## A tour of the most important Hamiltonians

position, momentum

i.e. functions on $M = T^*Q$ (coords $(x^i, p_j)$)

Organization by degree in vector space directions ($p_i$)

<u>Simplest (degree 0)</u>: functions which are constant in cotangent dir.

i.e. $f = V(x^1, \dots, x^n)$ (not a $f^n$ of $p_i$).

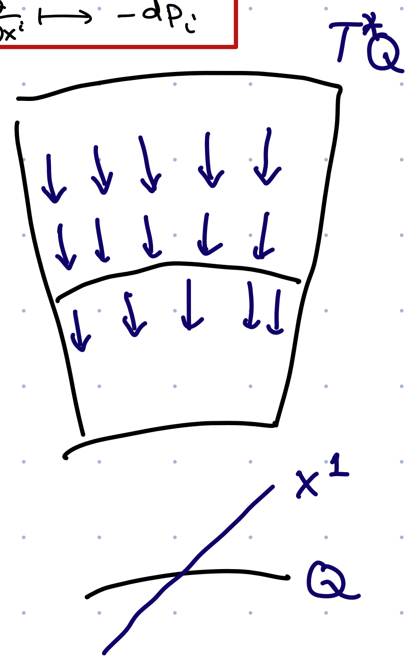
i.e. $f$ is pulled back from $V \in C^\infty(Q, \mathbb{R})$

$$\begin{tikzcd} M = T^*Q \arrow[d, "\pi"] & \\ Q \arrow[r, "V"] & \mathbb{R} \end{tikzcd}$$

$\pi^* V = V \circ \pi : M \to \mathbb{R}$

$$\boxed{\begin{aligned} \omega &= dp_i \wedge dx^i \\ \tfrac{\partial}{\partial p_i} &\mapsto dx^i \\ \tfrac{\partial}{\partial x^i} &\mapsto -dp_i \end{aligned}}$$

e.g. ① $V = x^1$




$$X_V = -\omega^{-1} dV = -\omega^{-1} dx^1$$

$$= -\frac{\partial}{\partial p_1}$$

(notice level sets of Ham. are preserved)

② $V = (x^1)^2 + (x^2)^2$

Next: homogeneous degree 1 in momenta:

$$f = V^1 p_1 + \cdots + V^n p_n \qquad V^i = V^i(x^1, \ldots, x^n) \text{ smooth } f^n \text{ of } x.$$

$$X_f = -\omega^{-1}\left(df = p_i(dV^i) + V^i dp_i\right)$$

$$= -\omega^{-1}\left(p_i \underbrace{dV^i} + V^i dp_i\right)$$

$$\left(dV^i = \frac{\partial V^i}{\partial x^j} dx^j\right)$$

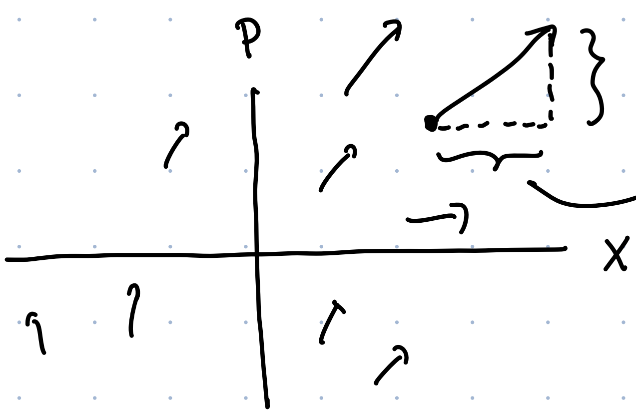
$$= -\omega^{-1}\left(p_i \frac{\partial V^i}{\partial x^j} dx^j + V^i dp_i\right)$$

$$= -p_i \frac{\partial V^i}{\partial x^j} \frac{\partial}{\partial p_j} + V^i \frac{\partial}{\partial x^i}$$

this generates Ham. flow:

$$\frac{d}{dt} x^i = V^i(x^1, \ldots, x^n)$$

$$\frac{d}{dt} p_j = -p_i \frac{\partial V^i}{\partial x^j}$$

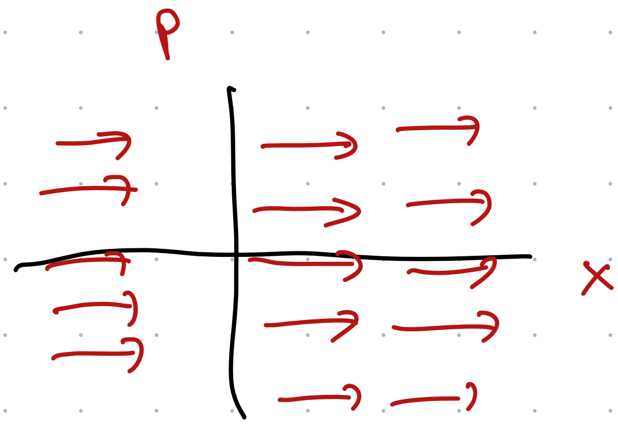



phase portrait

eg.: $f = p_1$  $X_f = \frac{\partial}{\partial x^1}$  motion at a constant speed in $x^1$ dir

$v^1 = 1$

$v^k = 0$ else




"momentum $f^n$ generates spatial translations"

flow $\varphi_t^{X_f} : (x, p) \longmapsto (x + (t, 0, \dots, 0), p)$

translation by $t$ in $x^1$ direction.

## Special feature of deg 1 Hamiltonians:

Since $f \in C^\infty(T^*Q)$ is linear on $T^*_x Q$ fibres, it defines a tangent vector at every pt in $Q$

i.e. Hamiltonians of deg 1 $= \mathfrak{X}(Q)$ vector fields on $Q$

$$f = V^1 p_1 + \cdots + V^n p_n$$

corresponds to vectorfield $\boxed{V = V^i \frac{\partial}{\partial x^i}}$

**Prop:** Ham. flow of $f_V = V^i p_i$ coincides with the flow on $T^*Q$ induced by the flow of $V$ on $Q$.

<u>So far:</u> - deg 0 $\quad C^\infty(Q)$

- deg 1 $\quad \mathfrak{X}(Q) = \Gamma(Q, TQ)$

vector fields = section of tangent bundle of $Q$

Most important case: deg 2

$$f = V^{ij} \, p_i p_j$$

$V^{ij}$ may be assumed to be symmetric $i,j$.

$$V^{ij} \in C^\infty(Q),$$

$\frac{n(n+1)}{2}$ smooth $f^{ns}$ of $x^1, \dots, x^n$

<u>e.g.:</u> Suppose we are given a Riemannian metric $g$ on $Q$

$$g = g_{ij}(x^1, \dots, x^n) \, dx^i \otimes dx^j$$

symm. $n \times n$ of $f^{ns}$ nondeg + pos. def.

$$g : TQ \xrightarrow{\cong} T^*Q$$
$$v \longmapsto g(v,-)$$

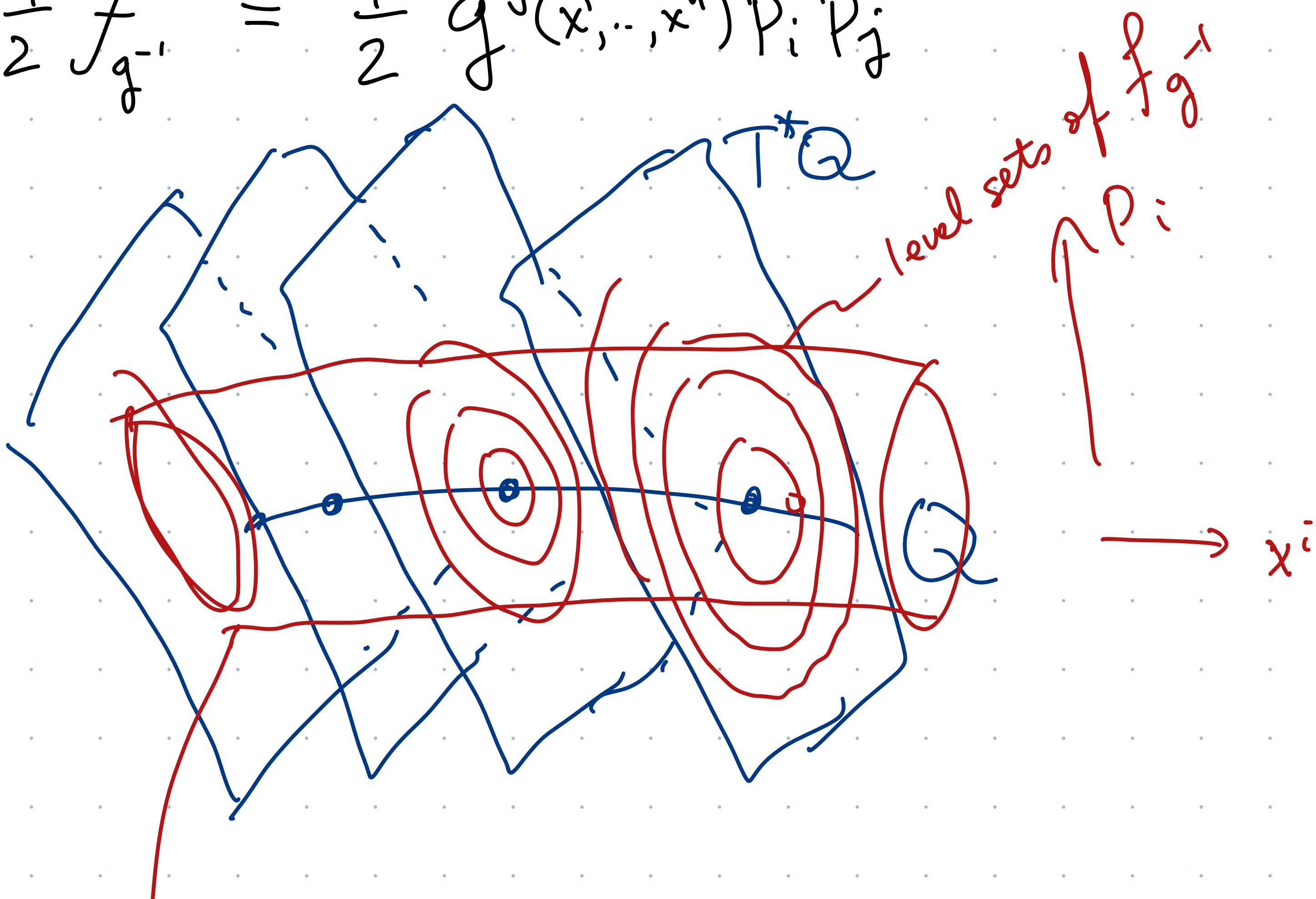
$$g^{-1} : T^*Q \xrightarrow{\cong} TQ.$$

$$g^{-1} = g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$$

$g^{ij}$ inverse matrix of $g_{ij}$

defines a quadratic $f^n$

$$\frac{1}{2} f_{g^{-1}} = \frac{1}{2} g^{ij}(x^1, \dots, x^n)\, p_i p_j$$

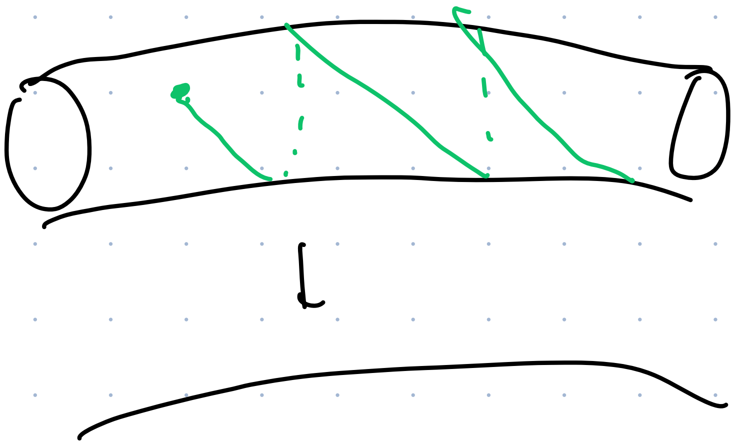


cylindrical level sets of $f_{g^{-1}}$

$=$ cosphere bundles inside $T^*Q$

What is the Ham. flow of $f_{g^{-1}}$

(it must preserve cospheres!)



norm of momentum must be preserved by flow

$\frac{1}{2} f_{g^{-1}} =:$ "The Kinetic energy of a particle on $Q$."

c.f. $K.E. = \frac{1}{2} m v^2 = \frac{1}{2} m^{-1} p^2$

$p = mv$

$\boxed{g \sim m}$

## Ham. flow of $\frac{1}{2} f_{g^{-1}}$:

$$H = \frac{1}{2} g^{ij} p_i p_j$$

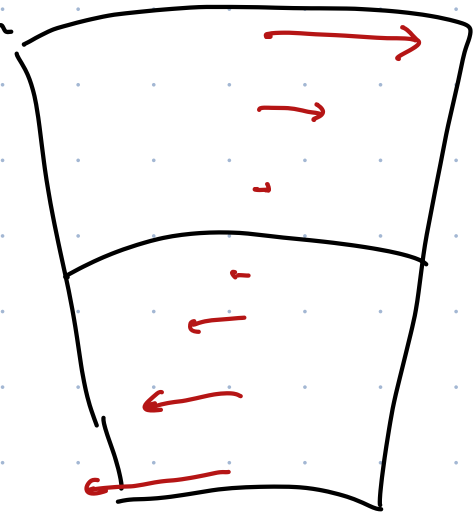
$$dH = \frac{1}{2} d(g^{ij}) p_i p_j + g^{ij} p_i \, dp_j$$

$$= \frac{1}{2} \frac{\partial g^{ij}}{\partial x^k} p_i p_j \, dx^k + g^{ij} p_i \, dp_j$$

$$-\omega^{-1} dH = -\frac{1}{2} \frac{\partial g^{ij}}{\partial x^k} p_i p_j \frac{\partial}{\partial p_k} + g^{ij} p_i \frac{\partial}{\partial x^j}$$

$$\boxed{\begin{aligned} \dot{x}^j &= g^{ij} p_i \\ \dot{p}_k &= -\frac{1}{2} \partial_k g^{ij} p_i p_j \end{aligned}}$$

geodesic equation



Like any flow of v. field, this is a system of nonlinear ODE, which (like Ricatti eqn) may have finite-time blowup. If flow exists for all time, we say $(Q, g)$ is a COMPLETE Riemannian mfld.

The sol$^{ns}$ $(x(t), p(t))$ determine

parametrized curves

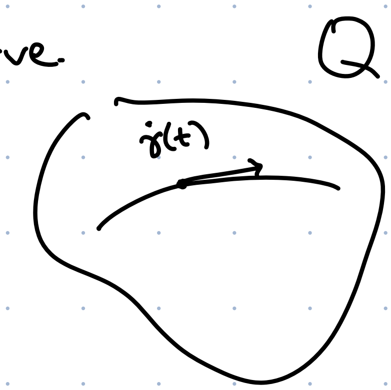
$x(t)$ in $Q$ ;

These curves are called <u>GEODESICS</u>!

(This approach to study of Geodesics is called Hamilton - Jacobi theory)

<u>Note:</u> Geodesic eq$^n$ is usually written as follows:

$\gamma : \mathbb{R} \to (Q, g)$ param. curve




$\dot\gamma \in \Gamma(\mathbb{R}, \gamma^* TQ)$

$$\boxed{\nabla_{\frac{\partial}{\partial t}} \dot\gamma = 0}$$ acceleration $= 0$ (Forces $= 0$)

$\uparrow$

$$\gamma(t) = \left(x^1(t), \dots, x^n(t)\right)$$

Geodesic eq$^n$:
$$\boxed{\ddot{x}^i + \Gamma^i_{jk}\, \dot{x}^j \dot{x}^k = 0}$$

$\Gamma^i_{jk}$ Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \left( \frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right.$$