω_2, H_2)
are each in equilibrium,
The joint system may not be in equilibrium



$$E_{tot} = E_1 + E_2$$

Warning: Without interaction terms in $H_{tot} = H_1 + H_2$,
H_1 and H_2 are separately conserved:

$$\{H_{tot}, H_1\} = \{H_{tot}, H_2\} = 0$$

Implicit in Equipartition for joint systems
is that there are small unknown interaction
terms $\Rightarrow$ we may wait longer for equilibrium.
(making the Hamiltonian system "Ergodic".

<u>Main question</u>: How to determine equilibrium of Joint system

i.e. at equilibrium, what are energies of systems I & II?

<u>Aside</u>:

$$
\begin{array}{c}
(\mathbb{R}^{2n}, v) \\
\Big\downarrow H \\
\mathbb{R}
\end{array}
$$

volume form

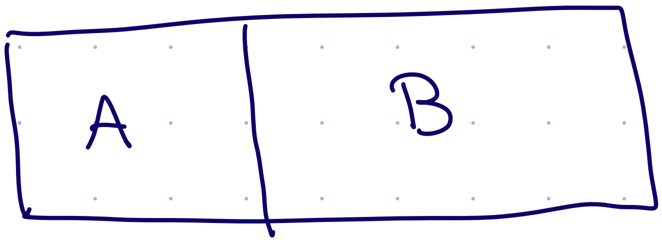
$\Rightarrow$ level sets inherit volume forms

$H^{-1}(c) = \Sigma_c \quad \exists!\, v_c \in \Omega^{top}(\Sigma_c)$

s.t. $\quad v_c \wedge dH = v_{Lio}\big|_{H^{-1}(c)}$

v_c defines uniform dist. on states of fixed energy

(assuming $H^{-1}(c)$ compact)



two systems, A & B joined together

density of states: $\Omega_A(x)$, $\Omega_B(x)$
by energy

total energy constant: $x + y = E_{tot}$

Q.: what is prob. that system A has energy x
(and that syst. B has energy $E_{tot} - x = y$)

$$P(H_A = x)\,dx$$

$P \propto$ number of states of joint system w/ Energy E_{tot} st. $H_A = x$

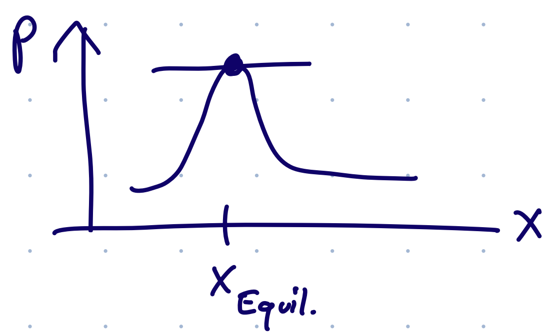
$$P \propto \Omega_A(x)\,\Omega_B(E_{tot} - x)$$

Q: for what value of x is P maximum?

i.e. $\left.\frac{\partial P}{\partial x}\right|_x = 0$

$\Updownarrow$

$\left.\frac{\partial \log P}{\partial x}\right|_x = 0$



i.e. $$\left.\frac{\partial \log \Omega_A^{(x)}}{\partial x}\right|_{x_{eq}} + \left.\frac{\partial \log \Omega_B(E_{tot} - x)}{\partial x}\right|_{x_{eq}} = 0$$

i.e. $$\boxed{\left.\frac{\partial \log \Omega_A}{\partial x}\right|_{x_{eq}} = \left.\frac{\partial \log \Omega_B}{\partial y}\right|_{y = E_{tot} - x}}$$

equality of derivatives of entropies wrt to subsystem energy.

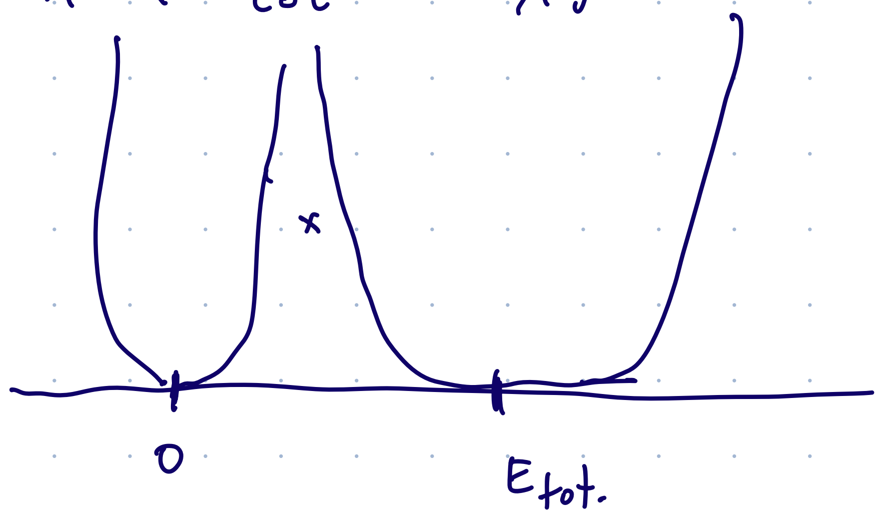
Def: The inverse temperature $\beta \left(= \frac{1}{k T} \right)$ is this quantity

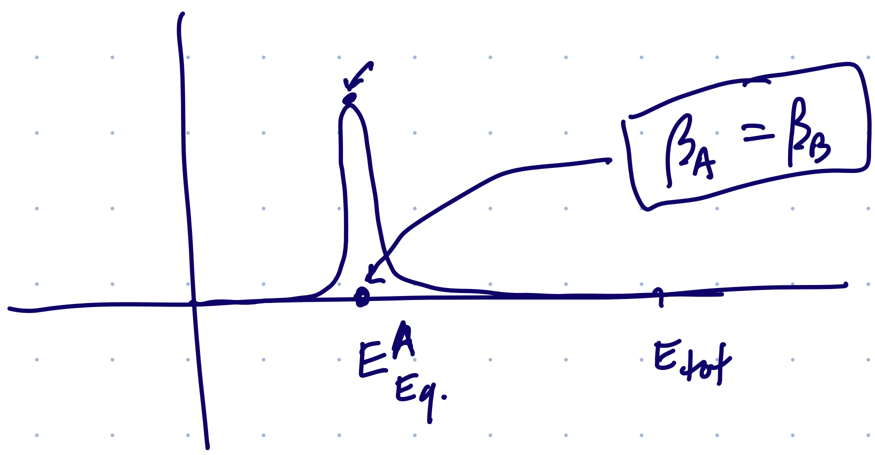
$T =$ temp.

$$\boxed{\beta = \frac{\partial \log \Omega(E)}{\partial E}}$$

" two subsystems are in equilibrium when they have same temperature " e.g. $$\left.\begin{aligned} \Omega_A(E) &= C_A E^m \\ \Omega_B(E) &= C_B E^n \end{aligned}\right\}$$

$$P \propto E_A^m (E_{tot} - E_A)^n$$





<u>Remark:</u> β can be negative for systems where energy bounded above, in this case as β passes through zero

$T \to +\infty = -\infty \to$ negative values

Boltzmann distribution

Pair of systems A, B $\qquad E_A \ll E_B$

("B is a heat bath" or "reservoir")

Assume (A, B) is in equilibrium.

Q.: What is prob. that A is in a specific state of energy E_A

$$P \propto \Omega_B(E_{tot} - E_A)$$

Taylor

$$\log \Omega_B(E_{tot} - \underbrace{E_A}_{small}) = \log \Omega_B(E_{tot}) - \left(\frac{\partial \log \Omega_B}{\partial E_B}\right)\Bigg|_{E_{tot}} E_A + \cdots$$

$$\Rightarrow \boxed{\Omega_B(E_{tot} - E_A) = \Omega_B(E_{tot})\, e^{-\beta_B E_A}}$$

$$\Rightarrow \boxed{P\left(A \text{ has specific state of energy } E_A\right) = C \cdot e^{-\beta E_A}}$$

Boltzmann "Canonical" distribution.

$\Rightarrow$ obtain Prob. density for E_A:

$$\boxed{P_A(E)\,dE = C \cdot \Omega_A(E)\, e^{-\beta E}\, dE}$$

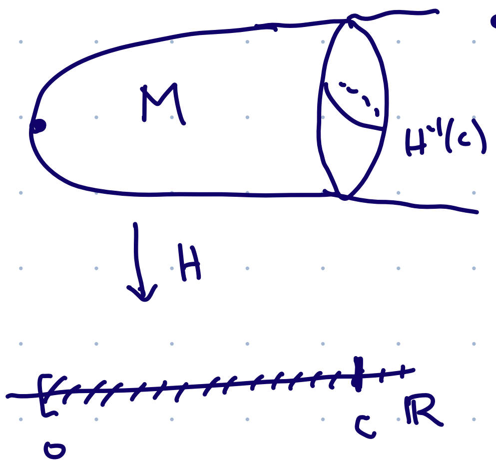
<u>Book:</u> (Souriau : structure of dynamical systems.

English by Cushman et al.)

in equil. w/ large bath

Push-forward:

- $H : M \longrightarrow \mathbb{R}$ $\quad C^\infty$



- Proper $\left(H^{-1}([a,b]) \text{ compact} \right)$.

  $\left(\text{e.g. } \sum (x_i^2 + p_i^2) \right)$

- $c \in \mathbb{R}$ regular value

  $\left(dH \text{ non-vanishing on } H^{-1}(c) \right)$.

- v volume form on M

$$H_* v = \int_{H^{-1}(c)} v$$

to define this: in a nbhd $\mathcal{U}$ of $H^{-1}(c)$,

$$v = \underbrace{dH}_{\text{normal}} \wedge \underbrace{u}_{\text{Longitudinal}} \qquad u \in \Omega^{2n-1}(\mathcal{U})$$

for each $H \in (c-\varepsilon, c+\varepsilon)$ define

$$\boxed{\Omega(H) = \int_{H^{-1}(c)} u}$$

requires $H^{-1}(c)$ to be oriented

then $H_* v = \Omega(H)\, dH$ $\quad$ (we use natural orient. defined by u)

$\Omega(H) \geq 0$

<u>More generally</u>:

$$\begin{array}{c} E \\ \downarrow \pi \\ M \end{array}$$

fibre bundle w/ compact oriented fibres of dim k

$$\pi_* : \Omega^p(E) \to \Omega^{p-k}(M)$$

"fiber integration"