Orion Nebula (visible (Hubble, NASA)

Hydrogen spectrum in visible

410 434nm 486nm 656 nm

δ γ β α

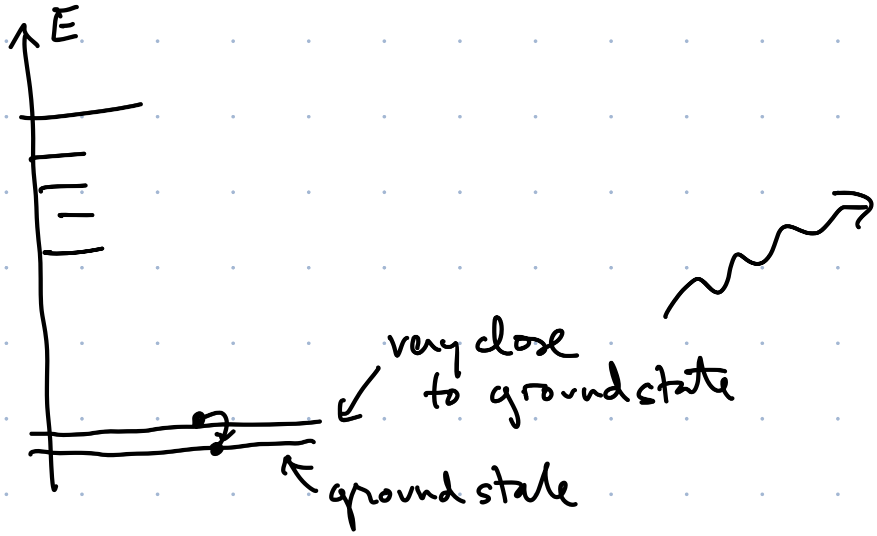
$m=2$

$n=3,4,5,6$

atoms, ions, etc emit light in a char. discrete spectrum use of spectra:

- chemical analysis on Earth
- analysis of stars + (exo) planets.
- recession velocity of parts of universe
- define unit of time (s)

Cesium atom Wt 133 (most common isotope)



transition $2^{nd} \to 1^{st}$

spectral line

Second := 9192631770 periods of the light emitted

In case of Hydrogen, 1800s culminate in formula

Rydberg formula

$$\frac{1}{\lambda} = R_H \left(\frac{1}{m^2} - \frac{1}{n^2} \right)$$

$m, n \in \mathbb{Z}$

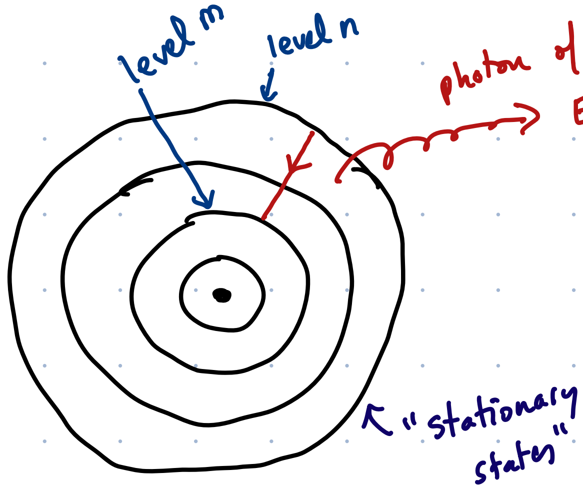
$\nu = \frac{c}{\lambda}$

$\sim 10967758 / m$

$E = h\nu$

$h = 6.626 \times 10^{-34}\ J \cdot s$

Bohr:



$E_n - E_m = h\nu_{mn}$

assume only orbits w/ ang. momentum $\hbar\,\mathbb{Z}_{>0}$ permitted

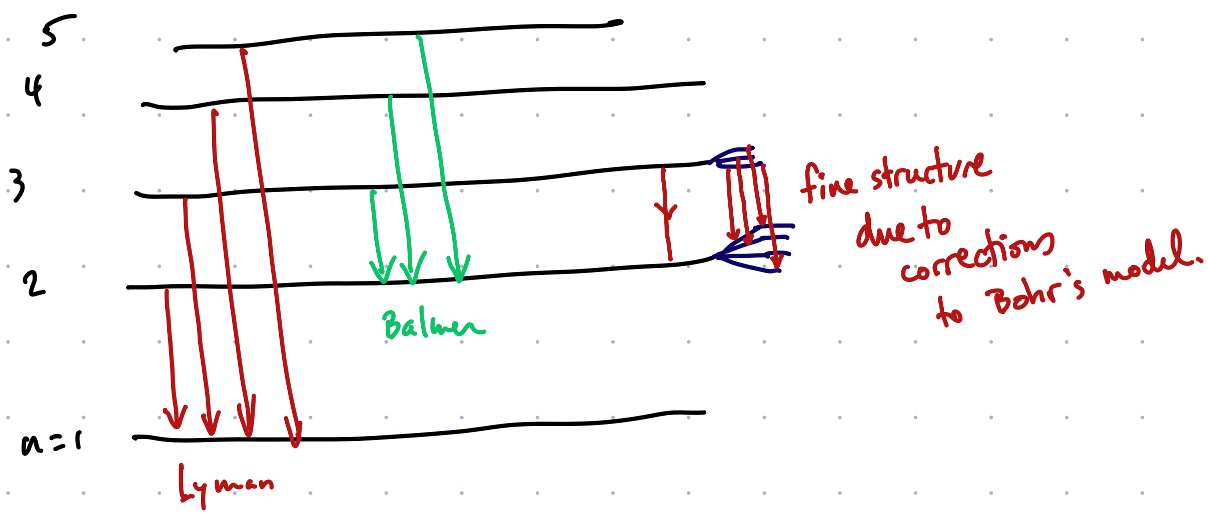
circular orbits, inverse square law (planets)

If we fix

$m = 1 \qquad n \in \{2, 3, 4, \dots\}$ Lyman series $\rightarrow$ 91 nm

$m = 2 \qquad n \in \{3, 4, \dots\}$ Balmer series $\rightarrow$ 364 nm UV

$\vdots$



(I) "Old QM" Bohr, Sommerfeld

(II) Born, Heisenberg – Jordan, Matrix Mechanics

(III) Dirac, Von Neumann

Bohr - Sommerfeld Quantization: (Geometric Quantization)

Start with completely Integrable Hamiltonian system

<u>Def</u>^{n} (Completely integrable) (M^{2n}, ω, H) is

completely integrable if $\exists \; J_1, \dots, J_n \in C^\infty(M)$

independent commuting conserved quantities

$dJ_1 \wedge \dots \wedge dJ_n$ almost everywhere nonzero

$\{J_i, J_j\} = 0 \quad \forall i, j$

$\{H, J_i\} = 0 \quad \forall i.$

(this includes toric symplectic manifolds)

<u>Def</u>^{n} (M^{2n}, ω) is a toric symplectic mfld

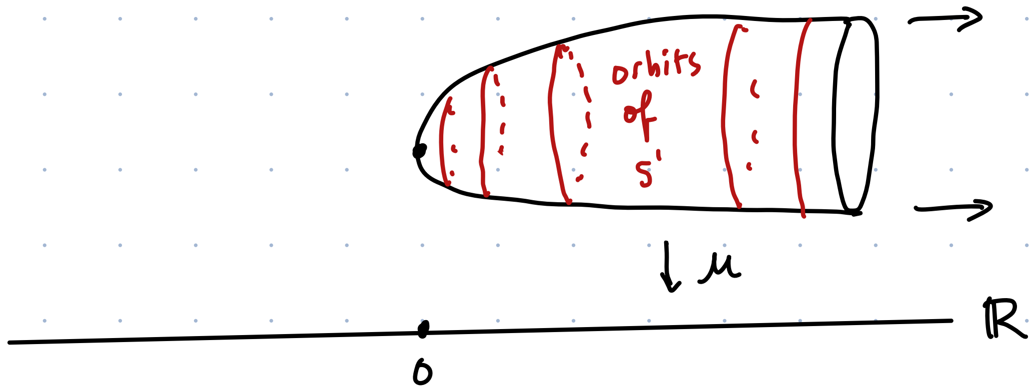
if $\exists \; T^n = (S^1)^n$ action $T^n \times M \to M$,

preserving ω, with moment map

$$\mu : M \longrightarrow \mathrm{Lie}(T^n) = \mathbb{R}^n$$

$\mu = (\mu_1, \dots, \mu_n)$ $\qquad$ X_{μ_i} generates the S^1 action of ith factor of T^n

Ex.: $(\mathbb{R}^2, \omega = dp \wedge dx) \xrightarrow{\mu} \frac{1}{2}(x^2 + p^2)$

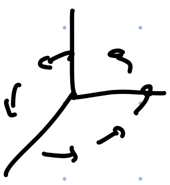


Ex: $(S^2, \omega) = \{x^2+y^2+z^2=1\} \subseteq (\mathbb{R}^3, x, y, z)$

$$\omega = \iota_{r\partial_r} dx \wedge dy \wedge dz \qquad r\frac{\partial}{\partial r} = x\partial_x + y\partial_y + z\partial_z$$

$$= x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy$$

choose axis of rot (z) $\quad$ S^1 action generated by rot. v.field



$x\partial_y - y\partial_x$

$$\iota_{x\partial_y - y\partial_x}\omega = x^2 dz - zx\,dx + y^2 dz - yz\,dy$$
$$= (x^2+y^2)\,dz - z(x\,dx + y\,dy)$$

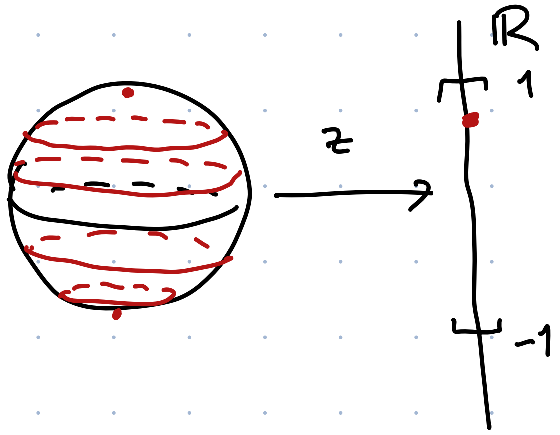
$-z\,dz$

$$x^2+y^2+z^2=1 \implies x\,dx + y\,dy + z\,dz = 0$$

$$= (x^2+y^2+z^2)\,dz$$

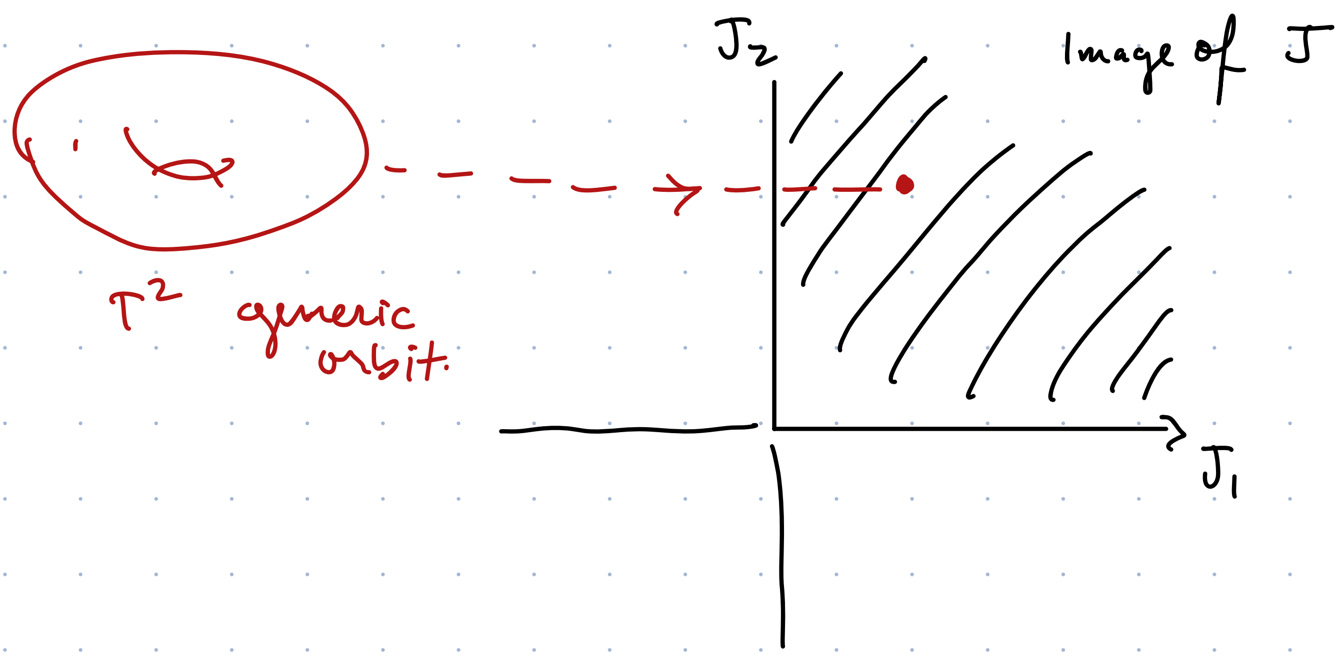
$$= d(z)$$

f^n generates rotation on z-axis



3) $\left(\mathbb{R}^2, dp_1\wedge dx_1, J_1 = \frac{1}{2}(x_1^2+p_1^2)\right) \times \left(\mathbb{R}^2, dp_2\wedge dx_2, J_2 = \frac{1}{2}(x_2^2+p_2^2)\right)$

$$\mathbb{R}^2 \times \mathbb{R}^2 \xrightarrow{J=(J_1,J_2)} \mathbb{R}^2$$



4) $(S^2, a\omega) \times (S^2, b\omega) \xrightarrow{(J_1, J_2)} \mathbb{R}^2$

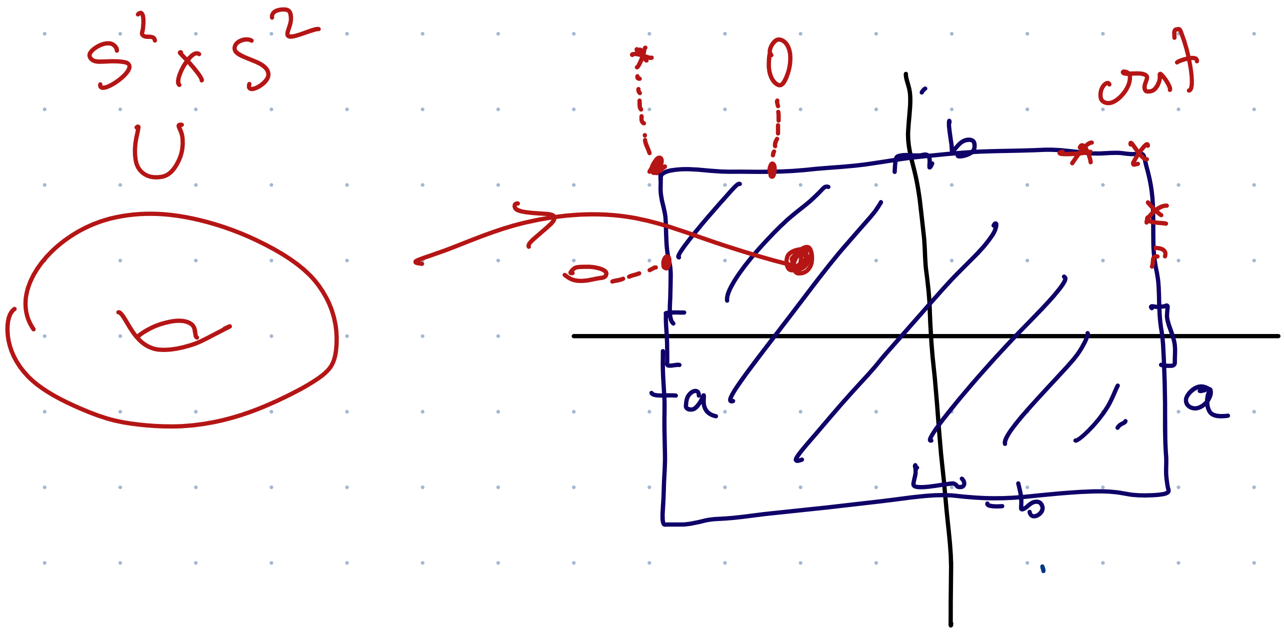
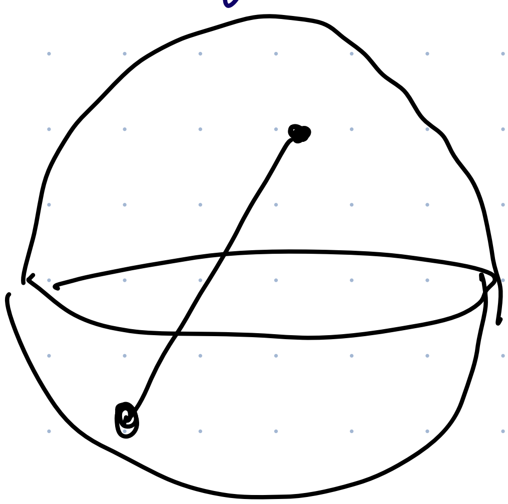


Image J

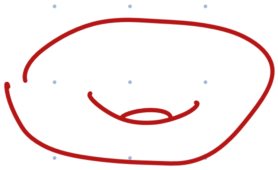
5) Spherical Pendulum:

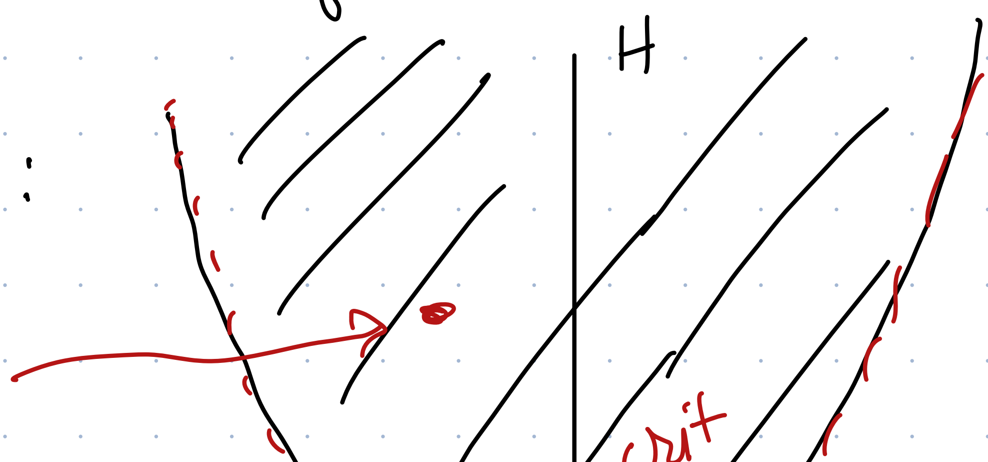
$(T^*S^2, H = K.E. + V_{grav})$

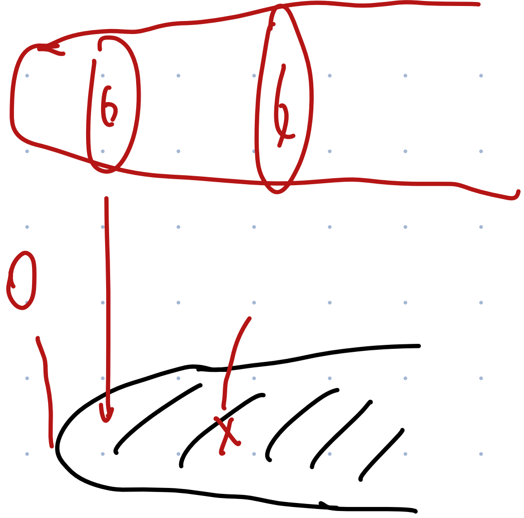


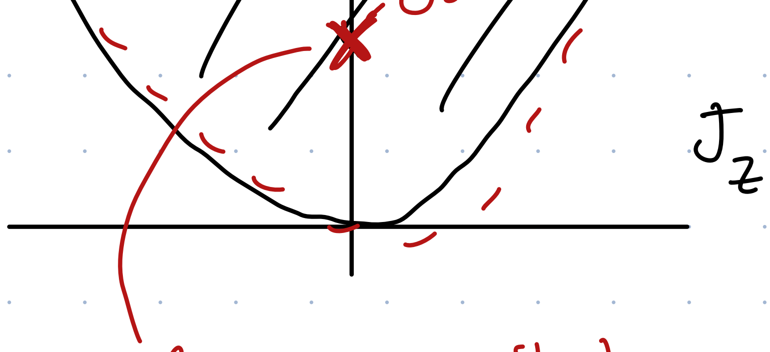
$J = (H, J_z)$ two conserved quantities

Image of J :

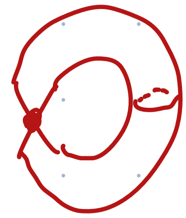








fibre over crit pt is pinched torus



Complete integrability $\Rightarrow$ separation of variables $(J_1, \dots, J_n, \underbrace{\theta_1, \dots, \theta_n}_{\text{angular variables for tori (or } \mathbb{R}^n)})$

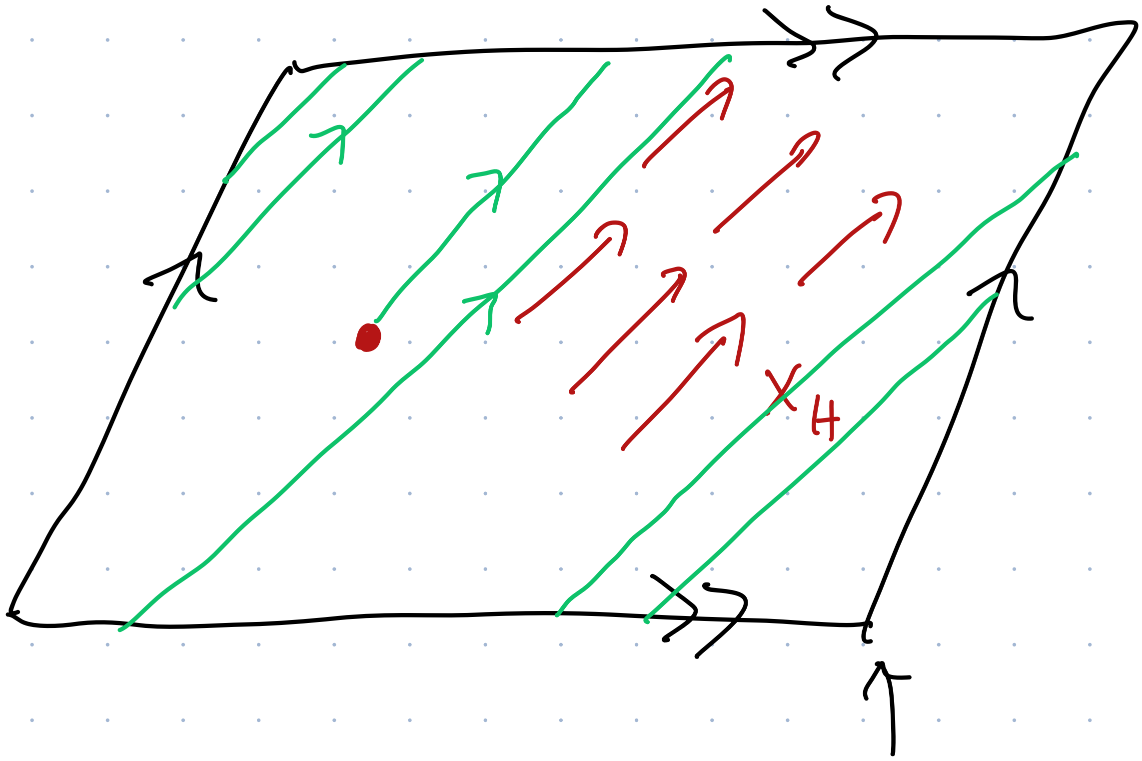
Action - Angle coordinates

$$\omega = \sum_{i=1}^{n} dJ_i \wedge d\theta_i$$

Importance of complete integrability:

obtain motion along $J^{-1}(c)$ fibres which are usually <u>tori</u> (if J proper).

Furthermore, motion on torus is linear

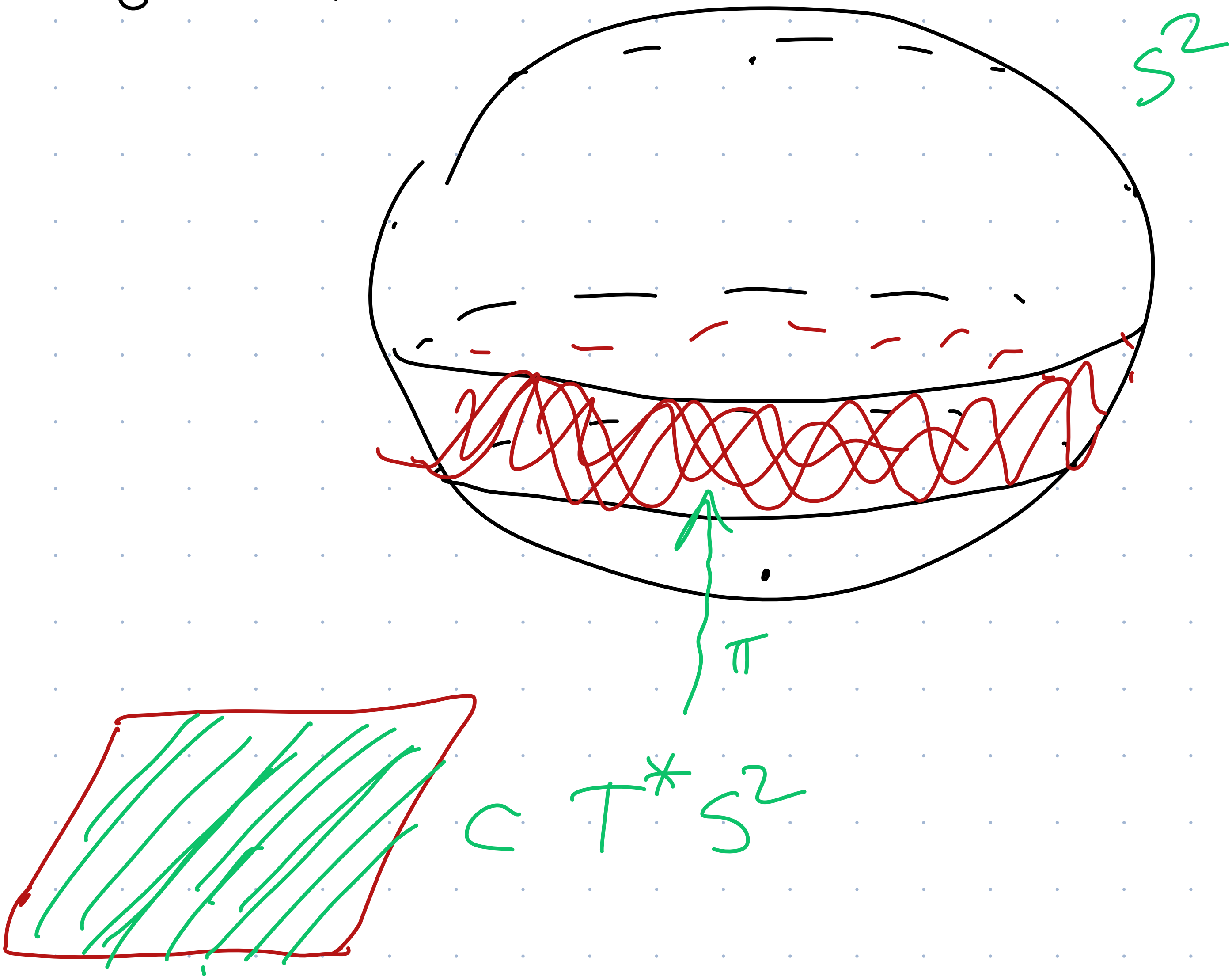


X_H Ham. flow is constant along $J^{-1}(c)$

torus with fixed (J_1, J_2)

leads to <u>quasi-periodic motion.</u>

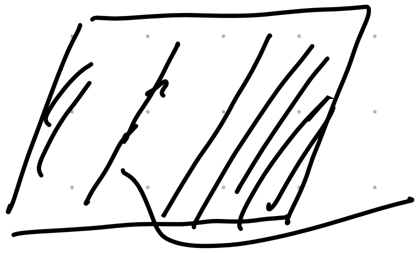
e.g. spherical pendulum



Heisenberg: — quasi-periodic motion — quantized energy } combine

<u>A. Connes</u> Non-commutative geometry

In an integrable system, quasi-periodic motion $\gamma(t)$



$\gamma : \mathbb{R} \to (M, \omega)$

$\searrow \quad \nearrow$

$J^{-1}(c)$ torus

Any f^n on phase space $f \in C^\infty(M, \mathbb{R})$

can be evaluated along traj:

"chord of n notes"

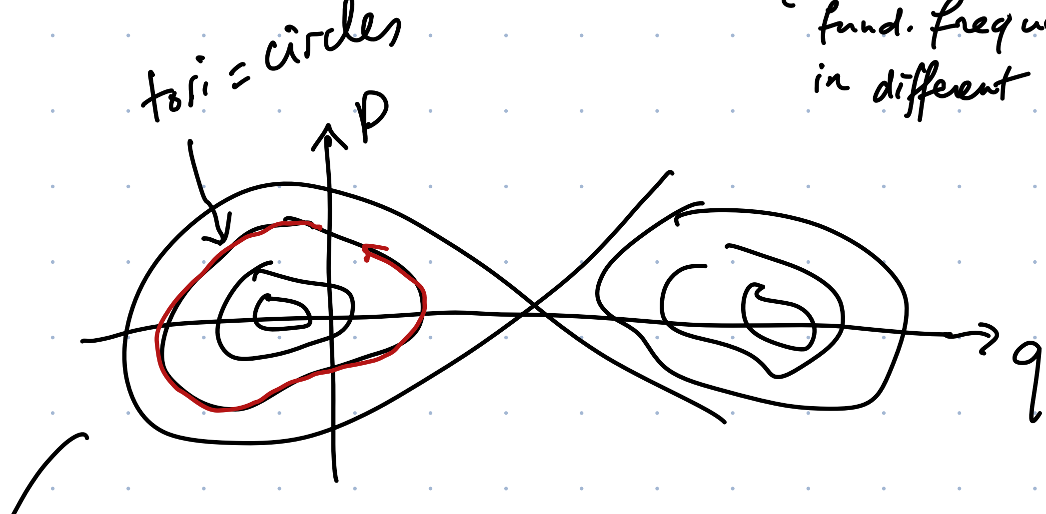
$$f(t) = \sum_{n \in \mathbb{Z}^n} f_n \, e^{2\pi i (n \cdot \nu) t}$$

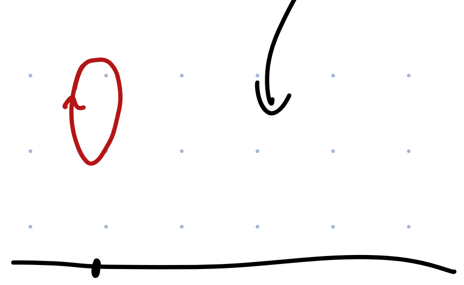
almost periodic f^{ns}

$n = (n_1, \dots, n_n) \in \mathbb{Z}^n$
$\nu = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n_{>0}$

fund. frequencies in different torus dir.

e.g. $n=1$





note: available frequencies form a group:

$$\mathbb{Z}\nu_1 \oplus \mathbb{Z}\nu_2 \oplus \cdots \oplus \mathbb{Z}\nu_n \xrightarrow{f_n} \mathbb{C}$$

Fourier transform is a complex f^n

s.t. $f_n = \overline{f_{-n}}$ (assuming f real)

Heisenberg: Due to quantization, available freq. for observables (E_x or electric mag. moment...) do not form a group!

Standard Fourier analysis not right tool (Gamelan) anharmoni

Instead, consider a groupoid