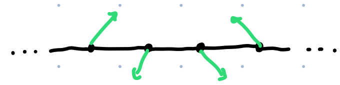
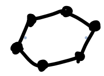
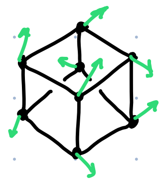
Joint quantum systems: Spin chains







classical phase space: $\underbrace{S^2 \times S^2 \times \cdots \times S^2}_{\text{number of vertices}}$

graph: vertices X (finite set)

edges $E \subset$ unordered distinct pairs of vertices

Phase space: $(S^2)^X$ = maps $X \to S^2$

Quantum system: each vertex x endowed w/ finite-dim^{l} Hilbert space V_x

$\Rightarrow$ Hilbert space of joint system $= \mathcal{H} = \bigotimes_{x \in X} V_x$

Hamiltonian is chosen to reflect structure of graph,

i.e. if x_1, x_2 are joined by edge,

then we want V_{x_1}, V_{x_2} to interact —

we want an interaction Hamiltonian

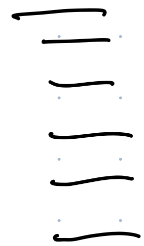
$$H_{x_1 x_2} \in \mathrm{Obs}\,(V_{x_1} \otimes V_{x_2})$$

The total Hamiltonian is then

$$H = \sum_{e \in E} \underbrace{H_e}_{\substack{\text{interaction between} \\ \text{vertices in } e}} \otimes \overbrace{\mathbb{1}_{e^\perp}}^{\text{identity on complement of } e.}$$

<u>Main problem:</u> figure out eigenspace decomp.

$\Rightarrow$ energy levels λ



$\Rightarrow$ eigenstates ψ_λ (the lowest being the "ground state")

(diagonalize H $\Rightarrow$ evolution in time $e^{i\lambda t}\psi_\lambda$)

<u>Difficult:</u> even for $V_{x_i} = \mathbb{C}^2$

$$\mathcal{H} = \bigotimes_{x_i \in V} V_{x_i} \text{ has dim } 2^n$$

(beyond current capab. $n > 30$)

main idea we will use to build these: representation theory of SU(2).

- each V_x will be a rep'n of SU(2)

helpful since su(2) provides us with self-adjoint ops $(\sigma_1, \sigma_2, \sigma_3) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$

Rep'n $\pi : SU(2) \longrightarrow GL(V_x)$

$$\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$$
$$\pi(e) = \mathbb{1}_{V_x}$$

$$\pi' : su(2) \longrightarrow L(V_x)$$
$$\sigma_i \longmapsto \pi'(\sigma_i) \in \text{Obs}(V_x)$$

- If V, W are representations of SU(2)

then a Homomorphism of rep'ns is a Linear map

$$V \xrightarrow{h} W \qquad \forall g \in SU(2)$$

which respects rep'n: $h \circ \pi_V(g) = \pi_W(g) \circ h$

$W \leftarrow V \leftarrow V \quad W \leftarrow W \leftarrow V$

Such morphisms are called Intertwiners

$\Rightarrow$ we can ask for the Interaction Hamiltonians

$$V_{x_1} \otimes V_{x_2} \longrightarrow V_{x_1} \otimes V_{x_2}$$

to respect representations

Such a choice results in a spin chain with $SU(2)$ symmetry

<u>Main example:</u> Heisenberg spin-$\frac{1}{2}$ chain

__Aside:__ Irreducible f.d. Rep'n of $SU(2)$.

Rep'ns of $su(2)$ $\Longleftrightarrow$ Rep'ns of $SU(2)$ by exp. map

$\pi_1(SU(2)) = \{1\}$.

$su(2) = so(3)$; Rep'ns of $SU(2)$ $\supset$ Rep'ns of $SO(3)$ proper subset.

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find $U(1) \subset SU(2)$ $\qquad \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} = e^{i\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$

Maximal torus (in general $U(1)^n$)

Infinitesimal gen. of $U(1) = i \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_3} \in su(2)$

If (V, π) is a rep'n of $SU(2)$ $\Longrightarrow$ is a rep'n of $U(1)$

$\Rightarrow$ $U(1)$ action on $\mathbb{C}$ vector space V

$$\Rightarrow \quad V = \bigoplus_{k \in \mathbb{Z}} V_k$$

weights/charges. Charge matrix

$\begin{pmatrix} k_1 & \\ & \ddots \end{pmatrix}$

$$\pi\begin{pmatrix}1 & \\ & -1\end{pmatrix} = \begin{pmatrix} k_2 & & \\ & \ddots & \\ & & k_n\end{pmatrix}$$

$$\pi(e^{i\theta}) = \exp\left(i\theta \begin{pmatrix} k_1 & & \\ & \ddots & \\ & & k_n\end{pmatrix}\right)$$

[each summand V_k is 1-dim^l irred rep'n of $U(1)$ $\quad e^{i\theta}\cdot v = e^{ik\theta} v$]

(note needede to complexify to get such decomp)

$\Rightarrow \quad \underbrace{su(2)\otimes\mathbb{C}}_{sl_2\mathbb{C}}$ acting on V

$$\sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = H$$

$$\tfrac{1}{2}(\sigma_1 + i\sigma_2) = \sigma_+ = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = E$$

$$\tfrac{1}{2}(\sigma_1 - i\sigma_2) = \sigma_- = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = F$$

$$\boxed{\langle H, E, F\rangle = sl_2\mathbb{C}}$$

"$sl_2\mathbb{C}$ triple"

$$\boxed{\begin{aligned} [H, E] &= 2E \\ [H, F] &= -2E \\ [E, F] &= H \end{aligned}}$$

$v \in V_k$ weight k

$(= \pi(H)v)$

$$Hv = kv$$

$$H(Ev) = EHv + 2Ev$$

$$= (k+2)Ev$$

E raises H-eigenval by 2

$$HFv = FHv - 2Fv$$

$$= (k-2)Fv$$

F lowers weight by 2.

rep'n

$$V = \quad \begin{array}{cccccc} V_{-2} & V_{-1} & V_0 & V_1 & V_2 & V_3 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

$$H \longrightarrow \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \cdots$$

E E E E

F F F

Argument for classification:

V finite-dim^{l} $\Longrightarrow$ $\exists$ highest weight vector

$v \in V_n$

i.e. $Ev = 0$

$$\begin{array}{c} n \\ \bullet \\ v \xmapsto{E} 0 \end{array}$$

apply F repeatedly

$$\begin{array}{cccc} \cdots & F^2 v & Fv & v \\ & V_{n-4} & V_{n-2} & V_n \end{array}$$

Claim: $\mathrm{Span}\left(F^j v,\ j \geq 0\right)$ is a rep'n i.e. H, E, F preserve the span.

$$EF^jv = EF(F^{j-1}v)$$
$$= (H+FE)F^{j-1}v$$
$$= (n-2(j-1))F^{j-1}v_n + F\underbrace{EF}_{(H+FE)}F^{j-2}v$$
$$= (n-2(j-1))F^{j-1}v + (n-2(j-2))F^{j-1}v + \cdots$$
$$= jn - 2(1+2+\cdots+j-1)F^{j-1}v$$
$$= j(n-j+1)F^{j-1}v \qquad \square$$

if j is first index st. $F^jv = 0$

then formula above $\Rightarrow$

$$0 = EF^jv = j(n-j+1)\underbrace{F^{j-1}v}_{\text{nonzero.}}$$

$$\Rightarrow \boxed{j = n+1}$$

$v \in V_n$

$n-2(n)$ ... $n-4$ $\quad n-2$ $\quad n$

rep'n has weights symmetric

$-n$

also π zero.

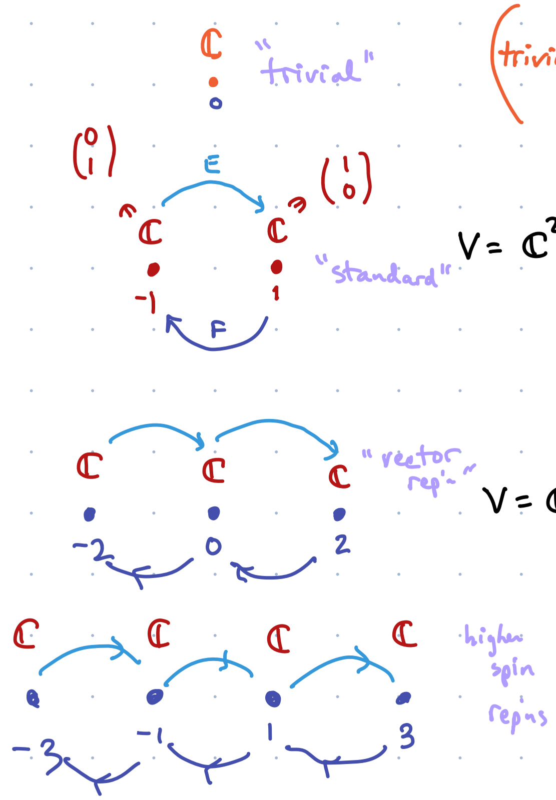
Ex.:

dim 1
spin 0

dim 2
spin $\frac{1}{2}$

dim 3
spin 1

dim 4



(trivial rep'n (1-d) — SU(2) acts by 1; $\pi(E), \pi(H), \pi(F) = 0$)

$V = \mathbb{C}^2$ $\quad H = \begin{pmatrix}1 & \\ & -1\end{pmatrix}$ $\quad E = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$

$V = \mathbb{C}^3$

$\pi(H) = \begin{pmatrix}2 & & \\ & 0 & \\ & & -2\end{pmatrix}$

$\pi(E) = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$

$\pi(F) = \begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$

Thm: F.dim. irreps of SU(2) have weights

$$-n, \; -n+2, \; \dots, \; n-2, \; n \qquad n \in \mathbb{Z}_{\geq 0}$$

each with multiplicity 1. classified.

note: $\frac{1}{2}\begin{pmatrix}1 & \\ & -1\end{pmatrix} = H$ alternative basis choice.

END

In general to study rep'ns of G

1) find maximal torus $U(1)^n \subset G$

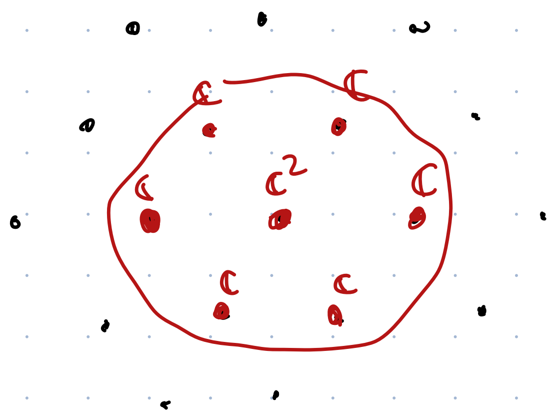
e.g. $SU(3)$ $\quad \begin{pmatrix} e^{i\theta_1} & & \\ & e^{i\theta_2} & \\ & & e^{-i(\theta_1+\theta_2)} \end{pmatrix}$

$U(1) \times U(1)$

$$\Rightarrow \underset{\text{rep'n}}{V} = \bigoplus_{(k_1,\dots,k_n)} V_{(k_1,\dots,k_n)}$$

common eigensp. of n generators of $U(1)$ actions.

2) use sl_2 triples for each $U(1)$ $\Rightarrow$ raising / lowering operators



$SU(3)$

Main tool needed from rep'n theory:

decomposition:

$$V_2 \otimes V_2 \otimes V_2 \otimes \cdots \otimes V_N = \text{direct sum of irreducibles on list}$$

2-d irred. rep'n $\mathbb{C}^2$

<u>which ones</u>?

Rmk (Rep'ns of $SO(3)$ vs $SU(2)$)

$$1 \to \{\pm \mathbb{1}\} \longrightarrow SU(2) \longrightarrow SO(3) \to 1$$

The $SU(2)$ irreps which descend to $SO(3)$ are those for which $\pi(-\mathbb{1}_{SU(2)}) = \mathbb{1}_V$

<u>e.g.</u>: $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$

vector (3d) rep'n can be described as quadratic polys in std rep'n

$$(a^2,\ ab,\ b^2)$$

$$(-\mathbb{1}) \cdot a^2 = a^2$$
$$\cdot\ ab = ab$$
$$\cdot\ b^2 = b^2$$

$\Rightarrow$ $\mathbb{C}^3$ is a $SO(3)$ rep'n (std rep'n of $SO(3)$).

($SU(2) = \text{spin}(2)$)

Classif. of complex Lie groups & Rep'ns

Most important ones: Simple groups, Semisimple groups

$A \leftrightarrow SL_n\mathbb{C}$

B, C, D: even/odd $>$ $\text{Spin}(n)$ Orthog.; $Sp(n)$ Symp

$E \leftrightarrow E_6, E_7\ E_8$ exceptional

$F \leftrightarrow F_4$

$G \leftrightarrow G_2$ 14