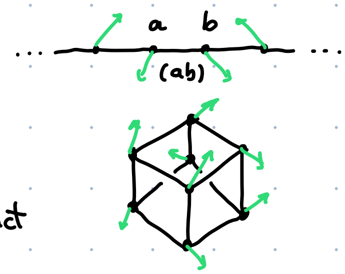
Spin chains cont'd



X finite (discrete) set of vertices

E edges (unordered pairs $\{a,b\}$ of distinct vertices)

<u>Quantum system:</u> $(\mathcal{H}, H \in \mathrm{Obs}(\mathcal{H}))$

$$\mathcal{H} = \bigotimes_{x \in X} V_x$$

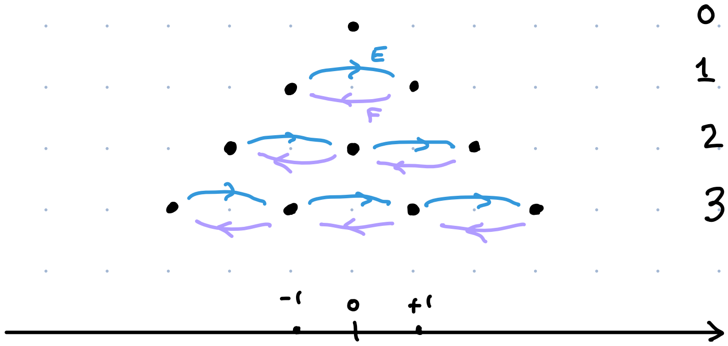
each vertex $x \in X$ assigned unitary irrep of $SU(2)$ (or G)

$$H = \sum_{(ab) \in E} H^{(ab)} \otimes \mathbb{1}_{E^\perp}, \qquad H^{(ab)} \in \mathrm{Obs}(V_a \otimes V_b)$$

<u>Idea:</u> use classific. of irreps of $SU(2)$ to provide example (Heisenberg spin chain: XXX model)

<u>Recall:</u> Irreps of $SU(2)$ classified

spin	highest wt	dim
0	0	1
$\frac{1}{2}$	1	2
1	2	3
$\frac{3}{2}$	3	4



weight of H

<u>Alternative approach to constructing irreps</u>

$SU(2) \hookrightarrow SL_2\mathbb{C}$ act on $\mathbb{C}^2$ (standard rep'n)

view $\mathbb{C}^2$ as a space with functions $\mathbb{C}[z_1, z_2]$

i.e. if $f \in \mathbb{C}[z_1, z_2]$ then $g \in SL_2\mathbb{C}$

acts :

$$(g \cdot f)(x \in \mathbb{C}^2) = f(g^T x)$$

transpose (or inverse) to obtain correct action law.

$$\cancel{\begin{aligned} (g_1 g_2) \cdot f &= f \circ (g_1 g_2) \\ &= (f \circ g_1) \circ g_2 \\ &= g_2 \cdot (g_1 \cdot f) \end{aligned}}$$

This has corresponding Lie algebra action

$$X \in \mathfrak{sl}_2\mathbb{C} = 2 \times 2 \text{ matrices w/ } \mathrm{Tr} = 0$$

$$X \cdot f = \frac{d}{dt}\Big|_{t=0} e^{tX} \cdot f = \frac{d}{dt}\Big|_{t=0} f \circ e^{tX^T} = f \circ X^T$$

$$= f \circ X^T \cancel{e^{tX^T}}\Big|_{t=0}$$

$$\text{basis of } \mathfrak{sl}_2\mathbb{C} = \langle \underset{H}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}, \underset{E}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{F}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} \rangle$$

$$H \cdot f = f \circ H^T = f \circ H$$

If $f = z_1$: $H \cdot z_1 = z_1 \circ (H^T = H)$: $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto z_1 \begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} = z_1$

$$H \cdot z_2 = -z_2$$

The Lie group action preserves multiplication:

$$g\cdot(f_1 f_2) = (g\cdot f_1)\cdot(g\cdot f_2)$$

The Lie alg. action is a <u>derivation</u> (Leibniz)

$$X\cdot(f_1 f_2) = (Xf_1)f_2 + f_1(Xf_2)$$

H

$\Rightarrow$ H acts as $z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2}$

$$H\left(z_1^p z_2^q\right) = (p-q)\, z_1^p z_2^q$$

$\Rightarrow$ monomials $z_1^p z_2^q$ are weight vectors = eigenvect of H

E

$$\left(E\cdot z_1 = z_1 \circ E^T\right)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$= z_1 \circ \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ z_1 \end{pmatrix}\right) = 0$$

$$\left((E\cdot z_2) = z_2 \circ E_1^T\right)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$= z_2 \circ \begin{pmatrix} 0 \\ z_1 \end{pmatrix} = z_1$$

$$E = z_1\frac{\partial}{\partial z_2}$$

Ⓕ $\boxed{F = z_2 \frac{\partial}{\partial z_1}}$

(H, E, F) acts by $(z_1\partial_1 - z_2\partial_2,\ z_1\partial_2,\ z_2\partial_1)$

on $\mathbb{C}[z_1, z_2]$

<u>notice:</u> v.fields are <u>linear</u> $\Rightarrow$ degree is preserved!

$\Rightarrow$ rep'n decomposes:

$$\mathbb{C}[z_1,z_2] = \bigoplus_{k \geq 0} \underset{\text{poly. deg } k}{\text{homogeneous}}\ \mathbb{C}[z_1,z_2]^{(k)}$$

deg k summand has basis

$$\begin{tikzcd}
z_2^k \arrow[r, bend left, "E"] & \cdots \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & z_1^{k-2}z_2^2 \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & z_1^{k-1}z_2 \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & z_1^k \arrow[l, bend left, "F"]
\end{tikzcd}$$

dim $k+1$

wt $\quad -k \quad \ldots \quad k-4 \quad k-2 \quad k$

To make $\mathbb{C}[z_1, z_2]$ into a Hilbert space

$$\left(\begin{array}{l} \text{recall for } L^2(\mathbb{R}, \mathbb{C}) \quad \langle f, g \rangle = \int_{-\infty}^{\infty} \bar{f} g \, dx \\ \\ \text{note polynomials are never in } L^2 \, ! \end{array} \right)$$

use "weighted L^2" inner product:

$$\langle f, g \rangle = \frac{1}{\pi^2} \int \bar{f} g \, e^{-|z_1|^2 - |z_2|^2} \, dx_1 dy_1 dx_2 dy_2$$

$$\|z_1\|^2 = 1 = \|z_2\|^2$$

$z_1 = x_1 + i y_1$
$z_2 = x_2 + i y_2$

(to be precise, L^2 is the <u>completion</u> of $\mathbb{C}[z_1, z_2]$ in this norm)

Last ingredient from Rep. Thy:

Clebsch - Gordan decomposition

<u>Ex.</u>: suppose V^2, V^3 are irreps of $SU(2)$ w/ highest wt 2, 3.

	-3	-2	-1	0	1	2	3
V_2		•		•		•	
V_3	•		•		•		•

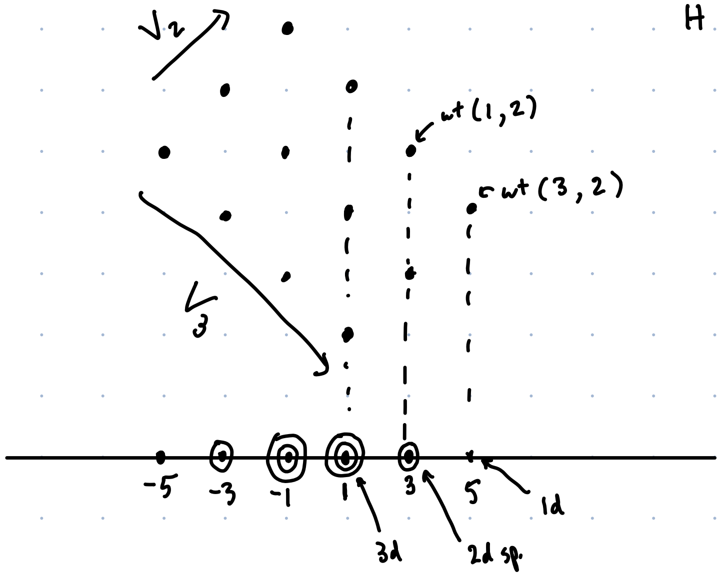
Q: how does $(\overset{\text{dim } 3}{V_2} \otimes \overset{\text{dim } 4}{V_3})^{\text{dim } 3 \cdot 4 = 12}$ decompose?

<u>Strategy</u>: use highest weights to decompose

H acts on $V_2 \otimes V_3$ by $H \otimes 1 + 1 \otimes H$

highest weight possible is $2 + 3 = 5$

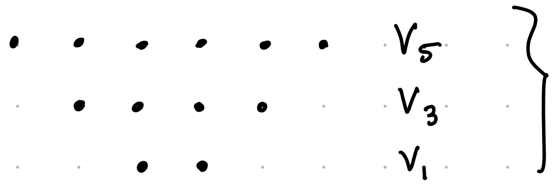
$$\Rightarrow \quad V_2 \otimes V_3 = \overset{6}{V_5} \oplus \cdots$$



$H = H \otimes 1 + 1 \otimes H$

physicists use $H/2$

"$\frac{3}{2} \otimes 1 = \frac{5}{2} \oplus \frac{3}{2} \oplus \frac{1}{2}$"



$$V_3 \otimes V_2 = V_5 \oplus V_3 \oplus V_1$$

__Clebsch-Gordan Theorem:__ $\boxed{V_m \otimes V_n = V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{|m-n|}}$

__Ex.:__ $V = V_1 = \mathbb{C}^2$

"$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$"

$$V_1 \otimes V_1 = \overset{3d}{V_2} \oplus \overset{1d}{V_0}$$

$$V_1 = \langle \underset{\text{highest wt}}{\overset{\binom{1}{0}}{|+\rangle}}, \overset{\binom{0}{1}}{|-\rangle} \rangle$$

$$V_1 \otimes V_1 = \langle \underset{\substack{\nearrow \\ \text{highest wt}}}{\overset{|+\rangle\otimes|+\rangle}{\overset{\shortparallel}{|++\rangle}}}, |--\rangle, |+-\rangle, |-+\rangle \rangle$$

$$F|++\rangle = |-+\rangle + |+-\rangle$$

$$F(|-+\rangle + |+-\rangle) = |--\rangle + |--\rangle = 2|--\rangle$$

$$\underset{\text{“triplet”}}{V_2} = \langle |++\rangle, \ |+-\rangle + |-+\rangle, \ |--\rangle \rangle$$ irred. subrep'n of $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$\underset{\text{“singlet”}}{V_0} = V_2^{\perp} = \langle |+-\rangle - |-+\rangle \rangle$$ irred subrep'n

$$V = \mathbb{C}^2 \qquad V \otimes V = \underbrace{\mathrm{Sym}^2 V}_{3} \oplus \underbrace{\Lambda^2 V}_{1}$$

<u>Schur's Lemma</u>: If (V, π) is a complex irred rep'n of G, then the only linear map $M: V \to V$ preserving rep'n (i.e. intertwiner) is

$$M = \lambda \mathbb{1} \qquad \lambda \in \mathbb{C}.$$

<u>Proof</u>: Any such M <u>has</u> eigenvalue $\lambda \in \mathbb{C}$ with eigensp $W = \ker(M - \lambda \mathbb{1})$

W is a sub-rep'n $\Rightarrow W = V \Rightarrow M = \lambda \mathbb{1}$

$$\left(\begin{array}{l} Mv = \lambda v \\ M(\pi(g) v) = \pi(g) M v = \lambda (\pi(g) v) \\ \Rightarrow \quad \pi(g) : W \to W. \end{array} \right)$$

Cor: If V, W are non-isomorphic irreps

then $\nexists$ intertwiner! $\quad V \xrightarrow{f} W$

(ker, Im of f would be sub-rep'ns)

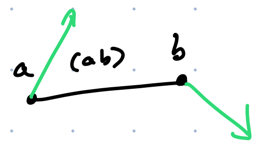
$\Rightarrow$ if $V = V_1 \oplus V_2 \oplus V_3$ irred.

$W = W_1 \oplus W_2 \oplus W_3$ irred.

intertwiners = mult. of $\mathbb{1}$ between equiv. irreps.

Spin Chain: $\quad X \ni x$ vertex $\xrightarrow{\text{assign}} V^1 = \mathbb{C}^2$

$$\mathcal{H} = \bigotimes_{x \in X} V^1_x$$



for each edge (ab) $\qquad H^{(ab)} \in \mathrm{Obs}\left(V^1_a \otimes V^1_b\right)$

i.e. need map $\quad V^1_a \otimes V^1_b \xrightarrow{H^{(ab)}} V^1_a \otimes V^1_b$

$$\begin{array}{ccc} V^1_a \otimes V^1_b & \xrightarrow{H^{(ab)}} & V^1_a \otimes V^1_b \\ \| & & \| \\ \underset{\substack{\uparrow \\ \text{triplet}}}{S^2_{ab}V} \oplus \underset{\substack{\uparrow \\ \text{singlet}}}{\Lambda^2_{ab}V} & & S^2_{ab}V \oplus \Lambda^2_{ab}V \end{array}$$

$$H^{(ab)}_{\lambda,\mu} = \lambda \mathbb{1}_{S^2V} \oplus \mu \mathbb{1}_{\Lambda^2 V}$$

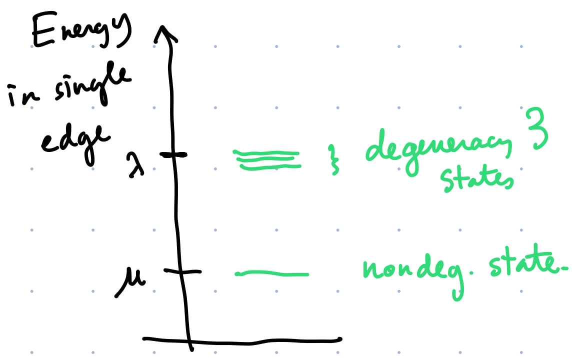
$$\begin{array}{cccc} & 3 & 1 & \\ 3 & \lambda \mathbb{1} & 0 & S^2_{ab}V \\ & & & \oplus \\ 1 & 0 & \mu \mathbb{1} & \Lambda^2_{ab}V \end{array}$$

energy of triplet states λ

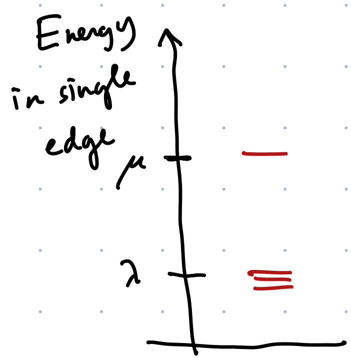
" " singlet μ

$\lambda, \mu \in \mathbb{R}$.
self-adjoint

$\lambda > \mu$



$\lambda < \mu$



Anti-Ferromagnetic

ground state
unique!

$|+-\rangle - |-+\rangle$

entangled!

Ferromagnetic system

ground state (lowest-energy)

is 3-dim^{l} space

contains

$|++\rangle$ $|--\rangle$

$(a|+\rangle + b|-\rangle)^2$

Spin chain $\quad H = \sum_{ab \in E} H^{(ab)}_{\lambda,\mu} \otimes \mathbb{1}_{E^\perp}$

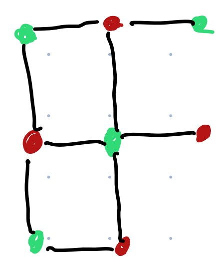
fix λ, μ for all

<u>Ferromagnetic case</u>: 3d space of ground states

including

$|{+}{+}{+}{+}{+} \cdots {+}\rangle$, $|{-}{-}{-}{-}{-}{-}\rangle$

<u>Anti-ferromagnetic</u>:



<u>Thm (Marshall-Lieb-Mattis)</u>

X connected bipartite graph

vertices A, B (two colors) $\quad |A| = |B|$

Then $\exists!$ ground state in Antiferro case,

$$\Omega = \sum_{\substack{\sigma \text{ balanced} \\ \text{word in } \pm}} \Big(\prod_{b \in B} \sigma_b \Big) c_\sigma \overbrace{|\sigma\rangle}^{|{+}{+}{-}{+}{-}{-}{+}{+}\rangle}$$

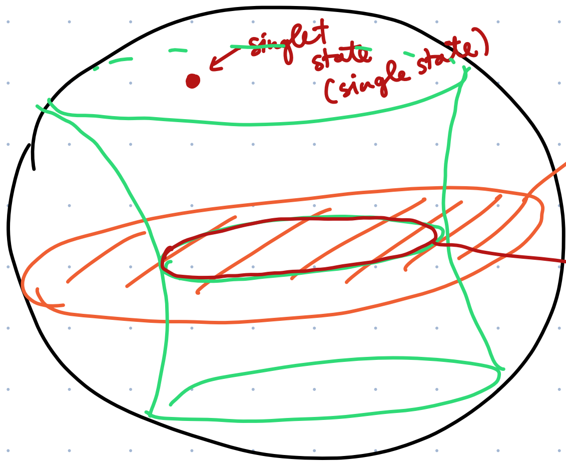
$c_\sigma > 0$

Hal Tasaki textbook on many-body systems
Youtube channel.

$V = \mathbb{C}^2$ $V \otimes V = \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$

$\mathbb{C}^2 \otimes \mathbb{C}^2 \underset{SU(2)}{=} \mathbb{C}^3 \oplus \mathbb{C}$ $\mathbb{P}(\mathbb{C}^4) = \mathbb{CP}^3$

Decomposables form Quadric hypersurf. in $\mathbb{CP}^3$



triplet states $\cong \mathbb{CP}^2$

decomposable triplet states form a quadric in $\mathbb{CP}^2$ $\cong \mathbb{CP}^1$

$\Downarrow$

$(a|+\rangle + b|-\rangle)^2$