Pure vs Mixed states $\longleftrightarrow$ Joint Quantum systems.

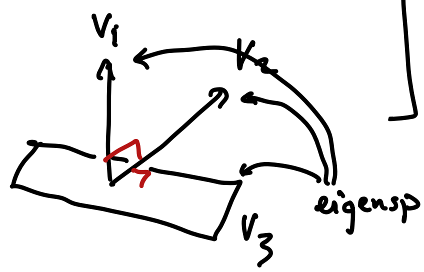
Pure states : $\mathbb{P}(\mathcal{H})$ $\quad \psi \in \mathcal{H} \setminus 0$ up to scale.

<u>Mixed states</u> : In case we have only a probabilistic knowledge of quantum state, we use <u>mixed</u> states.

Tool for encoding probability distrib. on quantum states :

<u>Def^n</u> A <u>Density operator</u> $\rho \in L(\mathcal{H})$ <u>self-adjoint</u> (so, an observable)

$\mathrm{Tr}\,\rho = 1$, non-neg. eigenvalues.



$\Rightarrow$ prob. distrib. on eigenspaces of ρ

$$\mathcal{H} = \bigoplus^{\perp} V_i \qquad \text{eigensp. decomp.}$$

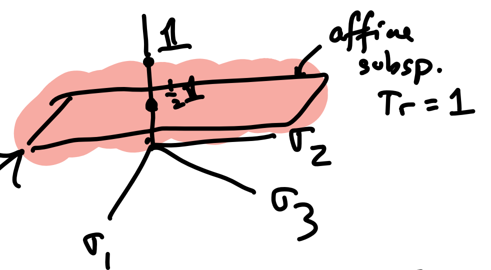
$$V_i = \ker(\rho - p_i \mathbb{1})$$

V_i assigned prob. p_i $\quad \left(\sum p_i = 1\right)$

<u>Ex.:</u> $\mathbb{C}^2$ $\quad \mathrm{Obs}(\mathbb{C}^2) = \left\langle \mathbb{1}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle_{\mathbb{R}}$

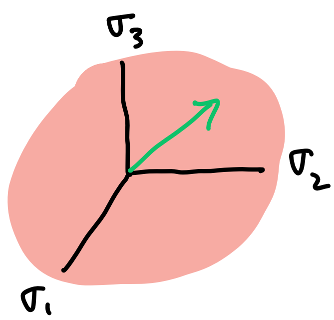
$\mathrm{Tr}\,\mathbb{1} = 2$

$\mathrm{Tr}\,\sigma_i = 0$



4d $\mathbb{R}$-vector space

3d space
$\mathrm{Tr}=1$ $\longrightarrow$



$\sigma_i^2 = \mathbb{1}$

$\mathrm{Tr}\,\sigma_i = 0$

eigenvals $\sigma_i = \pm 1$

any $Tr = 1$ operator:

$$\rho = \frac{1}{2}\left(\mathbb{1} + s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3\right)$$

$$= \frac{1}{2}\begin{pmatrix} 1+s_3 & s_1 - i s_2 \\ s_1 + i s_2 & 1 - s_3 \end{pmatrix}$$

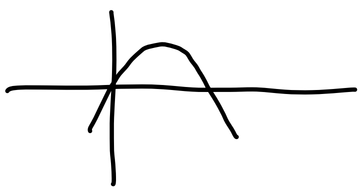
$$\det\rho = \frac{1}{4}\left(1 - (s_1^2 + s_2^2 + s_3^2)\right)$$

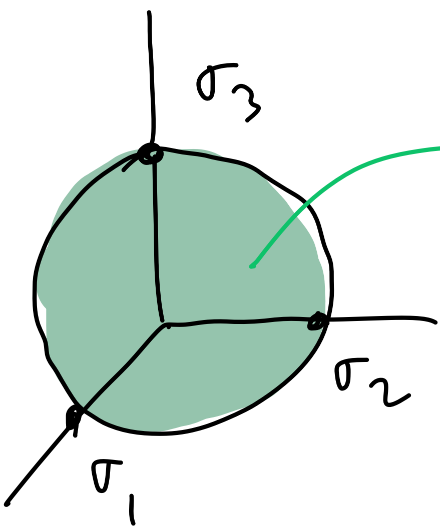
$$Tr\rho = 1 \Rightarrow \lambda_1 + \lambda_2 = 1$$

$$\lambda_1, \lambda_2 \geq 0 \Rightarrow \det\rho \geq 0 \Rightarrow \boxed{s_1^2 + s_2^2 + s_3^2 \leq 1}$$

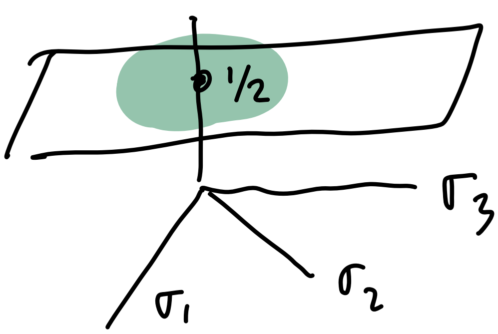
$$\det\rho = \lambda_1\lambda_2 = \underbrace{\lambda_1(1-\lambda_1)}_{} \geq 0 \Rightarrow \lambda_1 \in [0,1] \Rightarrow \lambda_2 \in [0,1]$$

non neg.





Block Ball: parametrizes all density operators "mixed states" in $\mathcal{H} = \mathbb{C}^2$



Boundary: pure states

Boundary of space of mixed states:

$\psi \in \mathcal{H} \setminus 0$

gives rise to a dual vector

$\ell_\psi \in \mathcal{H}^*$ (complex dual space)

$$\ell_\psi(v) = \langle \psi, v \rangle$$

$\mathcal{H}$ $\mathbb{C}$- vsp

$\langle \, , \, \rangle$ Herm. innerprod

$\mathbb{C}$-autilinear $\mathbb{C}$ - linear

$$\langle au, v \rangle = \bar{a} \langle u, v \rangle$$

$$\langle u, av \rangle = a \langle u, v \rangle$$

(modeled on $\bar{u}^T v$)

notation "bra-ket"

$\psi \longleftrightarrow |\psi\rangle \in \mathcal{H}$

$\ell_\psi \longleftrightarrow \langle \psi | \in \mathcal{H}^*$

$$\langle \psi, \phi \rangle = \langle \psi | \phi \rangle$$

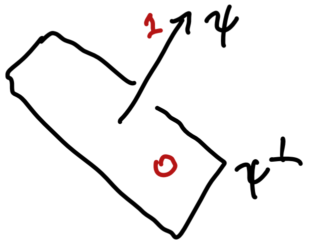
Def: The density operator $\mathbb{P}_\psi$ associated to $\psi \in \mathcal{H}$ is

$$\mathbb{P}_\psi(v) = \frac{\langle \psi, v \rangle}{\langle \psi, \psi \rangle} \psi$$

(projection to state ψ)

i.e. $$\mathbb{P}_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$$

$$\operatorname{Tr} \mathbb{P}_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} = \frac{\langle\psi|\psi\rangle}{\langle\psi|\psi\rangle} = 1$$



non-neg since spectrum $\mathbb{P}_\psi = \{0, 1\}$.

$\Rightarrow$ every state has corresponding density operator

$$\mathbb{P}(\mathcal{H}) \longrightarrow \operatorname{Obs}(\mathcal{H})$$

image: density operators of pure states

$$\psi \longmapsto \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$$

$$l_{\lambda\psi} = \bar{\lambda}\, l_\psi$$

$$\lambda \in \mathbb{C}^* \qquad \lambda\psi \longmapsto \frac{|\lambda\psi\rangle\langle\lambda\psi|}{\langle\lambda\psi|\lambda\psi\rangle} = \frac{\cancel{\lambda\bar{\lambda}}\, |\psi\rangle\langle\psi|}{\cancel{\lambda\bar{\lambda}}\, \langle\psi|\psi\rangle}$$

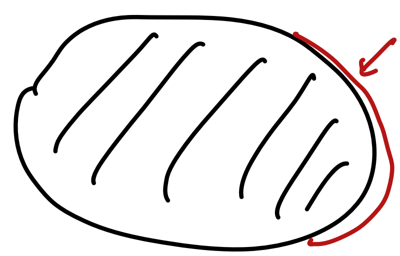
$\Rightarrow$ density operators of pure states $\Longleftrightarrow$ rank 1 density operators

<u>Prop</u>: The density operators form a <u>convex</u> set

i.e. ρ_1, ρ_2 density ops

$$\Rightarrow \quad \rho = t\rho_1 + (1-t)\rho_2 \quad \text{is a density op}$$

$$\forall\, t \in [0,1]$$



bdry pts are <u>pure</u>.

Ex.: if $\mathcal{H} = \mathbb{C}^2 = \langle |0\rangle, |1\rangle \rangle$ $\quad |\pm\rangle$

$|{-\tfrac{1}{2}}\rangle \quad |{+\tfrac{1}{2}}\rangle$

$\psi = \alpha|0\rangle + \beta|1\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$

normalize.

$$\rho_\psi = |\psi\rangle\langle\psi|$$

$$= |\alpha|^2 |0\rangle\langle 0| + \alpha\bar{\beta}\, |0\rangle\langle 1|$$

$$+ \bar{\alpha}\beta\, |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|$$

$$= \begin{pmatrix} |\alpha|^2 & \alpha\bar{\beta} \\ \bar{\alpha}\beta & |\beta|^2 \end{pmatrix}$$

$\mathrm{tr} = 1$

$\det = 0$

$\Downarrow$

eigenvalues $1, 0$.

unit sphere

$$\overset{\shortparallel}{S\mathcal{H}} \circlearrowleft \underset{\text{phase}}{S^1}$$

$$\downarrow$$

$$\mathbb{P}\mathcal{H}$$

$$\downarrow$$

$$\Delta$$

$$\rho = \frac{1}{2}\left(\mathbb{1} + s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3\right)$$

$$\det\rho = 0 \implies \boxed{s_1^2 + s_2^2 + s_3^2 = 1}$$

Rmk: ψ gives an observable ρ_ψ, could measure this obs. on any state:

if system is in $x \in \mathcal{H} \setminus 0$

ρ_ψ has eigenvals $\begin{cases} 0 \\ 1 \end{cases}$

$\langle a,b\rangle = \overline{\langle b,a\rangle}$

$$\langle \rho_\psi \rangle_\chi = \frac{\langle \chi | \rho_\psi | \chi \rangle}{\langle \chi | \chi \rangle}$$

$$= \langle \chi | \psi \times \psi | \chi \rangle / \langle \chi, \chi \rangle$$

$$= \frac{|\langle \chi | \psi \rangle|^2}{\langle \chi, \chi \rangle}$$

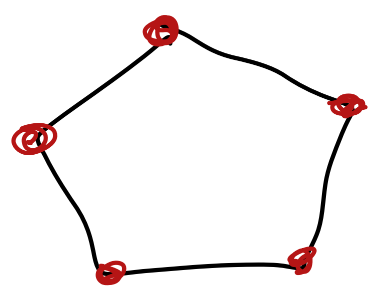
expected value of measuring ρ_ψ in state χ.

if $\chi = \psi$ $\quad = 1$

if $\chi \in \psi^\perp$ $\quad = 0$

In general dim, pure states are "extreme" states in convex set

: the points which are <u>not</u> convex linear combo of other points



Joint Quantum systems

If $(\mathcal{H}_1, \langle , \rangle_1, H_1)$ and $(\mathcal{H}_2, \langle , \rangle_2, H_2)$

are Quantum systems, then Joint (non-interacting)
system is defined by

$$\left(\mathcal{H}_1 \otimes \mathcal{H}_2 \,,\, \langle , \rangle_1 \otimes \langle \; \rangle_2 \,,\, H_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes H_2 \right)$$

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \ni u_1 \otimes v_1 + u_2 \otimes v_2 + \cdots + u_k \otimes v_k$$

$$\langle u \otimes v \,,\, u' \otimes v' \rangle = \langle u, u' \rangle_1 \langle v, v' \rangle_2 \,,$$ extend by (anti)-linearity

<u>key idea!</u> Entanglement:

There are joint states which do <u>not</u> correspond to states of $\mathcal{H}_1$, $\mathcal{H}_2$!

Makes no sense to ask: in what state is the subsystem $\mathcal{H}_1$

However, we <u>can</u> assign a mixed state to a subsystem ($\Rightarrow$ probabilistic description of the quantum state).

$$\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2 \qquad \text{(pure state)}$$

$$\rho_\psi \in \mathrm{Obs}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

$$\|$$

$$\mathrm{Obs}(\mathcal{H}_1) \otimes \mathrm{Obs}(\mathcal{H}_2)$$

$$\Big\downarrow \mathbb{1} \otimes \mathrm{Tr}$$

$$\mathrm{Obs}(\mathcal{H}_1) \otimes \mathbb{R} = \mathrm{Obs}(\mathcal{H}_1)$$

<u>Claim:</u> $$(\mathbb{1} \otimes \mathrm{Tr})(\rho_\psi) = \rho_{\psi,1}$$

is a mixed state of system 1.

Proof: Let b_i ON basis $\mathcal{H}_2$

$$\psi = \sum a_i \otimes b_i \qquad a_i \in \mathcal{H}_1$$

$$|\psi|^2 = 1 \Rightarrow \sum_i |a_i|^2 = 1$$

$$\rho_\psi = |\psi\rangle\langle\psi| = \sum_{i,j} |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|$$

$\mathrm{Tr}\, |y\rangle\langle z| = \langle z, y\rangle$

$$\Big\downarrow {\scriptstyle 1 \otimes \mathrm{Tr}_{\mathcal{H}_2}}$$

$$\sum_{i,j} |a_i\rangle\langle a_j| \, \cancelto{\delta_{ij}}{\langle b_j, b_i\rangle}$$

prob. ensemble of pure states.

$$\rho_{\psi,1} = \sum_i |a_i\rangle\langle a_i| = \sum_i |a_i|^2 \rho_{|a_i\rangle}$$

Reduced density op. of ψ. in 1st system.

Aside on tensor products

V vector sp.

$L(V) = V \otimes V^* \ni A$

$V \otimes W$

$L(V \otimes W) = (V \otimes W)^* \otimes (V \otimes W)$

$= V^* \otimes W^* \otimes V \otimes W$

$= V^* \otimes V \otimes W^* \otimes W$

$= L(V) \otimes L(W)$

$\underset{\sum A_i \otimes B_i}{\uplus}$

$\psi \in V \otimes W$

$\shortparallel$

$\sum v_i \otimes w_i$

e_i basis

e^i dual basis

$A = A^j_i \, \underset{\underset{V^*}{\uparrow}}{e^i} \otimes \underset{\underset{V}{\uparrow}}{e_j}$

$V = \mathrm{Span}(e_i)$ Bases

$W = \mathrm{Span}(f_i)$

$V \otimes W$ basis

$e_i \otimes f_j$

$\dim(V \otimes W)$

$\shortparallel$

$(\dim V)(\dim W)$

dual basis:

$e^i \otimes f^j$

how to act on $v \otimes w$?

$A_k \in L(V) \quad B_k \in L(W)$

$A_1 v \otimes B_1 w + A_2 v \otimes B_2 w + A_3 v \otimes B_3 w$

$\neq$

$A v \otimes B w$

Entangled vs decomposable states

$$\underset{(e_1,e_2)}{\mathbb{C}^2} \otimes \underset{(f_1,f_2)}{\mathbb{C}^2} \ni \overset{\text{decomposable}}{(ae_1+be_2)\otimes(cf_1+df_2)}$$

$$= \overset{\alpha}{ac}\; e_1\otimes f_1 + \overset{\beta}{ad}\; e_1\otimes f_2 + \overset{\gamma}{bc}\; e_2\otimes f_1 + \overset{\delta}{bd}\; e_2\otimes f_2 .$$

$\alpha\delta = \beta\gamma$ constraint satisfied by decomp. state

State space of joint system is $\mathbb{P}(\mathbb{C}^2\otimes\mathbb{C}^2 = \mathbb{C}^4) = \mathbb{CP}^3$ 3-dim cx proj space.

$\subset$ locus of states satisfy $\alpha\delta = \beta\gamma$

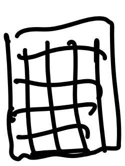
(quadric hypersurface)

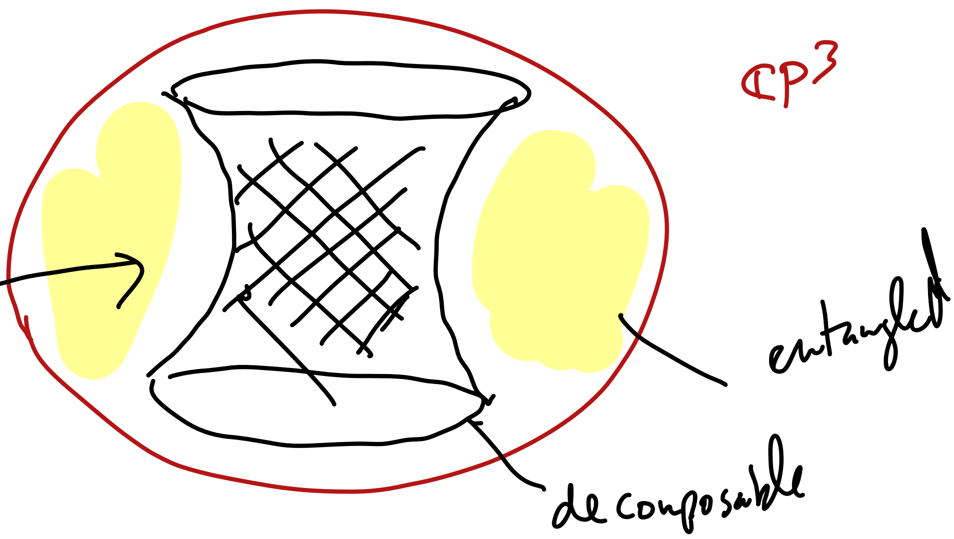
quadric: polynomial deg 2

hypersurface: single complex constraint.

decomposable states

$\mathbb{P}(\mathbb{C}^2)\times\mathbb{P}(\mathbb{C}^2) = \underbrace{\mathbb{CP}^1\times\mathbb{CP}^1}_{\text{dim } 2}$





k spin $\frac{1}{2}$ particles

state space $\quad \mathbb{CP}(\underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \mathbb{C}^2}_{k})$

$$\| $$

$$\subset \underbrace{\mathbb{CP}^{2^k-1}}_{2^k-1}$$

$$(\mathbb{P}^1)^k$$

$\nwarrow$ decomposables

k