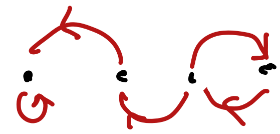
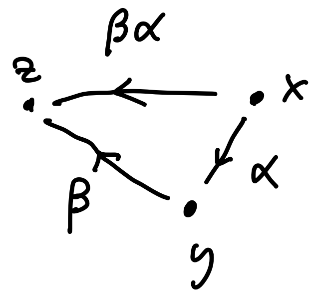
Intro to Groupoids:

- objects
- arrows between objects
- composition law on arrows (associative)
- every obj has identity arrow
- every arrow has an inverse





Objects: X "base"

arrows: G "groupoid"

each arrow has a source and target object

i.e. $G \underset{t}{\overset{s}{\rightrightarrows}} X$ maps

Identity $\mathrm{Id} : X \to G$

$x \mapsto \mathrm{Id}_x$

Inverse $i : G \to G$

composition: $G^{(2)} \xrightarrow{m} G$

$$G^{(2)} = \{ (g_2, g_1) : t(g_1) = s(g_2) \}$$

Ex.: Pair groupoid of set Γ

$$G = \Gamma \times \Gamma$$
$$X = \Gamma$$

$$(y,x) \in \Gamma \times \Gamma$$

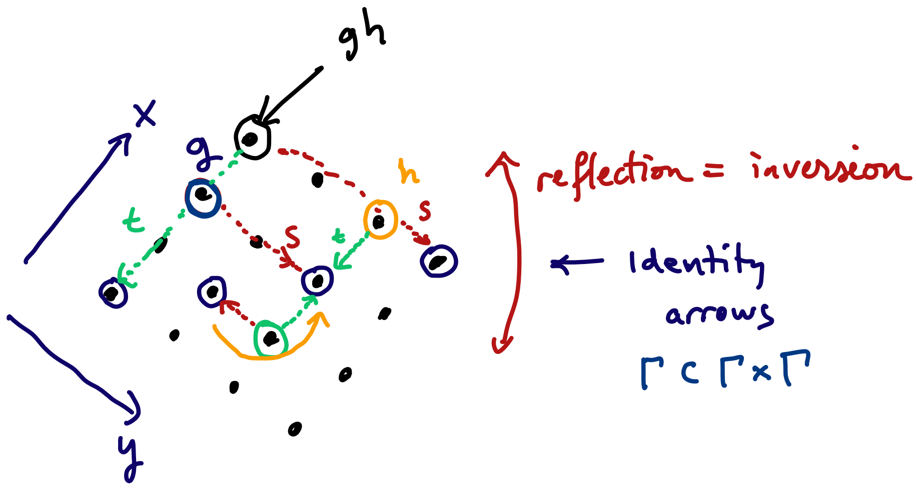
$$y \xleftarrow{t} (y,x) \xrightarrow{s} x$$

$$m\big((z,y)(y,x)\big) := (z,x)$$

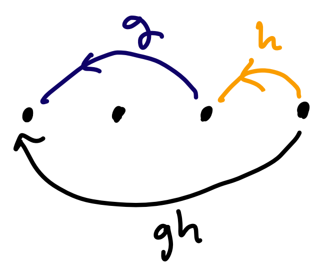
$$i(y,x) = (x,y)$$

$$\mathrm{Id}_x = (x,x)$$

$\Gamma \times \Gamma$



$\Gamma = \{ \cdot \ \cdot \ \cdot \ \cdot \}$



Rydberg-Ritz combination principle:

if ν_{mn} and $\nu_{\ell k}$ are observed frequencies

(corresp. to $\frac{1}{m^2} - \frac{1}{n^2}$ and $\frac{1}{\ell^2} - \frac{1}{k^2}$)

and if $n = \ell$ then $\nu_{mn} + \nu_{nk} = \nu_{mk}$
is observed. (almost always true)

<u>Heisenberg</u>: what we actually observe are "virtual oscillators" corresponding to $E_m - E_n$ differences.

Rydberg-Ritz $\Longleftrightarrow$ Composition in groupoid

arrows (n, m), (ℓ, k) are composable only if $s(n,m) = m$ coincides with $t(\ell, k) = \ell$ and comp. is (n, k)

$$n \longleftarrow m = \ell \longleftarrow k$$

$\Downarrow$ compose

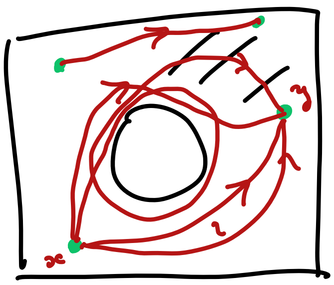
$$n \longleftarrow k$$

$$G \rightrightarrows X$$

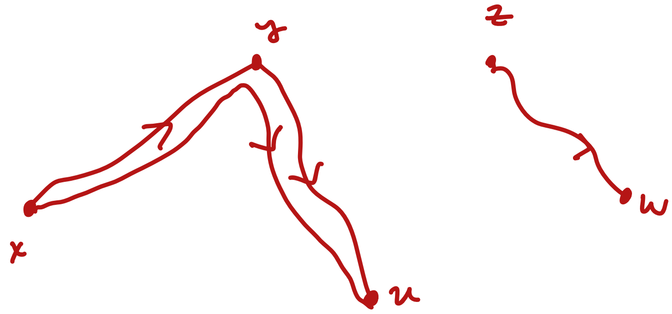
Ex.: $\Pi_1 X \rightrightarrows X$

objects = points in X

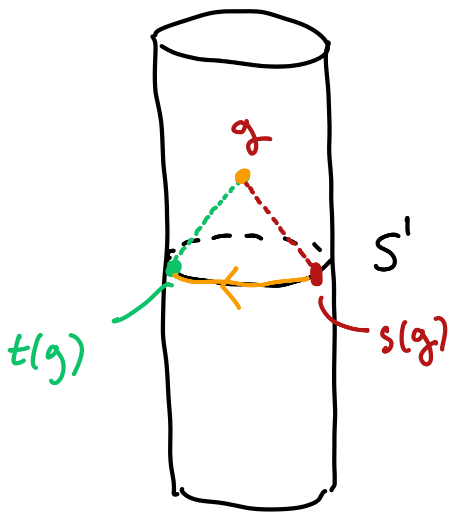
arrows = paths in X up to homotopy fixing endpoints



$X = \mathbb{R}^2 \setminus D(1)$



$\Pi_1(S^1)$



Heisenberg's main idea: observable quantities (f^{ns} of state) have Fourier representation involving frequencies $\nu_{ik} : \Gamma \times \Gamma \to \mathbb{R}$

$\nu_{ki} = -\nu_{ik}$

i.e. homomorphism $\mathrm{Pair}(\Gamma) \xrightarrow{\nu} (\mathbb{R}, +)$

i.e. given $(k, j)(j, i) = (k, i)$ in $\mathrm{Pair}(\Gamma)$

have $\nu_{kj} + \nu_{ji} = \nu_{ki}$

i.e. a f^n of state (observable)

has form $q : \mathrm{Pair}(\Gamma) \to \mathbb{C}$ (Fourier coeffs)

s.t. $q(i,j) = \bar{q}(j,i)$

i.e. $\boxed{i^* q = \bar{q}} \iff \overline{i^* q} = q$ $\quad \overline{i^*q} = q^*$

matrix version: self-adjoint.

To get dynamics: specify an observable

For the atoms above:

$$H(i,j) = \begin{cases} 0 & i \neq j \\ h\nu_i & i = j \end{cases}$$

ν_i freq. of i^{th} state

$$\begin{tikzcd} \mathrm{Pair}(\Gamma) \arrow[r, "H"] \arrow[d, shift right] \arrow[d, shift left] & \mathbb{C} \\ \Gamma & \end{tikzcd}$$

<u>Time evolution:</u> $\left(\text{c.f.} \quad \frac{dq}{dt} = \{H, q\} \right)$

$$\boxed{\frac{dq}{dt} = \frac{2\pi i}{h} \underbrace{\left(H * q - q * H \right)}_{[H,q]}}$$

$$(q_1 * q_2)(j,i) = \sum_k q_1(j,k)\, q_2(k,i)$$

convolution product of f^{ns} on a groupoid.

(automatically associative due to fact groupoid is)

matrix multiplication

$$(H * q)(j,i) = \sum_k H(j,k)\, q(k,i)$$

$$= h \nu_j\, q(j,i)$$

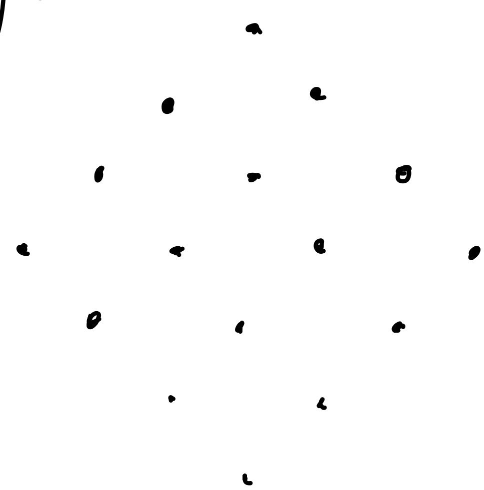
$$(q * H)(j,i) = \sum_{k} q(j,k)\, H(k,i)$$

$$= h\nu_i \, q(j,i)$$

$$[H,q] = h\underbrace{(\nu_j - \nu_i)}_{\nu_{ji}}\, q(j,i)$$

$$\dot{q}_{ji} = \frac{2\pi i}{} (\nu_{ji})\, q_{ji}$$

$$q_{ji} = q_{ji}(0)\, e^{i\nu_{ji} t}$$



In this way, Heisenberg identified quantum analogue of

- f^{ns} on state space
- Hamilton's evol. of such f^{ns}

What are the states themselves ?

<u>Born:</u> states = nonzero vectors in $\mathbb{C}$ v.space spanned by Γ objects

$= \mathbb{C}^{\Gamma} =$ functions $\Gamma \to \mathbb{C}$

if $\Gamma = \{x_1, \dots, x_n\}$ a state: $a_1 x_1 + \dots + a_n x_n$ $a_i \in \mathbb{C}$

$(a_1, \dots, a_n) \neq 0$

the coefficients $(a_1, \dots, a_n) \in \mathbb{C}^n$ are <u>probability amplitudes</u>

for finding system in corresponding basic state $\in \Gamma$

prob. amplitude determine prob. via $|a|^2$

<u>Details:</u> Equiv. relo on quantum states $f_1 \sim f_2$ in $\mathbb{C}^{\Gamma}$ when $\exists\, \lambda \in \mathbb{C}^*$ $\lambda f_1 = f_2$

True QM state space: Complex 1-d subsp. of $\mathbb{C}^n$

$= \mathbb{P}(\mathbb{C}^n)$ projective space.

Born rule: given a state

$$\tilde{\psi} = a_1 x_1 + \cdots + a_n x_n \qquad a_i \in \mathbb{C}$$

normalize: $$\psi = \frac{a_1 x_1 + \cdots + a_n x_n}{\sqrt{|a_1|^2 + \cdots + |a_n|^2}} \qquad |\psi|^2 = 1$$

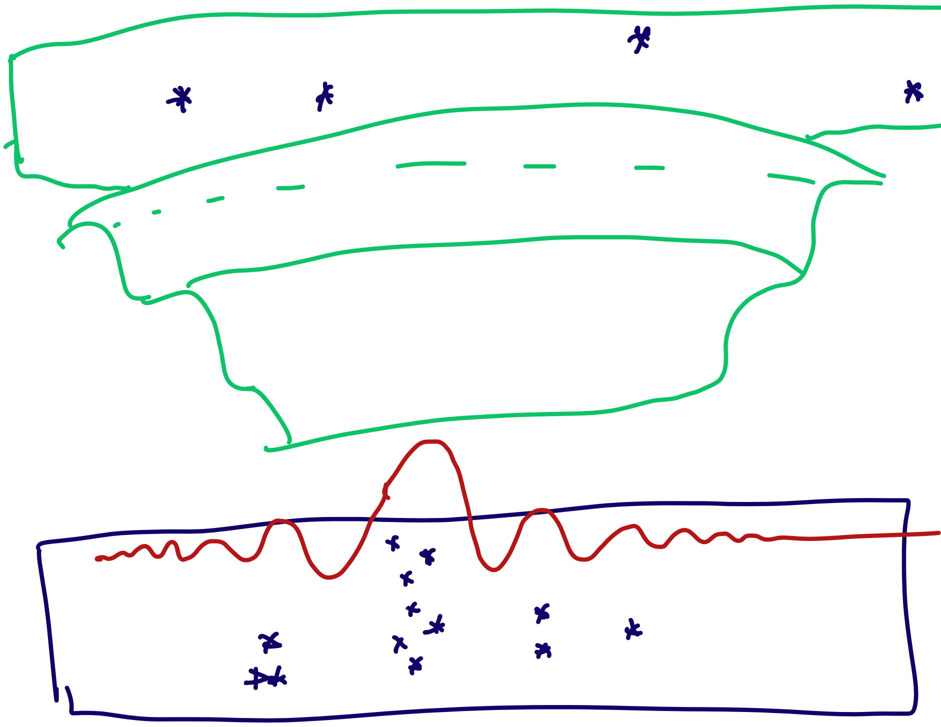
prob. amplitude for finding system in basic state x_i:

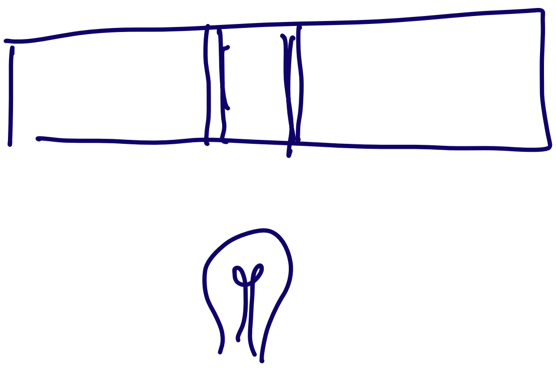
is $$\frac{a_i}{\sqrt{|a_1|^2 + \cdots + |a_n|^2}} \in \mathbb{C}$$

$\Rightarrow$ prob. for finding system in x_i is $\boxed{\dfrac{|a_i|^2}{\sum_k |a_k|^2}}$

probabilistic ideas derive from wave nature of light (sol^n to wave equation) in which waves can be added (vectors in a v. space)

$\left\{ \begin{array}{l} \text{amplitudes add} \\ \text{intensity} \quad |a_1 + a_2|^2 \end{array} \right.$





interference pattern (the EM wave)

has <u>intensity</u> predicting probability of detecting photon

<u>Born</u>: ① We can only predict probabilities ($\substack{\text{sometimes} \\ 100\%}$)

② there are <u>probability amplitudes</u> (beyond standard prob. theory)

can be added, govern observables, evolution equation controls them, we solve for them.

Main conceptual difficulty:

"Measurement" problem: when we observe a system, we always get a <u>basic</u> state

$x_i \in \Gamma$

<u>Kinematics</u>: states $\mathbb{C}^{\Gamma}$ $\longleftrightarrow$ vectors

observables (f^{ns} on state space) $\mathbb{C}^{\Gamma \times \Gamma}$ $\Longleftrightarrow$ matrices

dynamics det. by $H \in \mathbb{C}^{\Gamma \times \Gamma}$ st. $\overline{i^* H} = H$

via evol. eq^n

(Heisenberg eq^n).

$$\begin{tikzcd} G \arrow[r, "K"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & \mathbb{C} \\ X \arrow[r, "\psi"'] & \mathbb{C} \end{tikzcd}$$

f^n of 2 variables

f^n of one variable

$$\boxed{\begin{aligned} (K\psi)(\ell) &= \sum_k K(\ell, k)\psi(k) \\ K\psi &= t_*\left(K \; s^*\psi\right) \end{aligned}}$$

If H Hamiltonian is given

we can evolve states:

$$\frac{d}{dt}\psi = \frac{i}{\hbar} H \cdot \psi$$

and can evolve observables

$$\frac{d}{dt} q = \frac{i}{\hbar}[H, q]$$