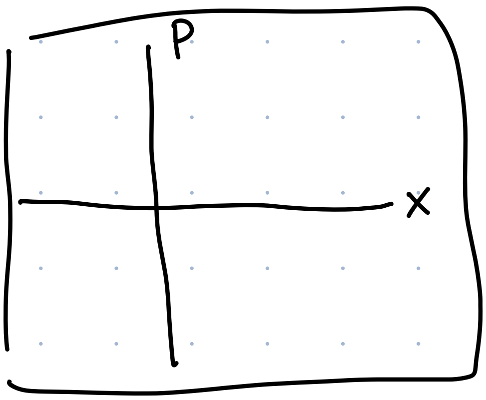
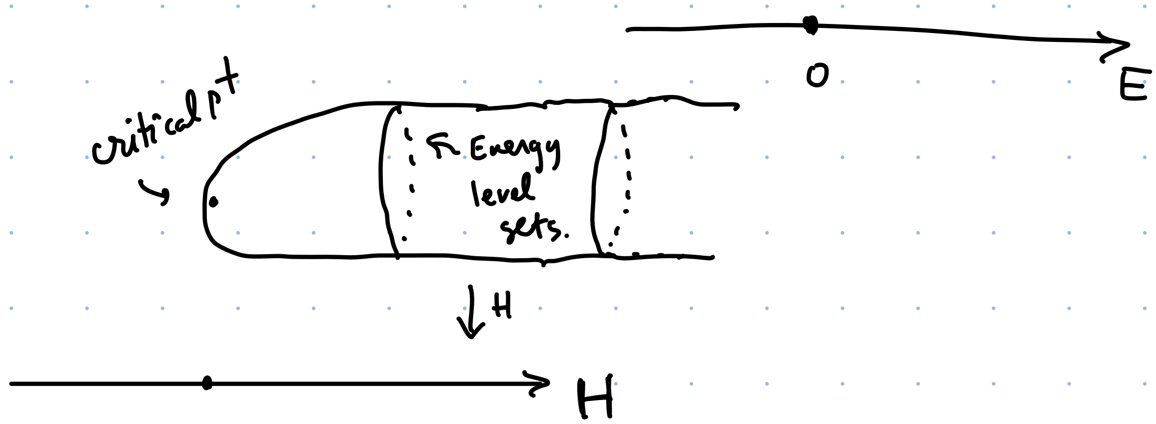
Energy distributions for Harmonic oscillators.



$$H = \frac{1}{2}(x^2 + p^2)$$

$$\mathbb{R}^2 \xrightarrow{H} \mathbb{R}$$



$\mathbb{R}^2$ has Liouville volume $v = dp \wedge dx$

(defines a measure

$$f \in C_c^\infty(\mathbb{R}^2) \longmapsto \mu(f) = \int_{\mathbb{R}^2} f v$$

compact support)

this induces a volume form $\underbrace{\Omega(H)}_{\text{function on } \mathbb{R}} dH$

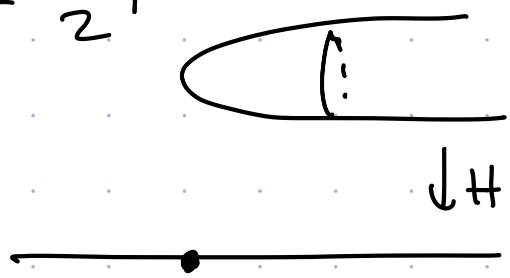
on image of H.

$H_*(v)$ "push-forward measure"

easy method to calculate this:

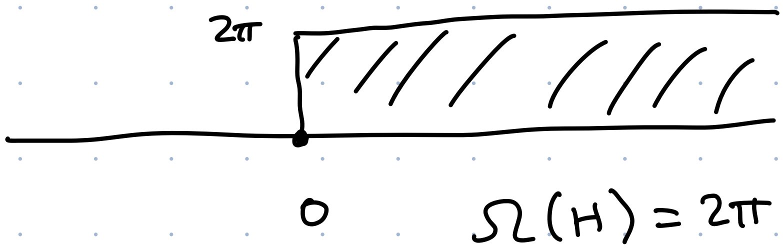
in polar coords $\quad dp \wedge dx = r\,dr \wedge d\theta$

$$H = \tfrac{1}{2}(x^2+p^2) = \tfrac{1}{2} r^2$$



$$v_{Liouville} = dH \wedge d\theta$$

$$H_* v_{Liouville} = \int_{\theta=0}^{2\pi} dH \wedge d\theta = \begin{cases} 2\pi\, dH & H > 0 \\ 0 & \text{else.} \end{cases}$$

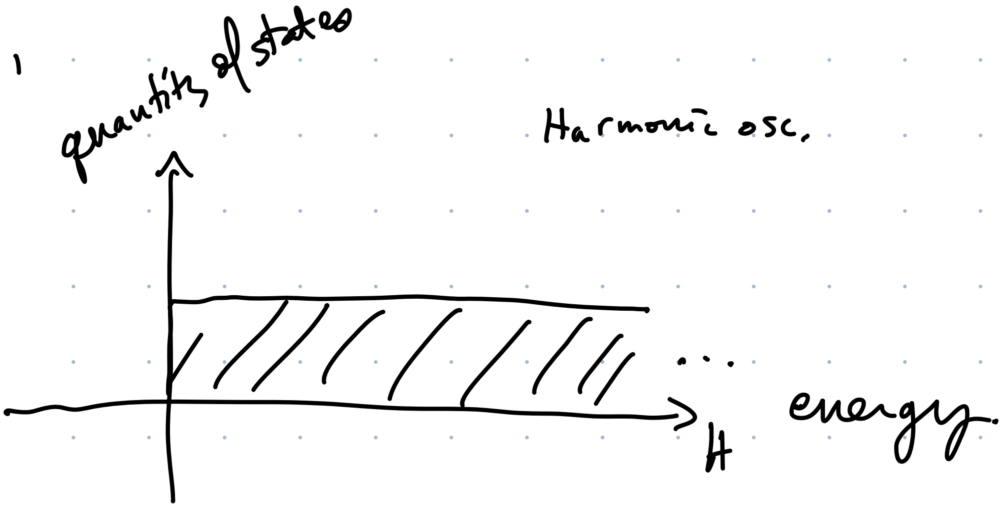


$\Omega(H) = 2\pi$

$\Rightarrow$ quantity of states between $0 < H_1 < H_2$

is $\displaystyle\int_{H=H_1}^{H_2} 2\pi\, dH = 2\pi(H_2 - H_1)$.

Harmonic osc.

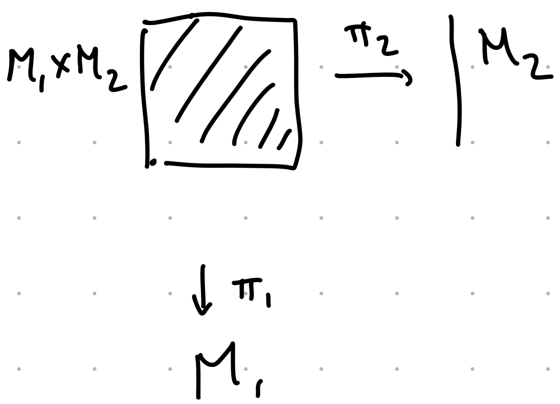


Major principle of Ham. mechanics:

How to combine two Ham. systems

$$(M_1, \omega_1, H_1) \qquad (M_2, \omega_2, H_2)$$
$$M_1 = T^*X_1 \qquad M_2 = T^*X_2$$

Def: The joint system is $(M_1 \times M_2, \ \pi_1^*\omega_1 + \pi_2^*\omega_2, \ \pi_1^* H_1 + \pi_2^* H_2)$, where $M_1 \times M_2 = T^*(X_1 \times X_2)$.



"joining two independent Hamiltonian systems"

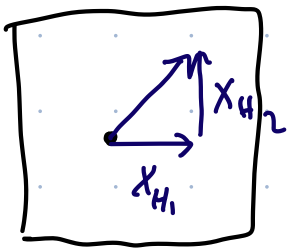
Ham. flow of joint system:

$$-(\omega_1 + \omega_2)^{-1} d(H_1 + H_2) = -(\omega_1^{-1} + \omega_2^{-1})(dH_1 + dH_2)$$

$=0$

$=0$ $\quad x_1, p_1$

$$\omega_1 + \omega_2 : \quad \begin{aligned} \partial_{p_1} &\longmapsto dx^1 \\ \partial_{x_1} &\longmapsto -dp_1 \\ \partial_{p_2} &\longmapsto dx^2 \\ \partial_{x_2} &\longmapsto -dp_2 \end{aligned} \qquad = \ \underset{\substack{\uparrow \\ \text{on } M_1}}{X_{H_1}} + \underset{\substack{\uparrow \\ \text{on } M_2}}{X_{H_2}}$$



$M_1 \times M_2$

simultaneous progress along each original flow.

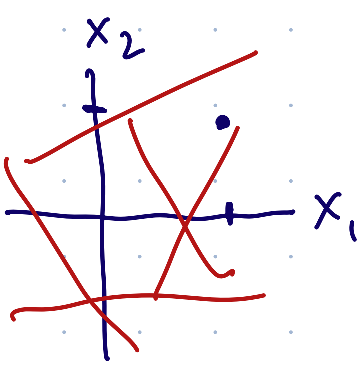
no interaction!

Q: How do we implement an interaction?

A.: by "turning on" a f^n on $M_1 \times M_2$ which is <u>not</u> of the form $H_1(x_1, p_1) + H_2(x_2, p_2)$

<u>Def:</u> Any such f^n is called "interaction term" in the Hamiltonian.

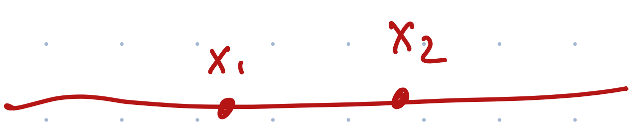
<u>Ex:</u> $\left(T^*\mathbb{R}_1,\ H_1 = \frac{1}{2}p_1^2\right) \times \left(T^*\mathbb{R}_2,\ H_2 = \frac{1}{2}p_2^2\right)$



$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(x_1, x_2)$$

((interaction potential

e.g. $\frac{1}{|x_1 - x_2|}$



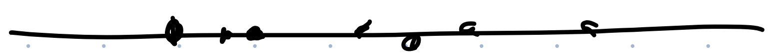
Gas (a large number of copies) of Harmonic oscillators (distinguishable)

Config. space : $\mathbb{R}^m$ $\qquad m = \#$ of particles

Phase space $(T^*\mathbb{R})^m = T^*(\mathbb{R}^m) = \mathbb{R}^m \times \mathbb{R}^{m^*}$

$$\omega = dp_1 \wedge dx^1 + dp_2 \wedge dx^2 + \cdots + dp_m \wedge dx^m$$

$$H = \sum_{i=1}^{m} \tfrac{1}{2}\left((x^i)^2 + (p_i)^2\right) \qquad \text{(non-interacting)}$$



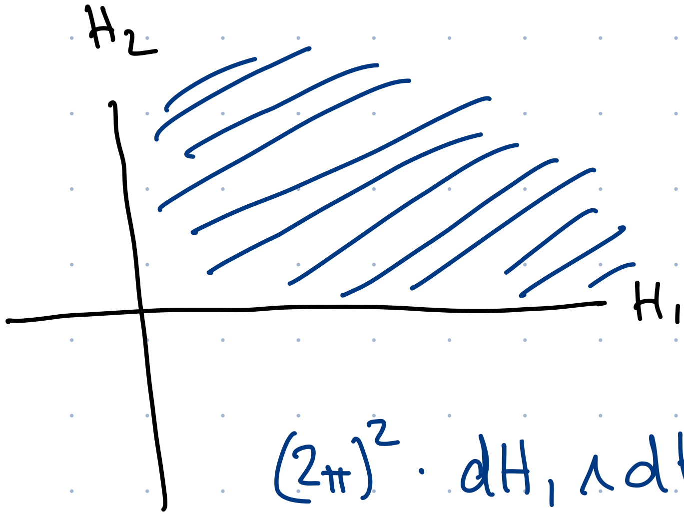
distribution of states as a f^n of total energy is very different :

$m=2$

$$\begin{tikzcd}
T^*\mathbb{R}^2 \arrow[rr, "H"] & & \mathbb{R} \\
\overset{\shortparallel}{\mathbb{R}^4} \arrow[dr, "{(H_1,H_2)}"'] & G & \\
& \mathbb{R}^2 \arrow[uur, "\text{a addition}"'] &
\end{tikzcd}$$

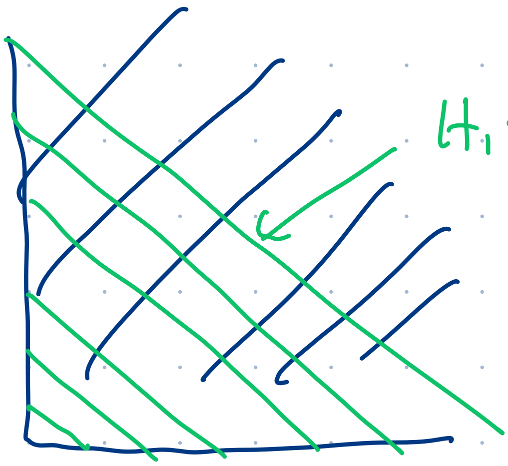
induced volume:

$$(H_1, H_2)_* \Big(\underbrace{\frac{\omega^2}{2!}}_{\text{Liouville volume}} \Big) =$$



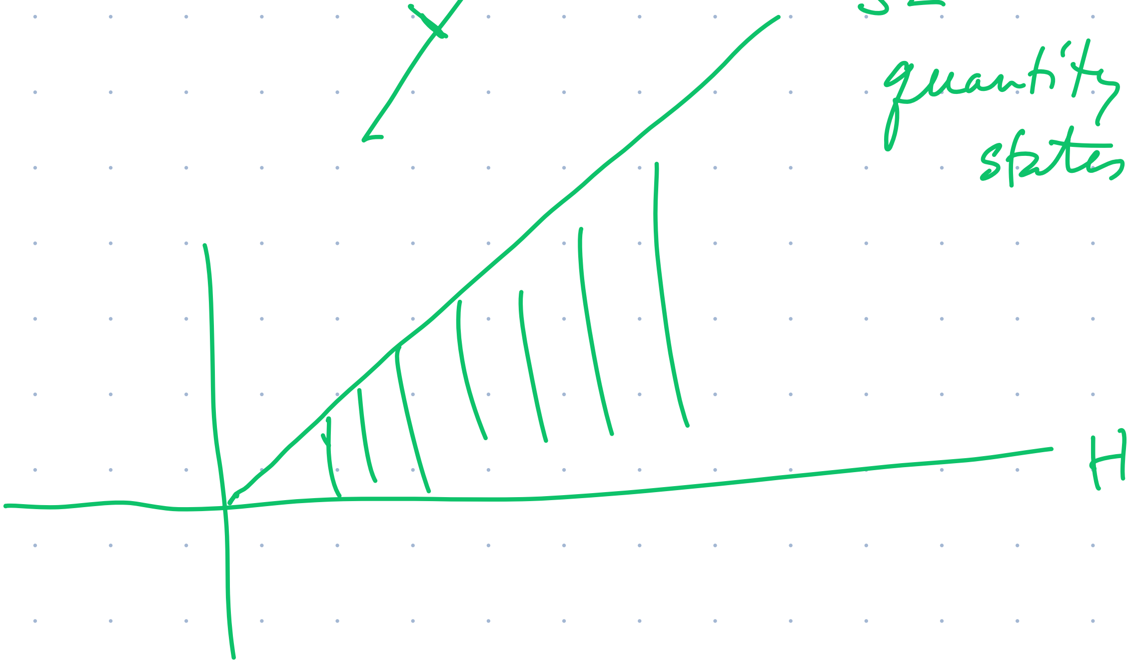
$(2\pi)^2 \cdot dH_1 \wedge dH_2$

$\big\downarrow a_*$



$H_1 + H_2 = \text{const}$

H



$\Omega(H)$

quantity of states!

Same argument for $\mathbb{R}^m \Rightarrow \Omega(H) \propto H^{m-1}$

$m \sim 10^{23}$ for air in our lungs.