Geometry of Quantum Mechanics 4

Review: $(\underset{\text{symplectic manifold}}{M} = T^*\overset{\text{configuration space}}{X},\ \omega,\ \underset{\text{Ham. } f^n \text{ in } C^\infty(M,\mathbb{R})}{H})$

eg.: $H \in C^\infty(M) \supseteq$ f^{ns} polynomial in cotangent fibres

$$\shortparallel \quad \bigoplus_{k=0}^{\infty} \Gamma(X, \mathrm{Sym}^k TX) \leftarrow \text{symmetric multivectors}$$

$k=0)$ $V \in C^\infty(X,\mathbb{R})$ "potential"

1) $Y \in \mathfrak{X}(X)$

2) $T_g = \frac{1}{2} f_{g^{-1}}$ g Riem. metric "kinetic energy"

Most important example: 0) + 2):

$$H = T_g + V \qquad V \in C^\infty(X,\mathbb{R}).$$

$$X_H = \underset{\text{geodesic v.field}}{X_g} + \underset{\text{deflection due to force gen. by } V.}{X_V}$$

Recall: Hamilton's eq^n of motion:

$$\boxed{\begin{aligned} \dot{x}^i &= \{x^i, H\} \\ \dot{p}_i &= \{p_i, H\} \end{aligned}}$$

flow of X_H

for X_g: ① $\begin{cases} \dot{x} = g^{-1} p \\ \dot{p}_i = \frac{1}{2}(\partial_k g^{ij}) p_i p_j \end{cases}$

sol^{ns} are geodesics

accel. $= \nabla_{\frac{\partial}{\partial t}} \dot{x} = 0$

for $X_V = -\omega^{-1}(dV) = -(dp_i \wedge dx^i)^{-1}(\partial_k V \, dx^k)$

$$\Rightarrow \text{②} \begin{cases} \dot{x} = 0 \\ \dot{p}_i = -\dfrac{\partial V}{\partial x^i} \end{cases}$$

<u>combining</u>: $\nabla_{\frac{\partial}{\partial t}} \dot{x} = -\underbrace{g^{-1}(dV)}_{\text{grad} V = \nabla V} = g^{-1} F$

$F = -dV \in \Omega^1(X)$

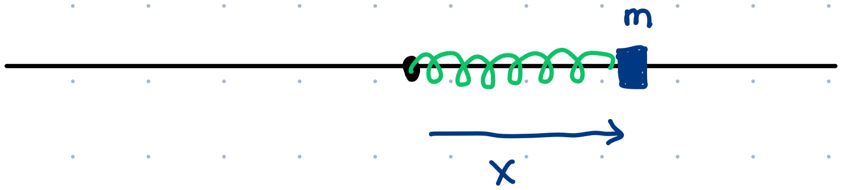
Force assoc. to potential V.

$$F = g\, a = g \nabla_{\partial_t} \dot{x}$$

g plays role of mass.

Ex.: Simple harmonic oscillator

$X = \mathbb{R}$

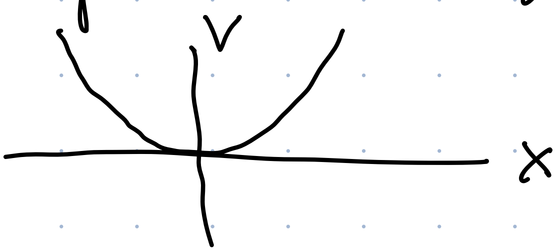


equip X w/ Riem. metric $g = m\, dx \otimes dx$

$$\Rightarrow T_g = \frac{1}{2} m^{-1} p^2$$

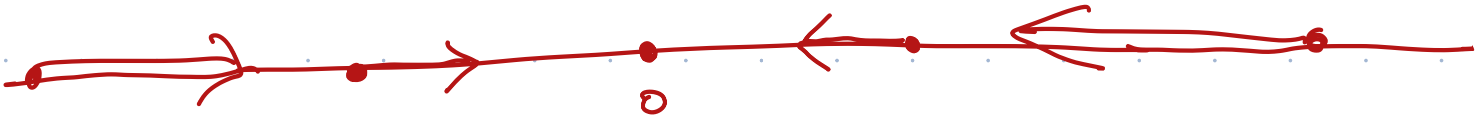
Provide Law of force for a spring (Hooke's Law)

$V = \frac{1}{2} k x^2$, $k \in \mathbb{R}_{>0}$



$$-\operatorname{grad} V = -g^{-1}(dV = kx\,dx)$$

$$= -kx\frac{\partial}{\partial x}$$

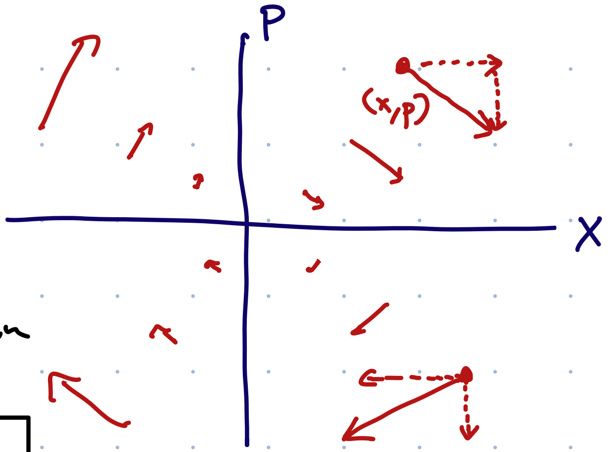


$$\boxed{H_{SHO} = \frac{1}{2}m^{-1}p^2 + \frac{1}{2}kx^2}$$

Ham. system for $X = (\mathbb{R}, x)$

$T^*X \cong (\mathbb{R}\times\mathbb{R}, (x,p))$

$$X_H = m^{-1}p\frac{\partial}{\partial x} - kx\frac{\partial}{\partial p}$$



special property: X_H linear!

⇒ express Hamilton's eqns in matrix form

$$\boxed{\frac{d}{dt}\begin{pmatrix}x\\p\end{pmatrix} = \underbrace{\begin{pmatrix}0 & m^{-1}\\ -k & 0\end{pmatrix}}_{A}\begin{pmatrix}x\\p\end{pmatrix}}$$

sol^n: $\begin{pmatrix}x(t)\\p(t)\end{pmatrix} = e^{tA}\begin{pmatrix}x(0)\\p(0)\end{pmatrix}$

$$A^2 = -\frac{k}{m}I = -\omega^2 I \qquad (\underbrace{\omega^{-1}A}_{J})^2 = -I$$

" operator behaves like i on $\mathbb{C}$ "

$$J = \left(\sqrt{\frac{k}{m}}\right)^{-1}\begin{pmatrix}0 & m^{-1}\\ -k & 0\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{\sqrt{km}}\\ -\sqrt{km} & 0\end{pmatrix}$$

$$\begin{pmatrix}x(t)\\p(t)\end{pmatrix} = \underbrace{e^{t\omega J}}\begin{pmatrix}x(0)\\p(0)\end{pmatrix} = \left(\cos(\omega t)\mathbb{1} + \sin(\omega t)J\right)\begin{pmatrix}x(0)\\p(0)\end{pmatrix}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

1^{st} component: $x(t) = \cos(\omega t)\, x(0) + \frac{1}{\sqrt{km}} \sin(\omega t)\, p(0)$

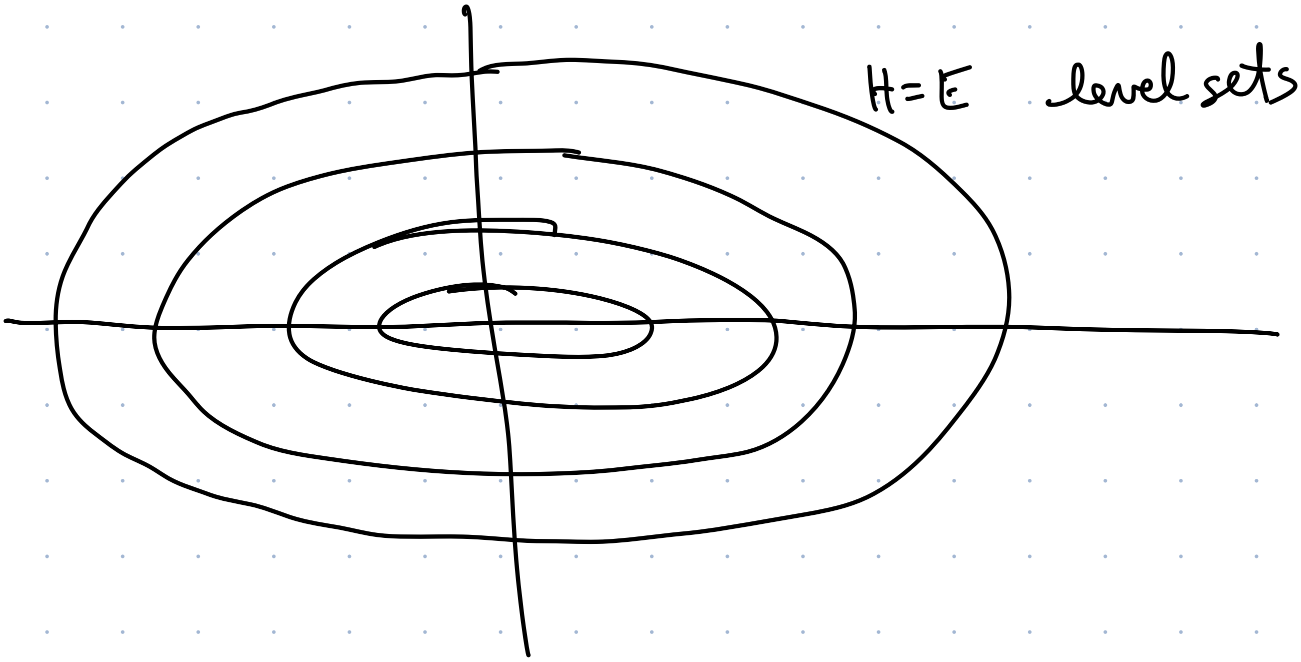
$$p = m\dot{x}$$

$$\boxed{x(t) = \cos(\omega t)\, x(0) + \omega^{-1} \sin(\omega t)\, \dot{x}(0)}$$

sinusoidal oscill. at freq ω

trajectories coincide w/ level sets of H:

$\frac{1}{2}m^{-1}p^2 + \frac{1}{2}kx^2 = E$ constant for each solution



what are the possible energies? $H \in \mathbb{R}_{\geq 0}$

whereas frequency is $\omega = \sqrt{\frac{k}{m}}$ fixed.

canonical transformation ~ symplectomorphis

notice : $\left\{\right.$ $H = \frac{1}{2} m^{-1} p^2 + \frac{1}{2} k x^2$

we could do a change of variables:

$$\tilde{x} = \sqrt{k}\, x \qquad \tilde{p} = \frac{1}{\sqrt{k}}\, p$$

$$H = \frac{1}{2} m^{-1}\left(k \tilde{p}^2\right) + \frac{1}{2} \cancel{k} \frac{\tilde{x}^2}{\cancel{k}}$$

$$= \frac{1}{2}\left(\frac{k}{m}\right)\tilde{p}^2 + \frac{1}{2}\tilde{x}^2$$

$$\tilde{X} = c\, x \qquad \tilde{p} = c^{-1} p$$

$$H = \frac{1}{2} \underbrace{m^{-1} c^2}\, p^2 + \frac{1}{2} \underbrace{k\, c^{-2}}\, x^2$$

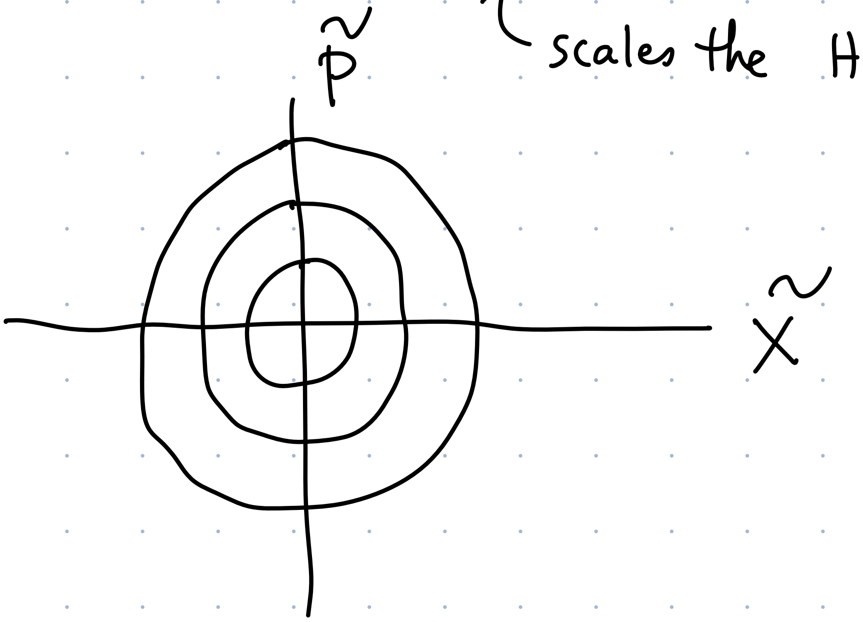
$$m^{-1} c^2 = k c^{-2}$$

$$c^4 = k m$$

$$\frac{c^2}{m} = \frac{\sqrt{km}}{m} = \sqrt{\frac{k}{m}}$$

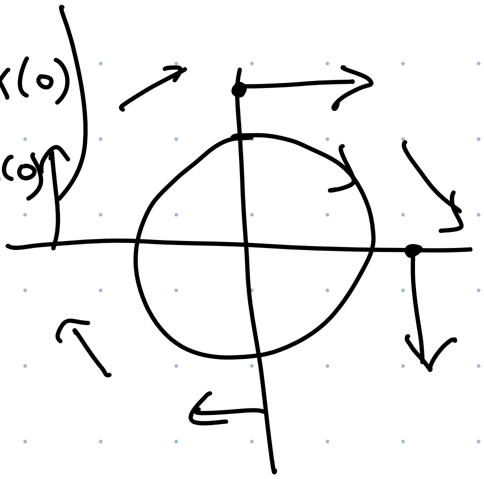
$$H = \frac{1}{2}\sqrt{\frac{k}{m}}\left(\tilde{p}^2 + \tilde{x}^2\right)$$

↑ scales the Hamiltonian + v. field



standard SHO $k = m = 1$ $\quad H = \frac{1}{2}\left(p^2 + x^2\right)$

has v. field $X_H = p\frac{\partial}{\partial x} - x\frac{\partial}{\partial p}$ (standard rotation in plane) flow : $\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x(0) \\ p(0) \end{pmatrix}$



ω enters as a scale factor determining angular speed.

This system has remarkable property :

- all its trajectories are closed (i.e. periodic)
- the periods all agree and are given by

period $\boxed{T = \frac{2\pi}{\omega}}$ ← transcendental number!

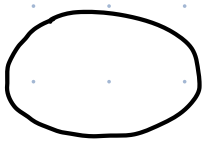
How is this period computed? $\displaystyle\int_{\text{one cycle}} dt = T$

$\shortparallel$

$H^{-1}(E)$ fix energy

$$\frac{dx}{dt} = \{x, H\}$$

$$\frac{dp}{dt} = \{p, H\}$$



curve.

$$\int_{H^{-1}(E)} \left(dt = \frac{dx}{\{x,H\}}\right)$$

if we use coordinate x on curve $\mathcal{C}$

$$T = \int \frac{dx}{\{x,H\}}$$

$$\boxed{T = \int_{\mathcal{C}} \frac{dx}{\sqrt{2E - x^2}}}$$

$$\frac{1}{2}(x^2+p^2) = E$$

$$\Downarrow$$

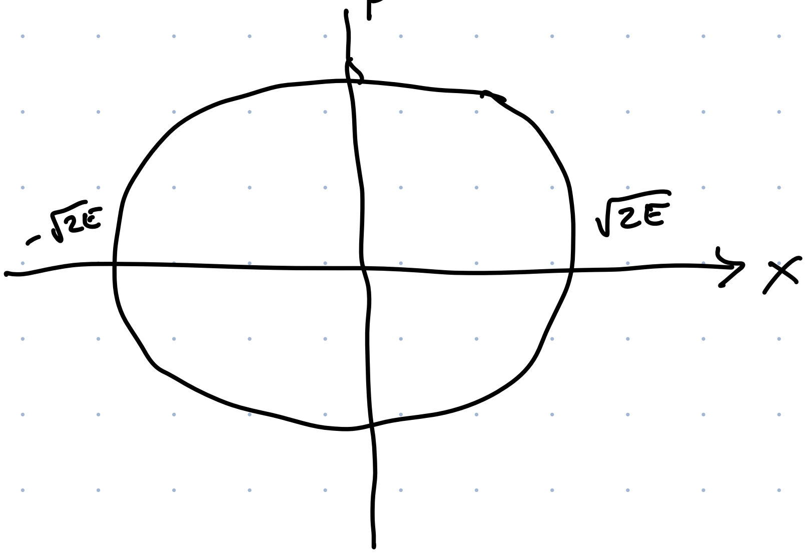
$$p = \sqrt{2E - x^2} \quad \left\{x, \frac{1}{2}(p^2+x^2)\right\}$$

$$'' \quad 2 \cdot \frac{1}{2} p \{x, p\}$$

$$'' \quad p$$

$$\{x, p\} = \frac{\partial x}{\partial x} \cdot \frac{\partial p}{\partial p} - \cancel{\frac{\partial x}{\partial p}}\, \cancel{\frac{\partial p}{\partial x}} = \underline{1}$$

$$\{x, x\} = 0$$



$$T = \int_{-\sqrt{2E}}^{\sqrt{2E}} \frac{dx}{\sqrt{2E - x^2}}$$

positive sqrt.

$$+ \int_{\sqrt{2E}}^{-\sqrt{2E}} \frac{dx}{-\underbrace{\sqrt{2E - x^2}}_{\text{pos sqrt.}}}$$

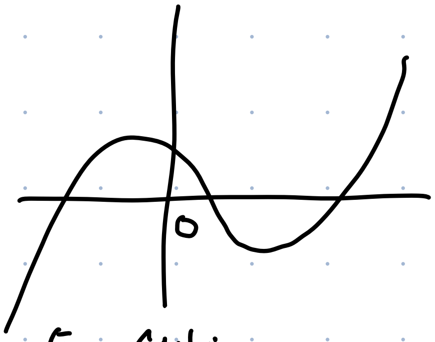
$$= 2 \int_{-\sqrt{2E}}^{\sqrt{2E}} \frac{dx}{\sqrt{2E - x^2}} = 2\pi$$

→ Algebraic geometry "periods"

Eg 2: $X = \mathbb{R}$

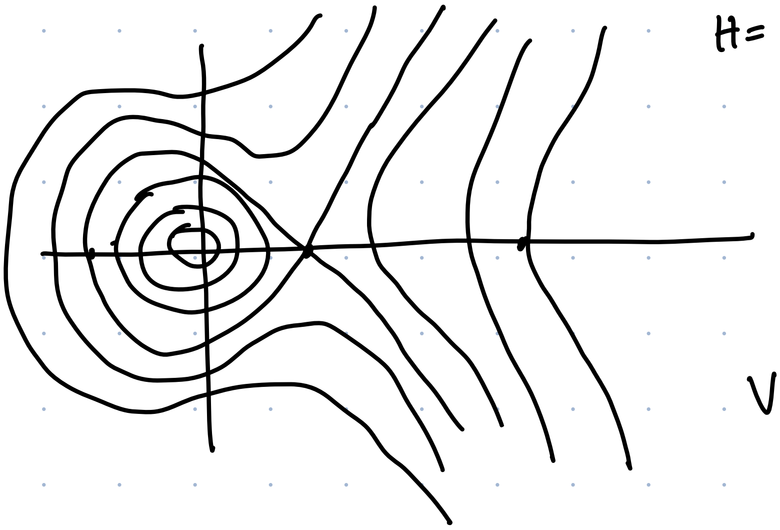
$g = g_{Euc} = dx \otimes dx$

V = polynomial in x



Eg. cubic

$$H = \frac{1}{2} p^2 + V$$



V cubic $\Rightarrow$ family of elliptic curves

$$T = \text{period} = \oint_{\text{trajectory}} \frac{dx}{\sqrt{2E - V}}$$

$\nwarrow$ elliptic integral

if $\deg V > 3$ "hyperelliptic integral"

$\dim X = 2$ $\qquad X = \mathbb{R}^2 \quad x^1, x^2$

$M = T^*X = \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$ 4-dimensional phase space

w/ coords p_1, p_2, x^1, x^2

$H = H(p_1, p_2, x^1, x^2)$

$\omega = dp_1 \wedge dx^1 + dp_2 \wedge dx^2$

$\underset{\text{e.g.}}{=} \frac{1}{2}\left(p_1^2 + p_2^2 + (x^1)^2 + (x^2)^2\right)$

$(x^1, x^2$ are any coords on $S^2)$

Level set: $H^{-1}(E) \subset T^*X$ $\qquad E$ const.

$$\overset{H^{-1}(E)}{\parallel} \quad \{(x^1, x^2, p_1, p_2) : (x^1)^2 + (x^2)^2 + p_1^2 + p_2^2 = 2E\}$$

$S^3 \subset \mathbb{R}^4$ $\qquad$ 3d sphere.

$\Rightarrow$ trajectory lies on $S^3(\sqrt{2E})$

3-sphere of radius $\sqrt{2E}$