<u>Hamiltonian Mechanics</u>: configuration space X

w/ coordinates $(q^1,\dots,q^n)$

a covector at $x \in X$ has form

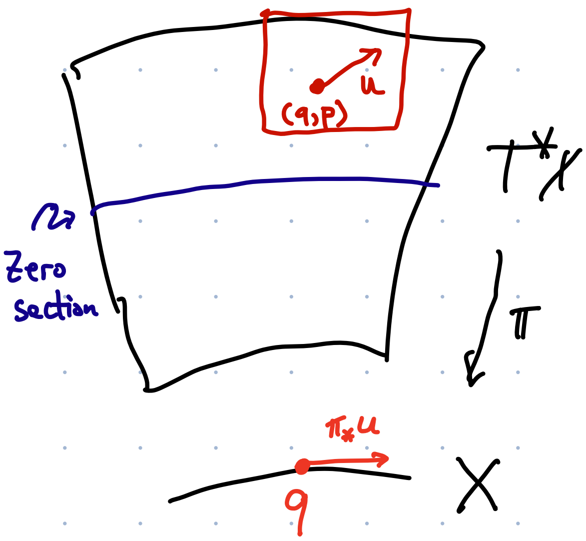
$$p_1 dx^1 + \dots + p_n dx^n$$

Phase space.

$\Rightarrow$ $(x^1,\dots,x^n, p_1,\dots,p_n)$ coordinates on $\boxed{T^*X = M}$

<u>Geometry of phase space</u>:

The canonical 1-form $\Theta \in \Omega^1(M)$



given a vector $u \in T_{(q,p)} M$

derivative of π

$$\boxed{\Theta(u) = p(\underbrace{\pi_* u}_{\text{vector at } q})}$$

covector at q.

<u>in coords</u>: $u = u^i \frac{\partial}{\partial q^i} + u_i \frac{\partial}{\partial p_i}$

$$p = p_i dq^i$$

$$\boxed{\Theta = p_i dq^i}$$

$\pi(p,q) = q$ $\quad \pi_*: \frac{\partial}{\partial p} \longmapsto 0$, $\quad \frac{\partial}{\partial q} \longmapsto \frac{\partial}{\partial q}$

$$\Theta = \underset{\substack{\shortparallel \\ p_i}}{a_i} dq^i + \underset{\substack{\shortparallel \\ 0}}{b^j} dp_j$$

$$\begin{cases} \Theta(u) = p_i dq^i \Big(\underbrace{\pi_* \Big(u^j \frac{\partial}{\partial q^j} + u_j \frac{\partial}{\partial p_j} \Big)}_{u^j \frac{\partial}{\partial q^j}} \Big) \\ \quad\;\; = p_i u^i \end{cases}$$

since $\Theta \in \Omega^1(M)$ can differentiate,

obtain 2-form Symplectic form on M

$$\omega = d\Theta = d(p_i dq^i) = dp_i \wedge dq^i$$

$$= dp_1 \wedge dq^1 + \cdots + dp_n \wedge dq^n$$

[indep. of orig. choice of coords q^i

i.e. $\tilde{q}^i = \tilde{q}^i(q^1, \ldots, q^n)$ given coord. change

$\tilde{p}_i$ canonically conj. to $\tilde{q}^i$

and in $(\tilde{p}_i, \tilde{q}^i)$ coords for M

$\omega = d\tilde{p}_i \wedge d\tilde{q}^i$]

view ω as a skew-symm. bilinear form

on $T_{(p,q)}M$

$u_1, u_2 \in T_{(p,q)}M$ $\quad \omega(u_1, u_2) = -\omega(u_2, u_1)$

has matrix in basis $\frac{\partial}{\partial q}, \frac{\partial}{\partial p}$ given by

$$
{}_\beta[\omega]_\beta = \left(\begin{array}{c|c} 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline \begin{matrix} -1 & & \\ & -1 & \\ & & \ddots \\ & & & -1 \end{matrix} & 0 \end{array}\right)
\qquad
\beta = \begin{pmatrix} \frac{\partial}{\partial p_1} \\ \vdots \\ \frac{\partial}{\partial p_n} \\ \frac{\partial}{\partial q^1} \\ \vdots \\ \frac{\partial}{\partial q^n} \end{pmatrix}
$$

$\longleftarrow n \longrightarrow \quad \longleftarrow n \longrightarrow$

order basis $\downarrow$

<u>key properties of ω:</u>

1. skew-symmetric $\quad \omega \in \Omega^2(M)$
2. closed $\quad d\omega = 0$
3. ω nondegenerate

$$
\omega : TM \xrightarrow{\cong} T^*M \qquad \text{isomorphism.}
$$

$$
u \longmapsto i_u \omega = \omega(u, -)
$$

$$
\frac{\partial}{\partial p_i} \longmapsto i_{\frac{\partial}{\partial p_i}} (dp_j \wedge dq^j) = \left(i_{\frac{\partial}{\partial p_i}} dp_j\right) dq^j + (-1)\, dp_j \wedge \cancel{\left(i_{\frac{\partial}{\partial p_i}} dq^j\right)}
$$

$$
= dq^i
$$

$$
\frac{\partial}{\partial q_i} \longmapsto -dp_i
$$

Main Consequence: we obtain a mechanism by which $f \in C^\infty(M, \mathbb{R})$ is converted to a vector field $X_f \in \mathfrak{X}(M)$

Def: The Hamiltonian vector field of f is given by:

$$C^\infty(M,\mathbb{R}) \longrightarrow \mathfrak{X}(M)$$

$$f \xrightarrow{d} df \xmapsto{-\omega^{-1}} -\omega^{-1} df = X_f$$

$$df \in \Omega^1(M), \qquad X_f \in \mathfrak{X}(M)$$

" Hamiltonian vector field associated to f. "

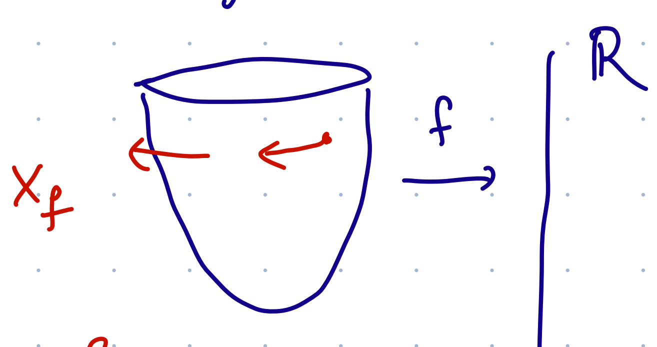
Main properties of Ham. v.f are

Prop: The Ham. v.f. of f is st.

① $X_f(f) = df(X_f) = df\left(-\omega^{-1}(df)\right)$

$= -\omega^{-1}(df, df)$

$= 0 \quad \omega$ is skew

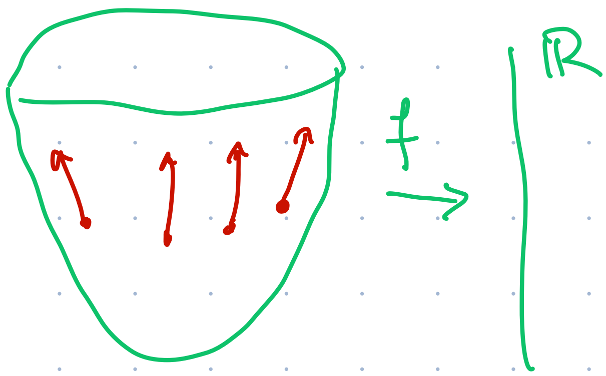


f is conserved by the flow of X_f

Warning: quite different from grad f

On a Riem. mfld $\nabla f = g^{-1}(df)$

$$(\nabla f)(f) = df(g^{-1}df) = g^{-1}(df, df)$$
$$= \|df\|^2$$



② $$L_{X_f}\omega = (d\,i_{X_f} + \cancel{i_{X_f} d})\,\omega$$

$d\omega = 0$

$$= d\, i_{X_f}\omega$$
$$= d\left(\omega(-\omega^{-1}df, -)\right)$$
$$= d(-df) = 0$$

ω is preserved!

Prop: The hamiltonian v.f. of a f^n f is a symmetry of (M, ω, f).

Def: A Hamiltonian system (M, ω, H) consists of

1. A manifold M $\left(\begin{array}{c}\text{for us}\\ M = T^*X\end{array}\right)$
2. ω symplectic form on M
   nondegenerate closed 2-form
3. A function H
   "The Hamiltonian"

Theorem: Near any point p of a symplectic mfld, it is possible to find "Darboux" coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$

s.t. $\omega = dp_i \wedge dq^i$

Note: There are degenerate versions of symp. forms

1. Poisson structure (ω^{-1} becomes degenerate)
2. quasi-symplectic (ω becomes deg.

(folded symplectic forms . . .

relevant to systems w/ constraints. or singularities.

(Also there are higher degree variants "Multi-symplectic" structures $\omega \in \Omega^{k \geq 2}(M)$)

Def: The Poisson bracket assoc. to ω is:

$$\{\ ,\ \} : C^\infty(M,\mathbb{R}) \times C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R}).$$

$$\{f,g\} = X_g(f) = df(-\omega^{-1} dg)$$
$$= \omega^{-1}(df, dg).$$
$$= -\omega^{-1}(dg, df)$$
$$= -X_f(g)$$

Prop: Poisson bracket satisfies Leibniz rule:

$$\{f_1 f_2, g\} = X_g(f_1 f_2) = (X_g(f_1))f_2 + f_1(X_g(f_2))$$

$$= \{f_1, g\} f_2 + f_1 \{f_2, g\}$$

Def: A Poisson algebra is a commutative $\mathbb{R}$-algebra A together w/ a bracket $\{ , \} : A \times A \to A$

s.t. (1) skew
(2) Leibniz $\{f, gh\} = \{f,g\}h + g\{f,h\}$
(3) Jacobi identity

$$\{\{f,g\},h\} = \{f,\{g,h\}\} - \{g,\{f,h\}\}.$$

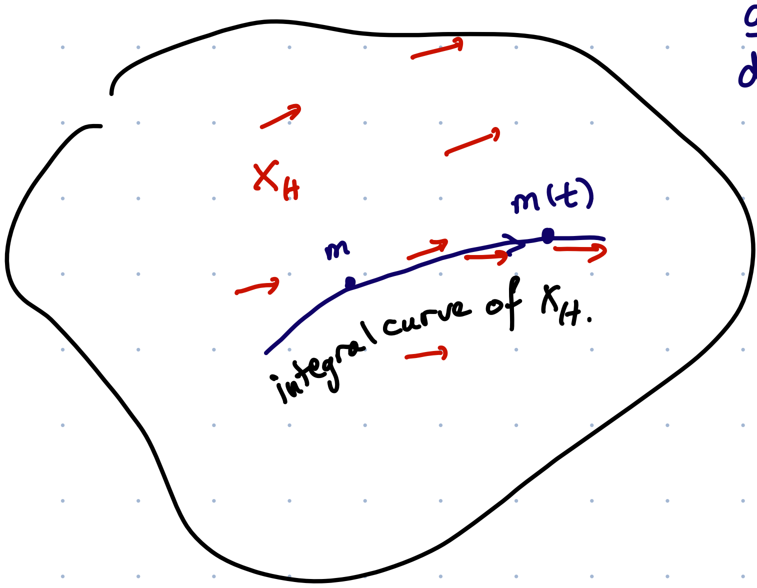
Verify that our Poisson bracket satisfies Jacobi identity.

Poisson bracket is helpful for describing Hamiltonian flow

$$C^\infty(M) \ni H \xmapsto{-\omega^{-1}d} X_H \quad \text{generates a flow}$$

$$\varphi_t^{X_H} : (\mathbb{R}, t) \times M \longrightarrow M$$
$$(t, m) \longmapsto \varphi_t^{X_H}(m) = m(t)$$

$$\frac{d}{dt} m(t) = X_H(m(t))$$



Alternatively we can flow a f^n $\quad f \in C^\infty(M, \mathbb{R})$

i.e. $$f(t) = (\varphi_t^{X_H})^* f$$

$$\boxed{\frac{d}{dt} f(t) = X_H(f)} = \{f, H\}$$

Equation of motion for a f^n f dragged along Ham. flow is

$$\frac{d}{dt} f = \{f, H\}$$ Hamilton's equation

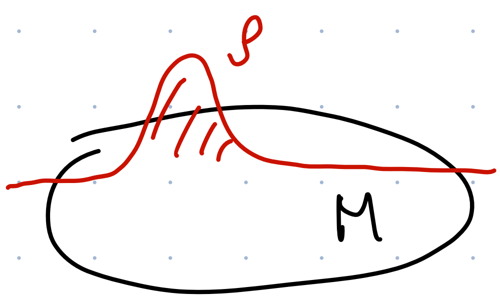
<u>Note:</u> In any Poisson algebra $(A, \{,\})$ can write similar eq^n: Fix $H \in A$

Then we can write a diff. eqn for a path $f(t)$ of elts in A:

$$\boxed{\dot{f} = \{f, H\}}$$

If we know init. state ^{m} exactly $\Rightarrow$ find evolution $\dot{m}(t) = X_H(m(t))$

If we only have probabilistic inform. about state



$$\dot{\rho} = \{\rho, H\}$$