Ham. mechanics $\to$ Planck's discovery of correct black body radiation law $\to$ QM

Harmonic oscillator: $X = (\mathbb{R}, x)$

$T^*X = (\mathbb{R}^2, (x,p))$ $\quad \omega = dp \wedge dx$

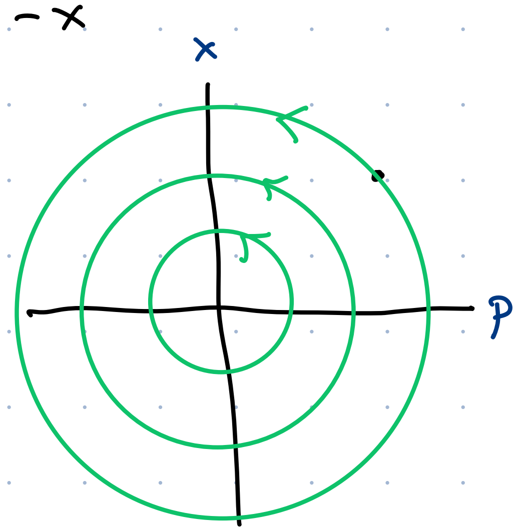
$H = \frac{1}{2}(x^2 + p^2)$

$T^*X = \mathbb{R}^2 = \mathbb{C}$, $z = p + ix$ $\quad \boxed{\omega = \frac{i}{2}\, dz \wedge d\bar{z}}$

Hamilton's equations: $\dot{x} = \{H, x\} = p$

$\dot{p} = \{H, p\} = -x$

$\dot{z} = iz \qquad z(t) = e^{it} z(0)$



X config sp.

T^*X phase space

$$H \in C^\infty(T^*X) \supset \bigoplus_{k \geq 0} \Gamma(X, \underbrace{\mathrm{Sym}^k(TX)}_{\text{polynomial } f^{ns} \text{ of } p \text{ of degree } k.}))$$

for $H = \frac{1}{2}(x^2 + p^2)$

x^2: degree 0 in p (f^n on config space) (potential)

p^2: deg 2 in p (inverse metric $g^{-1} \in \mathrm{Sym}^2 TX$)

g Riem. metric on $\mathbb{R}$.

Magnetic deformation: how to include Magnetic forces.

Electric forces come from potential $V(x)$ $\quad V \in \mathrm{Sym}^0(TX)$
Magnetic forces do not.

A magnetic field $B \in \Omega^2(X)$ $\quad B = B_{ij}(x)\, dx^i \wedge dx^j$

s.t. $dB = 0$.

skew-symmetric $n \times n$ matrix of f^{ns}.

to incorporate this into a Ham. system, we must <u>deform</u> the canonical symplectic form $\omega \in \Omega^2(T^*X)$

$$\begin{array}{c} T^*X \\ \downarrow \pi \\ X \end{array}$$

shift ω to $\omega + \pi^* B$

$$\omega_B = dp_i \wedge dx^i + B_{ij}\, dx^i \wedge dx^j$$

check that this is non-degenerate.

Recall ω_B defines a bundle map

$$\omega_B : TM \longrightarrow T^*M \qquad (M = T^*X)$$

$$v \longmapsto i_v \omega_B = \omega_B(v, -)$$

require this to be <u>iso</u>.

to check this iso. $\Leftrightarrow$ $\det (\omega_B)_{ij} \neq 0$

$$\| $$

$$\left(Pf\,((\omega_B)_{ij})\right)^2 \quad \text{Pfaffian.}$$

$\Updownarrow$

Pfaffian nowhere zero

$\|$

$\frac{1}{n!} \underbrace{\omega_B \wedge \cdots \wedge \omega_B}_{n \text{ copies}} \in \Omega^{2n}(M)$ top degree diff. form on M.

$\|$

$\frac{1}{n!}\omega_B^n$

i.e. ω nondeg iff. $\underbrace{\frac{1}{n!}\omega^n}_{\text{Liouville volume form.}}$ nowhere zero

$$\left(\omega_B = \omega + \pi^* B\right)^n \qquad \omega \underset{loc}{=} dp_i \wedge dx^i \qquad B \underset{loc}{=} B_{ij}\, dx^i \wedge dx^j$$

$$= \omega^n + \underbrace{n\, \omega^{n-1} \wedge B + \binom{n}{2} \omega^{n-2} \wedge B^2 + \cdots}_{\text{zero!}}$$

ω type $(1,1)$ $\quad$ 1 p $\quad$ 1 x

B type $(0,2)$ $\quad$ 0 p $\quad$ 2 x

ω^n type (n,n) $\quad$ n p $\quad$ n x

$\|$

$n!\, dp_1 \wedge dp_2 \wedge \cdots \wedge dp_n \wedge dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$

$$\omega^{n-1} \wedge B = 0$$

$(n-1, n-1) + (0,2)$ $\quad$ $(n-1, n+1)$ of $n+1$ dx^i must vanish!

problematic: wedge prod.

$$\omega^{n-k} \wedge B^k$$

$(n-k, n-k) + (0, 2k)$ $\quad$ $(n-k, \underline{\underline{n+k}})$ $\quad$ $\underbrace{dx^1 \wedge \dots \wedge dx^n}\wedge dx^* = 0$

<u>Thm:</u> ω_B is symplectic, with the same Liouville volume.

<u>Eg.:</u> i) (X, g) Riemannian mfld

$$(T^*X, \omega, H = \tfrac{1}{2} f_{g^{-1}}) \Rightarrow$$ motion of free particle via geodesics

i') $B \in \Omega^2(X)$ $\quad$ $dB = 0$ $\quad$ Magnetic field.

$$(T^*X, \omega_B = \omega + \pi^* B, H = \tfrac{1}{2} f_{g^{-1}})$$

$\Rightarrow$ motion of a charged particle in a magnetic field.

Poisson Bracket of ω_B: $\quad \omega_B^{-1}$

$$X_H = -\omega_B^{-1}(dH)$$

$$\omega_B^{-1} = (\omega + \pi^* B)^{-1} = (\omega(1 + \omega^{-1}B))^{-1}$$

$$= (1+\omega^{-1}B)^{-1}\omega^{-1}$$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$$= (1 - \omega^{-1}B + (\omega^{-1}B)^2 - \dots)\,\omega^{-1}$$

$$= \omega^{-1} - \omega^{-1}B\omega^{-1} + \cancel{\omega^{-1}\underbrace{B\omega^{-1}B}\omega^{-1} + \dots}$$

$$T \xrightarrow[\cong]{\omega} T^*$$

$$\frac{\partial}{\partial p} \longmapsto dx$$

$$\frac{\partial}{\partial x} \longmapsto -dp$$

$$T \xrightarrow{B} T^* \xrightarrow{\omega^{-1}} T \xrightarrow{B} T^*$$

$$\frac{\partial}{\partial x^i} \longmapsto B_{ij}\, dx^j \longmapsto -B_{ij}\frac{\partial}{\partial p^j} \longmapsto 0$$

$$\boxed{B\omega^{-1}B = 0}$$

$$B = B_{ij}\, dx^i \wedge dx^j$$

Prop:

$$\omega_B^{-1} = \omega^{-1} - \underbrace{\omega^{-1}B\omega^{-1}}_{\text{new Magnetic force}}$$

(Ham. vector fields do in fact change

$$\boxed{\begin{aligned} \omega_B^{-1} : \quad dx^i &\longmapsto \frac{\partial}{\partial p^i} \\ dp_i &\longmapsto -\frac{\partial}{\partial q^i} + B_{ij} \frac{\partial}{\partial p_j} \end{aligned}}$$

if $H = V(x)$ flow unchanged

$H = \frac{1}{2} f_{g^{-1}}$ geodesics will be modified!

$$\left.\begin{aligned} \dot{x} &= g^{-1} p \\ \dot{p}_k &= \frac{1}{2} \partial_k g^{ij} p_i p_j \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} \dot{x} &= g^{-1} p \\ \dot{p}_k &= \frac{1}{2} \partial_k g^{ij} p_i p_j + \underline{\underline{p_j g^{ij} B_{ik}}} \end{aligned}\right.$$

sol^{ns} to new system $=:$ magnetic geodesics

$$\nabla_{\dot{x}} \dot{x} = g^{-1}(\iota_{\dot{x}} B)$$

accel. $= \frac{1}{m} \cdot$ magnetic force.

Ex.: free particle in $(\mathbb{R}^2, (x,y)) = X$

$$T^*X = \mathbb{R}^4 \quad (x, y, p_x, p_y) \qquad \omega = dp_x \wedge dx + dp_y \wedge dy$$

$$B = b\, dx \wedge dy \qquad \left(b \overset{\text{e.g.}}{\text{const}}\right).$$

Euclidean metric on $\mathbb{R}^2$ $g = dx^2 + dy^2$

$H = \frac{1}{2}(p_x^2 + p_y^2)$ Kinetic energy of particle in $\mathbb{R}^2$

Magnetic def. $\omega_B = dp_x \wedge dx + dp_y \wedge dy$
$+ b\, dx \wedge dy.$

What is the Ham. flow? $\qquad B_{ij} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$

$$X_H = -\omega_B^{-1}(p_x dp_x + p_y dp_y)$$

$$= \left(p_x \left(\frac{\partial}{\partial x} - b \frac{\partial}{\partial p_y} \right) + p_y \left(\frac{\partial}{\partial y} + b \frac{\partial}{\partial p_x} \right) \right)$$

$$\left.\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} p_x \\ p_y \end{pmatrix} \\ \begin{pmatrix} \dot{p}_x \\ \dot{p}_y \end{pmatrix} &= \begin{pmatrix} p_y b \\ -p_x b \end{pmatrix} \end{aligned}\right\} \quad \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} p_y b \\ -p_x b \end{pmatrix} = b \begin{pmatrix} \dot{y} \\ -\dot{x} \end{pmatrix}$$

$$\begin{pmatrix}\ddot x\\ \ddot y\end{pmatrix} = -b \underbrace{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}}_{J}\begin{pmatrix}\dot x\\ \dot y\end{pmatrix} \qquad J^2 = -1.$$

$$\begin{pmatrix}\dot x\\ \dot y\end{pmatrix} = e^{-bJt}\begin{pmatrix}\dot x(0)\\ \dot y(0)\end{pmatrix}$$

rotation at ang. speed b for $\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}$

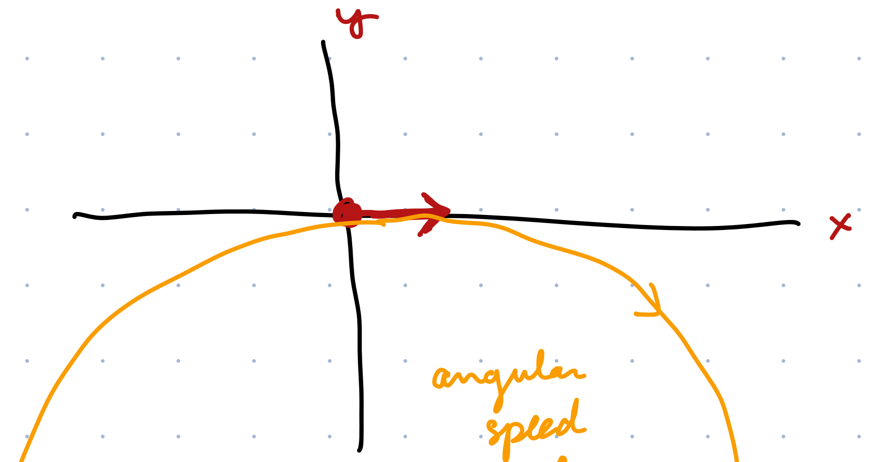
$z = x + iy \qquad J = \text{mult. by } i$

$$\dot z = e^{-ibt}\,\dot z(0)$$

$$z = \frac{1}{-ib}\left(e^{-ibt}\,\dot z(0) - \dot z(0)\right) + z(0)$$

$$\boxed{z = \left(\frac{e^{-ibt} - 1}{-ib}\right)\dot z(0) + z(0)}$$

for $z(0) = 0$
$\dot z(0) = 1$



b

Relativity: How to incorporate Special / General relativity into Ham. formalism?

X config space "space"

$\mathbb{R}$ time

extended config. space $(X \times \mathbb{R},\ (x,t))$

extended phase space : $T^*(X \times \mathbb{R}) = \underset{(x,p)}{T^*X} \times \underset{(s,t)}{T^*\mathbb{R}}$

Lorentzian metric : $g_X - dt^2$ ← standard metric on $\mathbb{R}$

↑ usual Riem. metric on X

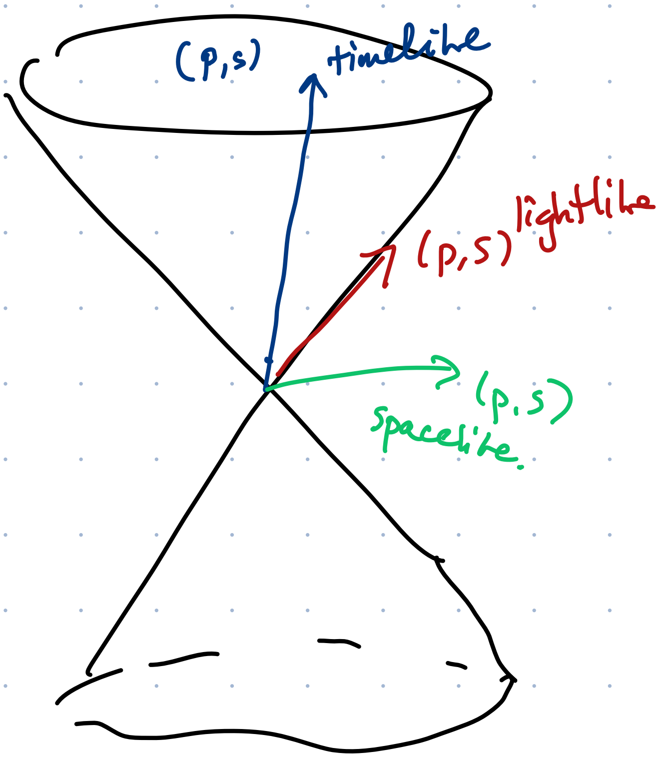
signature $\left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & \mathbb{1}_{n \times n} \end{array}\right)$ bilinear form at each event (x,t) in spacetime

Key idea: Lorentzian metric $g_X - dt^2 = g_{Lor}$ determines a quadratic f^n

$$f_{g_{Lor}} = \underbrace{\frac{1}{2} g^{ij}(x) p_i p_j}_{\text{K.E. along space}} - \underbrace{\frac{1}{2} s^2}_{\text{K.E along time.}}$$

If we use $H = f_{g_{Lor}}$ to generate flows

we obtain Lorentzian geodesics



Value of H on initial cond.

$$\big((x,t), \underbrace{(p,s)}_{\text{momentum in spacetime}}\big)$$

norm $\|p\|^2 - s^2$

could be pos spacelike.

0 light-like.

neg timelike.

Def: given a state $((x,t),(p,s))$ with timelike momentum

i.e. $\|p\|^2 - s^2 < 0$

$m = \sqrt{-(\|p\|^2 - s^2)}$ is called the rest mass of state.

if $X = \mathbb{R}^3$, $g_X = g_{Euc} = dx^2 + dy^2 + dz^2$

then the system $\left(\hat{X} = \mathbb{R}^3 \times \mathbb{R},\ g_{Lor} = g_X - dt^2 \right)$

has enlarged group of symmetries

including Poincaré group

(translations in space + time,

rotations in space

+ Boosts (change velocity))

$$SO(3,1) \ltimes \mathbb{R}^4$$

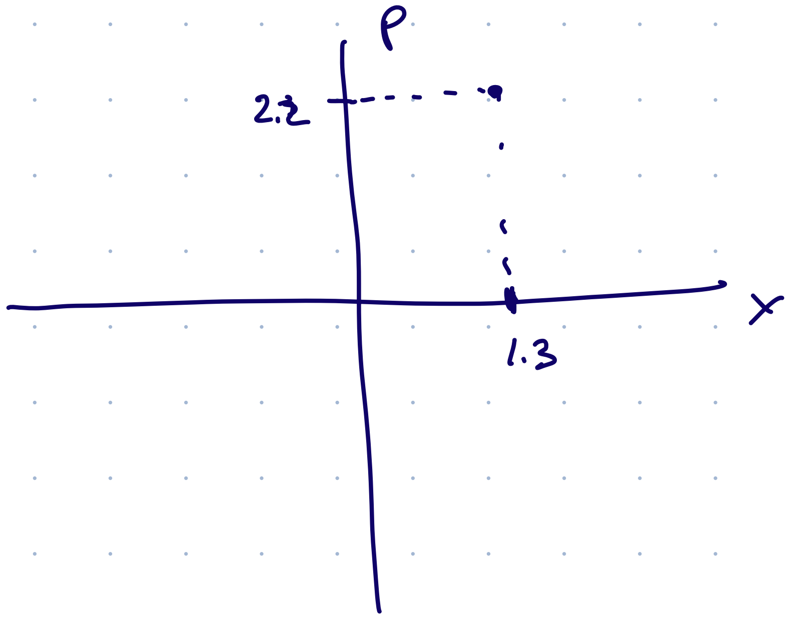
rotation preserving Lorentz metric

translation

$X = (\mathbb{R}, x)$ $\qquad$ $T^*X = (\mathbb{R}^2, (x,p))$

$\omega = dp \wedge dx$

$H = \frac{1}{2}(x^2 + p^2)$



$M = T^*X$ phase space

"Observables" = functions on phase space.

"algebra of classical observables" = $(C^\infty(T^*X), \cdot, +)$

e.g. $x, p \in C^\infty(T^*X)$

$\mathbb{R}[x,p] \subset C^\infty(T^*X)$

H Most Hamiltonians would be elts of subalgebra

the alg. of observables sits in a larger algebra

$C^\infty(T^*X, \mathbb{C})$ $\qquad$ complex-valued f^{us}

$z = p + ix$

complex conjugation

analogy w/ QM

$$C^\infty(T^*X, \mathbb{R}) \hookrightarrow C^\infty(T^*X, \mathbb{C})$$

$\updownarrow$ $\qquad\qquad\qquad\qquad$ $\updownarrow$

self-adjoint operators $\subset$ other operators in the complex operator algebra

$\Downarrow$

raising / lowering operator

$$\boxed{\nabla_{\dot x}\dot x = g^{-1}\, i_{\dot x} B}$$

$$i_{(\dot x, \dot y)} B = \dot x\, dy - \dot y\, dx$$

$$\Big\downarrow g^{-1}$$

$$\dot x \frac{\partial}{\partial y} - \dot y \frac{\partial}{\partial x}$$

$$g = dx^2 + dy^2$$

$$dx \wedge dy = B$$