QM axioms, Amplitudes, probability, pure + mixed states

Basic insight of QM: Enhancement of probability theory

normally probability of event is $p \in [0,1]$

in QM have something more fundamental than p: amplitude $a \in \mathbb{C}$

which implies prob. $p = |a|^2$

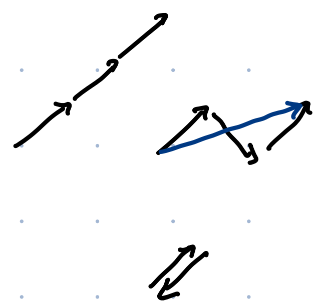
<u>Slogan:</u>

- If something can happen in different ways $1, 2, \ldots n$

  then $\cancel{p = p_1 + p_2 + \cdots + p_n}$

  $$a = a_1 + \cdots + a_n$$

  $$p = |a|^2$$



- If outcome is a result of a succession of events

  $$p = p_1 p_2 \cdots p_n$$

  $$a = a_1 \cdots a_n$$

Axioms I) Kinematics : states & observables

State of system is described by nonzero vector

$$\psi \in \mathcal{H} \setminus \{0\}$$

in a $\mathbb{C}$ v.space equipped w/ hermitian inner product
$\left(\mathbb{C}^n, \langle u,v \rangle = \bar{u}^T v \right)$ (in infinite-dim^l case require completeness .... Hilbert space).

we consider $\psi \sim \psi'$ when $\exists\, z \in \mathbb{C}^*$ $\quad z\psi = \psi'$

State space: $\left(\mathcal{H} \setminus \{0\}\right) / \mathbb{C}^* = \mathbb{P}\mathcal{H}$ complex projective space.

if $\mathcal{H} = \mathbb{C}^n$, $\quad \mathbb{P}(\mathbb{C}^n) = \mathbb{C}P^{n-1}$

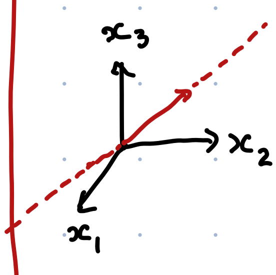
$$\mathbb{C}P^{n-1} \ni [z_1 : z_2 : \cdots : z_n] = [(3+i)z_1 : (3+i)z_2 : \cdots : (3+i)z_n]$$

Idea: suppose $X = \{x_1, \dots, x_n\}$ are possible outcomes of measurement

then $\mathcal{H} = \mathbb{C}^X$

$\mathbb{P}\mathcal{H} \ni [z_1 : z_2 : \cdots : z_n]$ state

amplitudes for finding $x_1, \dots, x_n$ outcomes



probability p_i for finding outcome x_i is

$$p_i = \frac{|z_i|^2}{|z_1|^2 + \cdots + |z_n|^2}$$

Observables are self-adjoint operators

$$X : \mathcal{H} \to \mathcal{H} \qquad\qquad X^* = X$$

and give rise to decomposition

$$\mathcal{H} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$$

orthogonal eigenspace decomposition

for simplicity assume X is regular (distinct eigenvals)

$$\mathcal{H} = \bigoplus_{i=1}^{n} \mathbb{C}\, \underbrace{|x_i\rangle}$$

vector in $\mathcal{H}$
eigenvector of X w/ e-val x_i

$$\boxed{X\,|x_i\rangle = x_i\,|x_i\rangle}$$

Interpret $\{|x_1\rangle, \dots, |x_n\rangle\}$ as different outcomes of measurement of observable X.

Measurement axiom:

Suppose system in state $\psi \in \mathcal{H}$.

If we "measure" observable X (Born Rule)

i) If ψ is e-vector of X $\quad X\psi = \lambda\psi$

then outcome is $\underline{\underline{\lambda}}$

ii) If $\psi = \alpha_1 |\lambda_1\rangle + \alpha_2 |\lambda_2\rangle + \cdots + \alpha_n |\lambda_n\rangle$

e-vectors of X

then outcome uncertain:

λ_1 with prob. $\dfrac{|\alpha_1|^2}{|\alpha_1|^2 + \cdots + |\alpha_n|^2}$

λ_2 " " $\dfrac{|\alpha_2|^2}{|\alpha_1|^2 + \cdots + |\alpha_n|^2}$

$\vdots$

λ_n " " $\dfrac{|\alpha_n|^2}{|\alpha_1|^2 + \cdots + |\alpha_n|^2}$

Kinematics	Classical	QM
state space	(M, ω)	$\mathbb{P}\mathcal{H}$
Observables	$C^\infty(M)$	Self-adjoint operators.
Dynamics	$H \in C^\infty(M)$; $\boxed{\frac{d}{dt}(p,q) = X_H(p,q)}$	H; $\boxed{\frac{d}{dt}\psi = -iH\psi}$

$$(z_1, \dots, z_n) \neq (0, 0, \dots, 0)$$

1) given observable X (assume regular and order eigenvalues (natural order on $\mathbb{R}$)).

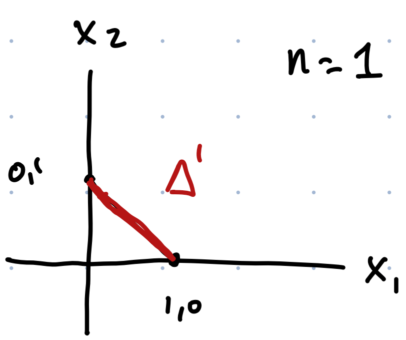
can identify $\mathcal{H} = \mathbb{C}^n$

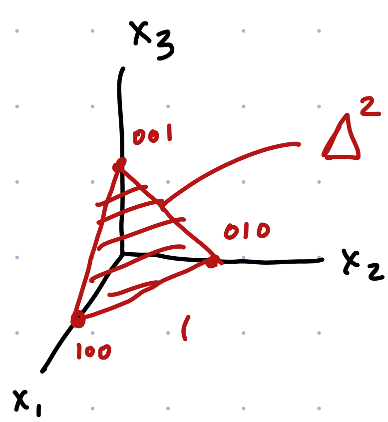
$$\mathbb{P}\mathcal{H} = \mathbb{CP}^{n-1} \ni [z_1 : \cdots : z_n]$$

Then there is a map $\mathbb{CP}^{n-1} \xrightarrow{\pi} \Delta^{n-1} = \{ (x_1, \dots, x_n) \in \mathbb{R}^n$ st. $x_i \geq 0$ and $\sum x_i = 1 \}$

set of probability dist. on n outcomes.

n-simplex

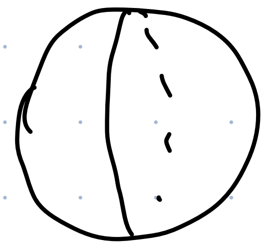




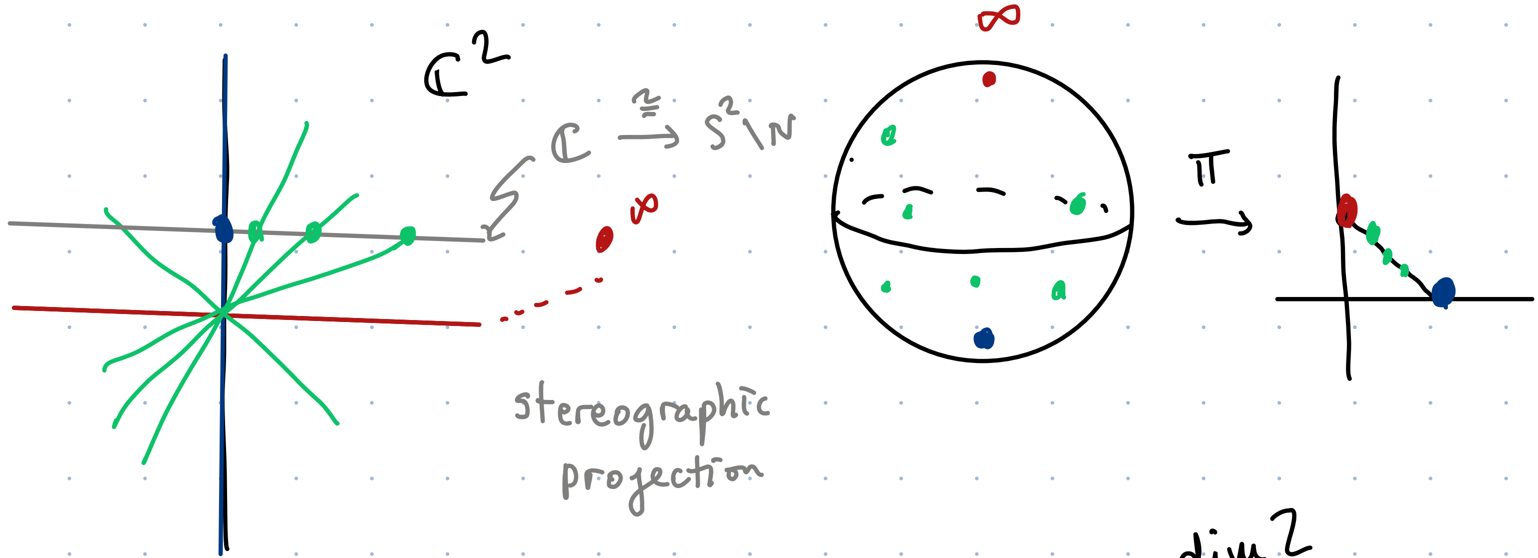
Geom. interpretation of Born Rule:

$$\pi([z_1 : \cdots : z_n]) = \frac{1}{\sum_{i=1}^{n} |z_i|^2} \left(|z_1|^2, |z_2|^2, \ldots, |z_n|^2 \right)$$

e.g. $n = 2$

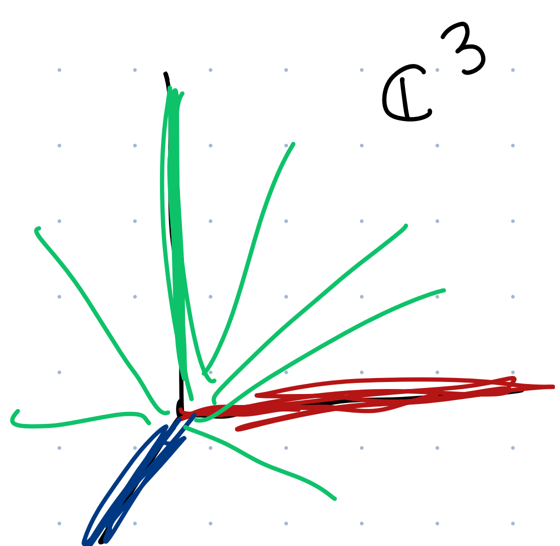
$\mathbb{CP}^1 =$  Bloch sphere = Riemann sphere

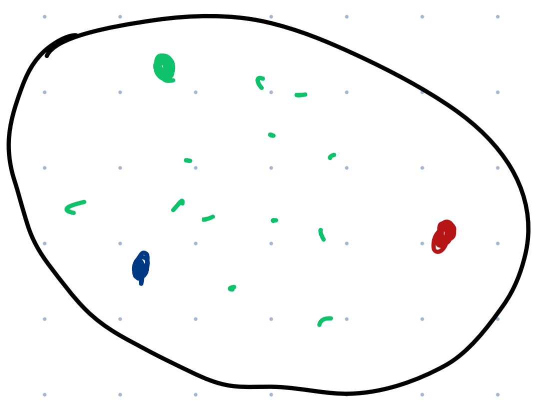
$\mathbb{C}^2 = \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\rangle$ eivectors of σ_3



$\mathbb{CP}^2 \leftarrow$ cx dim 2, $\mathbb{R}$ dim 4

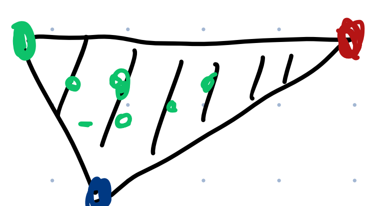
e.g. $n = 3$





quantum states

$\downarrow \pi$



probability distributions

Rmks: If X is a reg. observable

$$\Rightarrow \mathcal{H} = \bigoplus_{i=1}^{n} \mathbb{C}\,|x_i\rangle \qquad \text{eigenspaces}$$

X diagonal in $|x_i\rangle$ basis

$$X = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}$$

$\Rightarrow \exists$ a collection of Observables

$$H_1 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \\ & & & & 0 \end{pmatrix} \quad H_2 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & \ddots \\ & & & & 0 \end{pmatrix} \cdots H_n$$

which
- commute w/ X $\qquad [X, H_i] = 0$
- commute with each other $\qquad [H_i, H_j] = 0$

each of these defines Lie alg. element

$$-iH_1,\ -iH_2,\ \ldots,\ -iH_n \in \mathfrak{u}(n)$$

$U(n)$ = unitary operators on $(\mathbb{C}^n, \langle , \rangle)$
= symmetries of Quantum State space

$\Rightarrow$ unitary rotations

$$e^{-it_1 H_1} = \begin{pmatrix} e^{-it_1} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

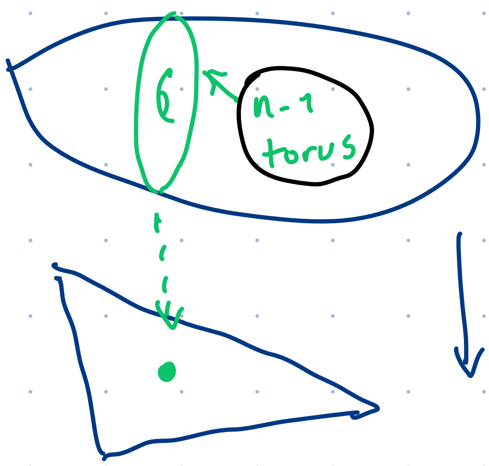
$$e^{-it_2 H_2} = \begin{pmatrix} 1 & & & \\ & e^{-it_2} & & \\ & & 1 & \\ & & & \ddots \; 1 \end{pmatrix}$$

$$\vdots$$

$$e^{-it_n H_n} = \ddots$$

$\Rightarrow$ action of $\overbrace{U(1) \times U(1) \times \cdots \times U(1)}^{n}$

on $\mathbb{CP}^{n-1}$, respecting π



$\mathbb{CP}^{n-1}$ $\leftarrow$ real dim $2n-2$

$\downarrow \pi$

Δ^{n-1} $\leftarrow$ real dim $n-1$

Toric manifold

$\downarrow \pi$ momentum map

Delzant polytope

II Dynamics:

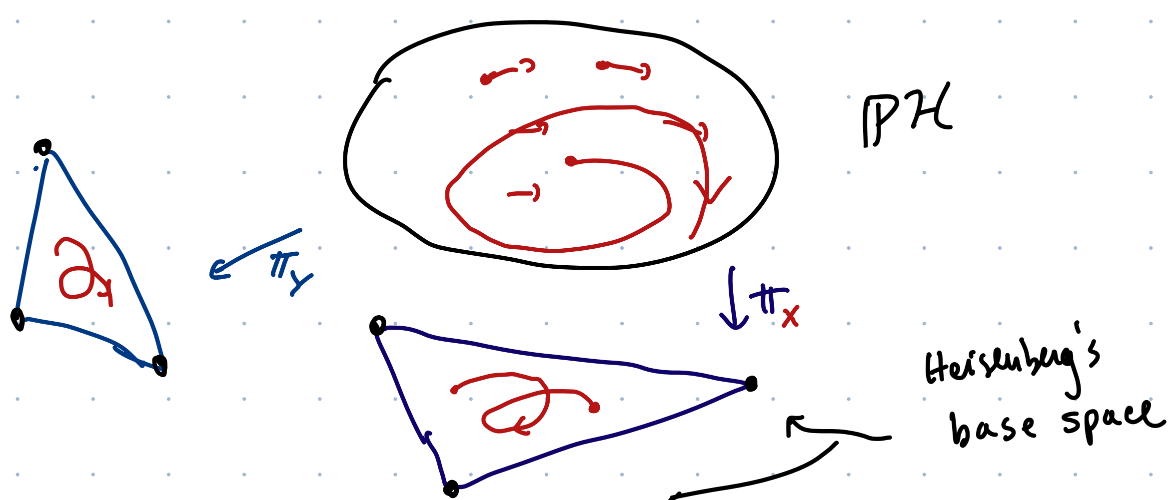
to get motion of states, must select an observable H (Hamiltonian operator)

states evolve via the eq. of motion

$$\frac{d\psi}{dt} = \overbrace{-iH}^{\mathfrak{u}(n)} \psi$$

this gives unitary evolution

$$\boxed{\psi(t) = e^{-itH}\,\psi(0)}$$



Preview:

$$(\mathcal{H}_1, H_1) \qquad (\mathcal{H}_2, H_2)$$

joint system: $\mathcal{H}_1 \otimes \mathcal{H}_2 \ni u_1 \otimes v_1 + u_2 \otimes v_2 + \cdots + u_k \otimes v_k$

joint Hamiltonian: $H_1 \otimes \mathbb{1} + \mathbb{1} \otimes H_2 = H$

$$H(u \otimes v) = (H_1 u) \otimes v + u \otimes (H_2 v)$$

if $\left.\begin{aligned} H_1 u &= \lambda u \\ H_2 v &= \mu v \end{aligned}\right\}$ $\quad H(u \otimes v) = \lambda\, u \otimes v + \mu\, u \otimes v$

$$= (\lambda + \mu)\, u \otimes v$$

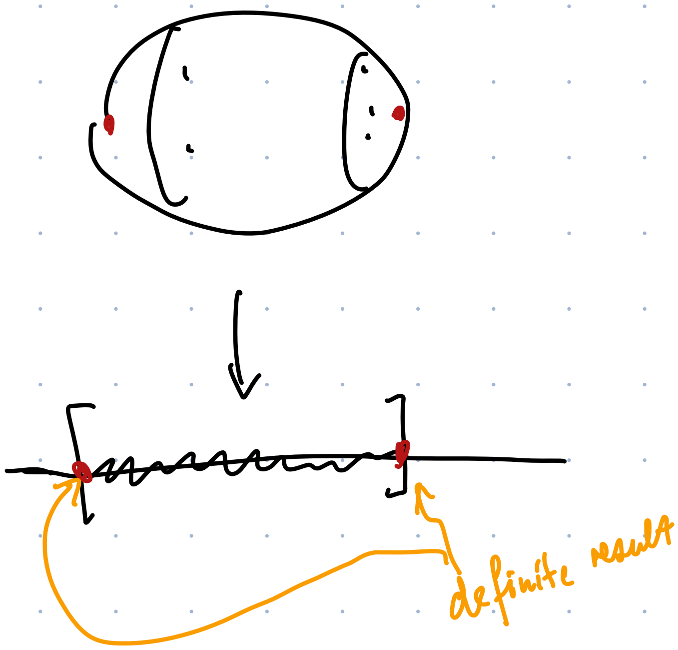
if $X \in \mathrm{Obs}(\mathcal{H}_1)$ then $X \otimes \mathbb{1} \in \mathrm{Obs}(\mathcal{H}_1 \otimes \mathcal{H}_2)$

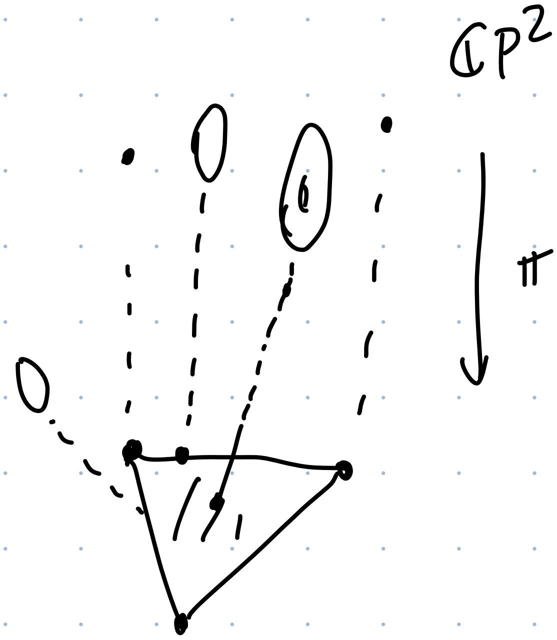
X_1 set outcomes of system 1

X_2 " " " " 2

set outcomes joint syst. is $X_1 \times X_2$

$$\mathbb{C}^{X_1 \times X_2} = \left(\mathbb{C}^{X_1}\right) \otimes \left(\mathbb{C}^{X_2}\right)$$





c.f. Heisenberg: focus on one observable, outcomes set X

$$\begin{array}{ccc} X \times X & \mathrm{Obs} = \mathbb{C}^{X \times X} & \ni A \\ t \downarrow \downarrow s & \circlearrowleft & \\ X & \text{states } \mathbb{C}^X & \ni \psi \end{array}$$

$$A\psi = t_*(A\, s^*\psi) = A\psi$$

operator acting on vector.