Rigid body dynamics:

(X, g) Riem n-mfld "space"

$P \circlearrowleft SO(n)$ Principal frame bundle

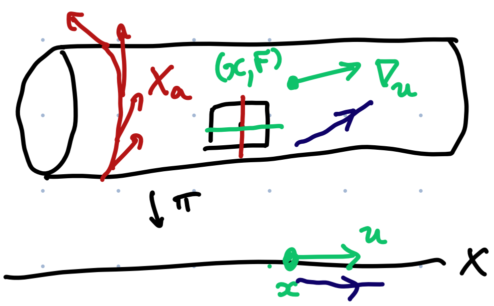
$$P \xrightarrow{\pi} X$$

infinitesimal rotations

$$0 \to \underbrace{P \times \mathfrak{so}(n)}_{V} \to TP \to \pi^* TX \to 0 \qquad \{a \in \mathbb{R}^{n \times n} : a^T = -a\}$$

∇

V vertical subbundle with basis $\{X_a : a \in \mathfrak{so}(n)\}$



$a \in \mathfrak{so}(n)$

$e^{ta} \in SO(n)$

$$g_P = g_X^{\nabla} \overset{\perp}{\oplus} K_{\mathfrak{so}(n)}$$

$\rightsquigarrow T_g = \frac{1}{2} g^{ij} p_i p_j$

$a \in \mathfrak{so}(n)$ $n \times n$ skew matrix

to make it pos. def.

$$K(a_1, a_2) = -(n-2)\, \mathrm{Tr}(a_1 a_2) \underset{n=3}{=} \mathrm{Tr}(a_1 a_2)$$

example of pos def. metric on $\mathfrak{so}(n)$ (isotropic body, Moment of inertia $= c \cdot \mathbb{1}$)

(K, ∇^{LC}) defines a Riemannian metric on P

$$TP = V \oplus H$$

$V \leftarrow \mathfrak{so}(n) \leftarrow K$, $H \leftarrow TX \leftarrow g_X$

metric defines K.E. (quadratic f^n on T^*P)

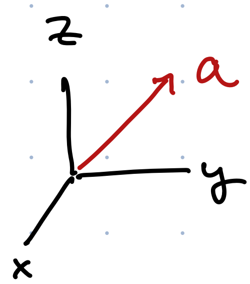
$\Rightarrow$ free particle motion on P (motion of rigid body)

$so(n)$ Lie algebra of skew-symmetric matrices.

$\parallel$

span $(\, \Sigma_{ij} \quad i<j \,)$ $\quad$ Σ_{ij} infinitesimal rotation in $x_i - x_j$ plane

$$\Sigma_{ij} = \begin{array}{c|c} & \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{array}{c} i \\ \\ j \end{array} & \begin{pmatrix} \cdots & 0 & & -1 & \\ & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 0 & \end{pmatrix} \end{array}$$



<u>for $n=3$</u> :

$$\Sigma_{12} = \Sigma_3 = \begin{pmatrix} & -1 & \\ 1 & & \\ & & 0 \end{pmatrix}$$ z-axis rot

$$-\Sigma_{13} = \Sigma_2 = \begin{pmatrix} & & 1 \\ & 0 & \\ -1 & & \end{pmatrix}$$ y-axis rot

$$\Sigma_{23} = \Sigma_1 = \begin{pmatrix} 0 & & \\ & & -1 \\ & 1 & \end{pmatrix}$$ x-axis

basis for $so(3)$

Inf. rotation about $\vec{a} = (a_1, a_2, a_3)$

would be $\quad a_1 \Sigma_1 + a_2 \Sigma_2 + a_3 \Sigma_3$

$$so(3) \xrightarrow{\exp} SO(3)$$ surjective, not injective.

$$\vec{a} \cdot \vec{\Sigma} \longrightarrow e^{\vec{a}\cdot\vec{\Sigma}}$$

Note: $e^{t\,\vec{a}\cdot\vec{\Sigma}}$ $\quad t \in \mathbb{R}$ is a hom $(\mathbb{R}, +) \longrightarrow SO(n)$

$$e^{(t_1+t_2)X} = e^{t_1 X} e^{t_2 X}$$

Lie bracket on $\mathfrak{so}(n)$ $\qquad e^X e^Y \neq e^{X+Y}$

given $X, Y \in \mathfrak{so}(n)$ $\qquad e^{tX} e^{tY} \neq e^{tY} e^{tX}$

measure failure: $\quad e^{tX} e^{tY} e^{-tX} e^{-tY} = \gamma(t)$ path in $SO(n)$, $\gamma(0) = \mathbb{1}$.

expanding:

$$\left(1 + tX + \tfrac{t^2}{2}X^2 \cdots\right)\left(1 + tY + \tfrac{t^2}{2}Y^2 \cdots\right)\left(1 - tX + \tfrac{t^2}{2}X^2 \cdots\right)\left(1 - tY + \tfrac{t^2}{2}Y^2 \cdots\right)$$

$$= \mathbb{1} + t\,(0) + t^2\Big(\cancel{(X^2 + Y^2)} + \cancel{XY} - \cancel{X^2} - \cancel{XY}$$
$$- YX - \cancel{Y^2} + XY \Big) + \cdots$$

first nontrivial term: $[X, Y] = XY - YX$.

<u>Lemma:</u> if $[X, Y] = 0$, i.e. inf. symmetries commute,

then $\quad e^{tX} e^{sY} = e^{sY} e^{tX}$

and we obtain $\mathbb{R} \times \mathbb{R}$ action via

$$(s, t) \overset{\varphi}{\longmapsto} e^{tX} e^{tY}$$

$$\varphi\big((s_1, t_1) + (s_2, t_2)\big) = \varphi(s_1, t_1)\, \varphi(s_2, t_2)$$

$$U(2) = \{ g \in \mathbb{C}^{2\times 2} \text{ s.t. } g\bar{g}^T = \mathbb{I} \}$$

$$\mathfrak{u}(2) = \{ \dot{g} \text{ s.t. } \dot{g}\bar{g}^T + g\dot{\bar{g}}^T = 0 \Big|_{g=\mathbb{I}} \}.$$

$$= \{ X : X + \bar{X}^T = 0 \}$$

skew-Hermitian

$$= -i \{ X : \bar{X}^T = X \}$$ Hermitian matrices.

Real vector space of dim 4

$$\sigma_0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

Pauli basis

$SU(2) \subset U(2)$ unitary matrices w/ det $= 1$.

$\mathfrak{su}(2) \subset \mathfrak{u}(2)$ skew-herm. matrices w/ $\mathrm{Tr} = 0$.

$$\mathfrak{su}(2) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \cong \mathbb{R}^3$$

$$[\sigma_1, \sigma_2] = 2i\sigma_3 \qquad \left[\frac{-i\sigma_1}{2}, \frac{-i\sigma_2}{2}\right] = \frac{-2i\sigma_3}{4} = \frac{-i\sigma_3}{2}$$

$$\hat{\sigma}_1 = -\frac{i\sigma_1}{2} \qquad [\hat{\sigma}_1, \hat{\sigma}_2] = \hat{\sigma}_3$$

$$\hat{\sigma}_2 = -\frac{i\sigma_2}{2} \qquad \text{c.p.}$$

$$\hat{\sigma}_3 = -\frac{i\sigma_3}{2} \qquad \text{c.p.}$$

$$\mathfrak{so}(3) = \langle \Sigma_1, \Sigma_2, \Sigma_3 \mid [\Sigma_1, \Sigma_2] = \Sigma_3 + \text{c.p.} \rangle$$

$$\uparrow \qquad\qquad \updownarrow \quad \updownarrow \quad \updownarrow$$

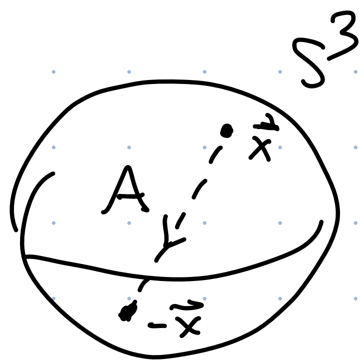
$$\mathfrak{su}(2) = \langle \hat\sigma_1, \hat\sigma_2, \hat\sigma_3 \mid [\hat\sigma_1, \hat\sigma_2] = \hat\sigma_3 + \text{c.p.} \rangle.$$

<u>Prop:</u> This induces a homomorphism of groups

$$SU(2) \xrightarrow{\varphi} SO(3). \qquad e^{\vec a \cdot \hat{\vec\sigma}} \longmapsto e^{\vec a \cdot \vec\Sigma}$$

<u>Thm:</u> If $\mathrm{Lie}(G) = \mathfrak{g}$, $\mathrm{Lie}(H) = \mathfrak{h}$, $\varphi : \mathfrak{g} \to \mathfrak{h}$

<u>and</u> G is simply-connected, then obtain $e^{\varphi} : G \to H$ hom. of groups.



$SU(2) \cong S^3$ is simply connected

$SO(3) \cong S^3 / \mathbb{Z}_2 \leftarrow$ antipodal A

$\cong \mathbb{RP}^3$

$$\begin{array}{c} S^3 \\ \big\downarrow {\scriptstyle 2:1} \\ \mathbb{RP}^3 \end{array} \quad \text{double cover.}$$

$\pi_1(\mathbb{RP}^3) = \mathbb{Z}/2\mathbb{Z}$

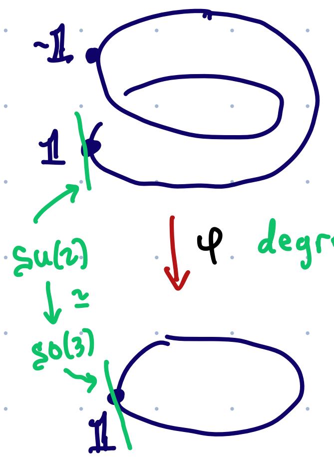
$$e^{t \hat\sigma_3} = e^{t \frac{-i}{2} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}} = \begin{pmatrix} e^{-i\frac{t}{2}} & \\ & e^{i\frac{t}{2}} \end{pmatrix}$$

at time 2π $\begin{pmatrix} e^{-i\pi} & \\ & e^{i\pi} \end{pmatrix} = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \in SU(2)$

$$e^{t\Sigma_3} = e^{t\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

at $t = 2\pi$ $\quad e^{2\pi\Sigma_3} = \mathbb{1}$

$SU(2)$



$\downarrow \varphi$ degree 2 covering space

$SO(3)$

$SO(n)$ $\quad n \geq 3$ $\qquad \pi_1(SO(n)) \cong \mathbb{Z}/2\mathbb{Z}$

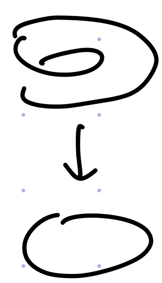
$n = 2$ $\quad SO(2) = U(1)$ $\quad \pi_1(SO(2)) = \mathbb{Z}$

$n = 1$ $\quad SO(1) = \{1\}$

double cover of $SO(n)$ is a group : $Spin(n)$

$n = 2$

$$\begin{array}{ccc} Spin(2) & \xrightarrow{2:1} & SO(2) \\ \| & & \| \\ U(1) & & U(1) \\ z & \longmapsto & z^2 \end{array}$$



$n = 3$

$$\begin{array}{ccc} Spin(3) & \longrightarrow & SO(3) \\ \| & \nearrow_{\varphi} & \\ SU(2) & & \end{array}$$

$n=4$ $\qquad \mathrm{Spin}(4) \longrightarrow SO(4)$

$\qquad\qquad \mathrm{Spin}(4) = SU(2)\times SU(2)$

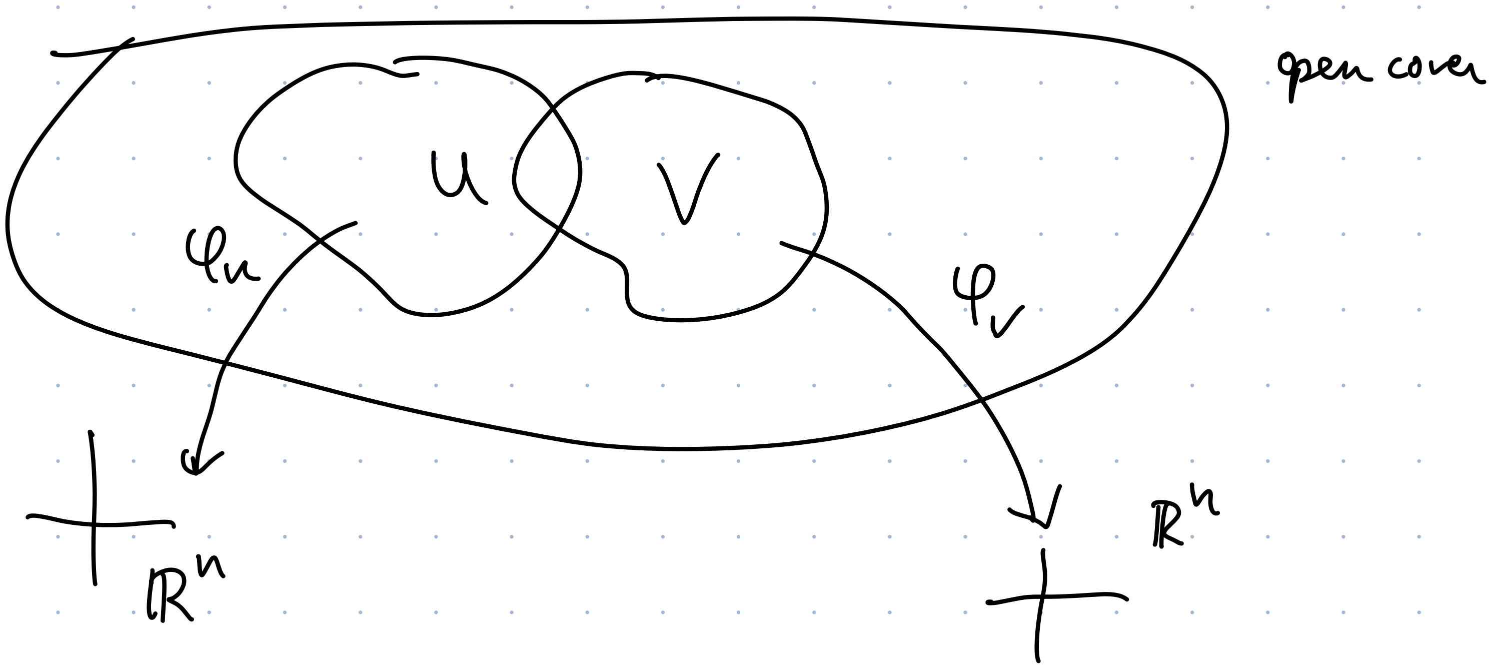
$n \geq 5$ $\qquad \mathrm{Spin}(n)$ $\qquad$ (to define it, use Clifford alg.)

3d objects (chair) : $SO(3)$ $\quad$ as $\quad$ electron : $SU(2)$.

$$1 \to \mathbb{Z}/2 \to SU(2) \xrightarrow{\varphi} SO(3) \to 1$$

$$1 \to SO(3) \longrightarrow O(3) \longrightarrow \mathbb{Z}/2 \to 1$$

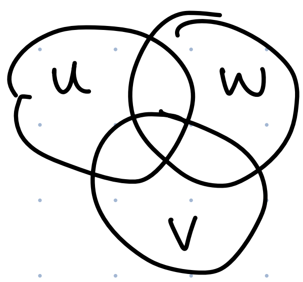
to build frame bundle, we could locally trivialize:



$$P|_U = U \times SO(n) \longrightarrow P|_V = V \times SO(n)$$

glue using $g_{UV} : U \cap V \longrightarrow SO(n)$

$$F_U \longmapsto g_{UV} \cdot F_U$$



$g_{WU}\, g_{VW}\, g_{UV} = \mathbb{1}$ in $U \cap V \cap W$

Principal bundle of frames is defined in open cover $\mathcal{U} = (U_i)_{i \in I}$

by transition f^{ns} $g_{ij} : U_i \cap U_j \to SO(n)$

st. $g_{ki}\, g_{jk}\, g_{ij} = 1 \quad \forall i, j, k$

1-cocycle $[g_{ij}] \in H^1(X, SO(n))$.

Q.: We know 2:1 cover of $SO(n)$ is $Spin(n)$. Is it possible to form a 2:1 cover of frame bundle, to create a $Spin(n)$ principal bundle?

(Spin frame bundle)

Given an oriented Riemannian mfld with frame bundle P, If it admits a Spin frame bundle we say (X, g, or) is <u>Spin</u> (or has a spin structure). There may be many such Spin frame bundle (form a torsor for $H^1(X, \mathbb{Z}_2)$)

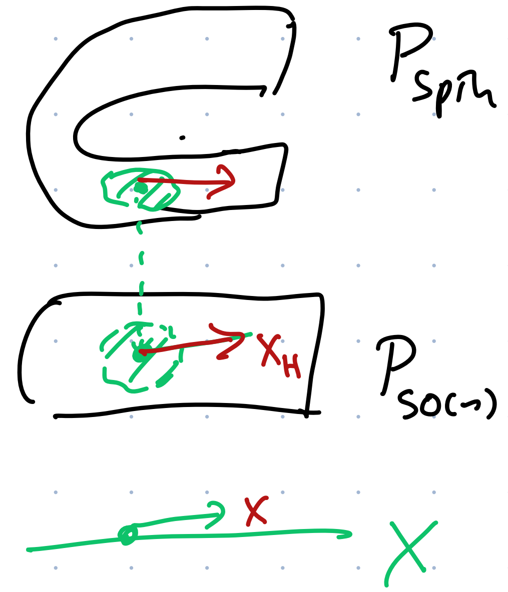
The sequence $\mathbb{Z}_2 \to Spin(n) \to SO(n)$

gives Long exact seq $H^1(X, \mathbb{Z}_2) \to H^1(Spin(n)) \to H^1(SO(n)) \xrightarrow{w_2} H^2(X, \mathbb{Z}_2)$

Second Stiefel-Whitney class.

Suppose we choose a spin str. for (X, g, or)

$$\begin{array}{c} P_{spin} \circlearrowleft Spin(n) \\ \downarrow \\ X \end{array}$$



P_{spin} inherits the Levi-Civita conn

$$TP_{Spin} \overset{D^{LC}}{=} \underset{TX}{H} \oplus \underset{so(n)}{V}$$

$\Rightarrow$ same choice of K gives a Hamiltonian here

$\Rightarrow$ dynamics of spin particle on P_{spin}

(360° rot. not $\mathbb{1}$ rather 720° is)

$$SO(2n) \supset U(n)$$

$$\begin{tikzcd}
P_{SO} \supset P_{U(n)} \arrow[r, mapsto, "c_1"] \arrow[dr, bend right, "w_2"'] & H^2(X,\mathbb{Z}) \arrow[d, "\text{mod } 2"] \\
& H^2(X,\mathbb{Z}_2)
\end{tikzcd}$$

$$w_2 = c_1 \mod 2$$

$$1 \to SO(n) \longrightarrow O(n) \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 1$$

$$g_{ij} : U_{ij} \to O(n)$$

$$[\tilde{g}_{ij}] \in H^1_X(SO(n)) \longrightarrow [g_{ij}] \in H^1(X, O(n)) \longrightarrow H^1(X, \mathbb{Z}/2\mathbb{Z})$$

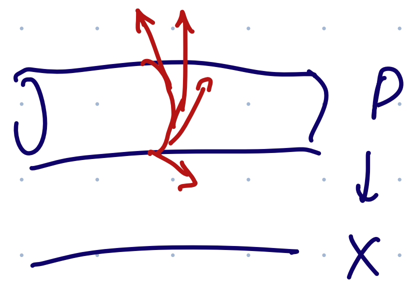
$$[g_{ij}] \longmapsto w_1(P_{O(n)})$$

$$\begin{array}{c} P \circlearrowleft SO(n) \\ \downarrow \\ X \end{array}$$

Principal frame bundle

P config. space

action provides vertical v.fields



$$\rho : \mathfrak{so}(n) \longrightarrow V \subseteq TP$$
$$a \longmapsto V_a \in \mathfrak{X}(P)$$

$\Rightarrow$ $\mathfrak{so}(n) \longrightarrow$ linear f^{ns} on T^*P

i.e.

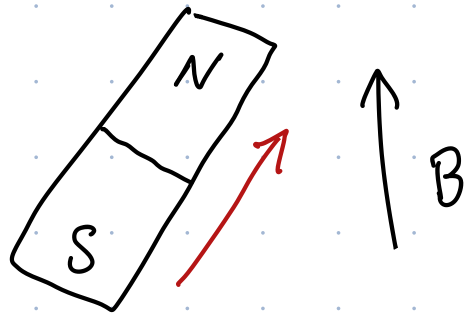
$$\begin{tikzcd} T^*P \arrow[r, "J"] \arrow[dr, bend right, "J_a"'] & \mathfrak{so}(n)^* \arrow[d, "a \in \mathfrak{so}(n)"] \\ & \mathbb{R} \end{tikzcd}$$

(angular) momentum map

J_a Hamiltonian generates (cotangent lift of) V_a

Magnetic Moment:

Magnetic Dipole:



A magnetic field B exerts a <u>torque</u> on a magnetic dipole

i.e. the rigid body has an intrinsic "magnetic dipole moment" which combines with $\vec{B}$ to produce rotational force

i.e. body has m mag. moment which, combined with B, gives a f^n on T^*P
(add this f^n to Hamiltonian)

Recall: $B \in \Omega^2(X) = \Gamma(\Lambda^2 T^* X)$

using g_x Riemannian metric $g_x : TX \to T^*X$

$$T_x \xrightarrow{B} T_x^* \xrightarrow{g_x^{-1}} T_x$$

$$v \longmapsto i_v B = B(v, -)$$

$g^{-1}B : T_x \to T_x$ endom. of T.

this is an infinitesimal rotation! $A \in \mathfrak{so}(T)$

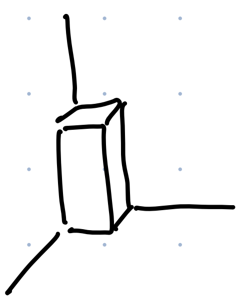
$$g(Ax, y) + g(x, Ay) = 0$$

$$\underbrace{g(g^{-1}Bx, y)}_{\substack{g(g^{-1}Bx)(y) \\ \| \\ B(x,y)}} + \underbrace{g(x, g^{-1}By)}_{\substack{g(g^{-1}By, x) \\ \| \\ B(y,x)}} = 0$$

$$B(x,y) + B(y,x) = 0$$

$$\left[(V, \langle\,,\,\rangle) \qquad \mathfrak{so}(V) \overset{\langle,\rangle}{\cong} \Lambda^2 V^* = \Lambda^2 V \right]$$

$\uparrow$ nondeg. symm.



$\mathbb{R}^n$ standard frame

$$T^*P \xrightarrow{J} \mathfrak{so}(n)^*$$

angular momentum.

Def^n The magnetic moment of a body (in standard frame in $\mathbb{R}^n$) is an element $m \in \mathfrak{so}(n)$

($n=3$ coincides w/ a vector)

Given a config. $(x, F_x) \in P$

we can transport $m \in \mathfrak{so}(n)$ to $m_x \in \mathfrak{so}(T_x)$

$$m \circlearrowright \mathbb{R}^n \xrightarrow[\cong]{F_x} T_x \circlearrowleft m_x = F_x\, m\, F_x^{-1}$$

now $\quad g^{-1}B, \; m_x \in \mathfrak{so}(T_x)$

so $\quad H_B = \underset{\substack{\uparrow \\ \text{coupling constant.}}}{\overset{\text{change}}{qc}} \langle g^{-1}B, m_x \rangle_g \in C^\infty(P) \xrightarrow{\pi^*} C^\infty(T^*P)$

Ham. for interaction mag. mom. and B.

$B = B_{ij}\, dx^i \wedge dx^j \qquad m_x = m_i^j\, dx^i \otimes \frac{\partial}{\partial x^j}$ skew

$g = g_{ij}\, dx^i dx^j$

$$g^{kp} g_{iq} \, g^{ij} B_{jk} \, m^q_p = g^{kp} B_{qk} m^q_p$$ ← Hamiltonian.

Note: ① electron is a charged particle and influenced by B

→ in standard way (magnetic deform. of symp. form.)

influencing linear momentum $\vec{p}$

influences the angular momentum.

→ It also has a magnetic dipole moment

② Neutron does not require mag. deform. but it does have magnetic dipole moment.

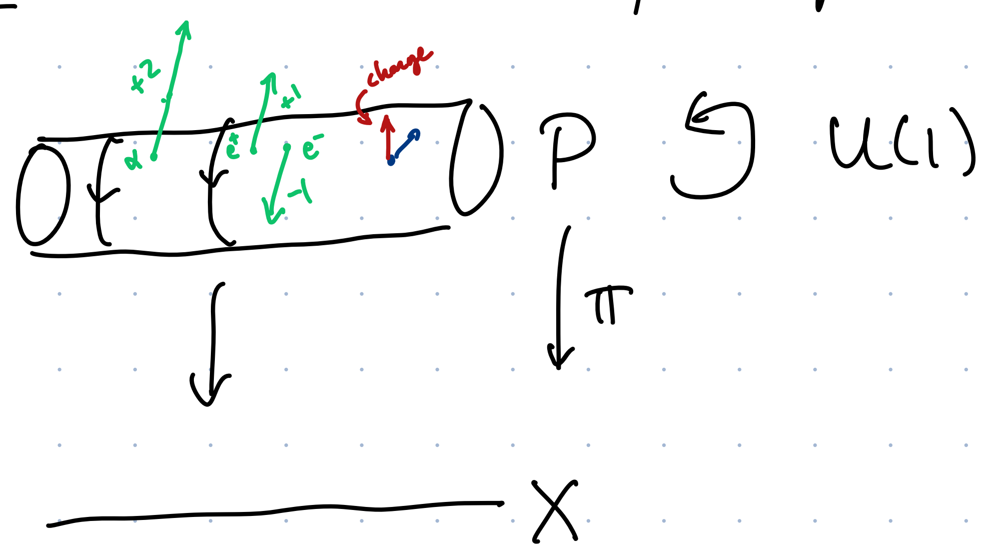
Gauge Theory

U(1) gauge theory for $-$ magnetism
($-$ electromagnetism)

Can we avoid magnetic deformation and obtain a Hamiltonian f^n capturing B?

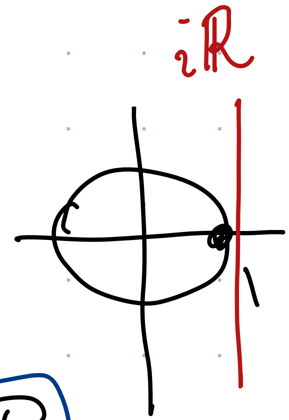
Clever sol^n: Kaluza-Klein, Weyl

Idea: consider a principal U(1)-bundle



The motion of magnetically charged particle in X is simply geodesic motion on P. (Need Hamiltonian on P)

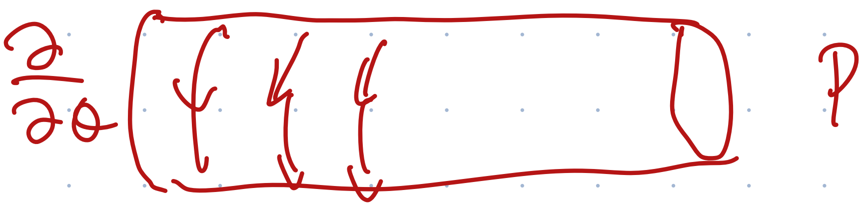
The Hamiltonian H (which includes g_X and B) should be $U(1)$-invariant.



Why?: $U(1)$ action on P gives momentum

$$\mu : T^*P \longrightarrow \mathrm{Lie}(U(1))^* = \mathbb{R}$$

μ is the f^n on T^*P whose ham. v.f. is action v.field



The fact that H is $U(1)$-inv't means

$$\partial_\theta H = 0$$

$$\Longleftrightarrow \{H, \mu\} = 0$$

$\Rightarrow$ μ is conserved in dynamics gen. by H

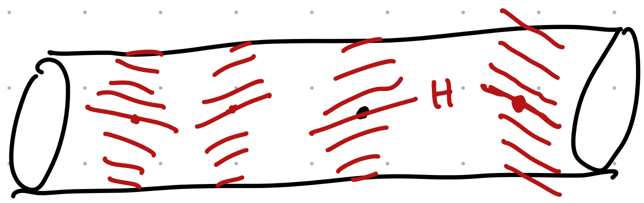
$\Rightarrow$ μ is the charge

To build Ham: • choose a conn. ∇ on P

i.e. $H \subset TP$ subbundle

which is - horizontal, i.e. $H \oplus V = TP$

- $U(1)$ - invariant



$$\pi^* TX \qquad V$$

$$\cong \qquad \|$$

Once horiz. is chosen, get $TP = H \oplus \mathbb{R}$

$\Rightarrow$ Riem. metric on P: $g_p = g_X \overset{\perp}{\oplus} \underset{\uparrow k}{d\theta^2}$ $\Rightarrow$ Hamiltonian $f_{g_p^{-1}}$

If we locally trivialize P: cover of X $\{U_i\}$

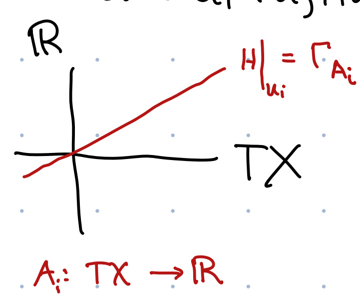
$$\left[\begin{array}{l} P|_{U_i} = U_i \times U(1) \\ \downarrow \\ P|_{U_j} = U_j \times U(1) \end{array}\right.$$

$g_{ij} : U_i \cap U_j \to U(1)$

$g_{ik} = g_{ij} g_{jk}$

over $U_i \cap U_j \cap U_k$

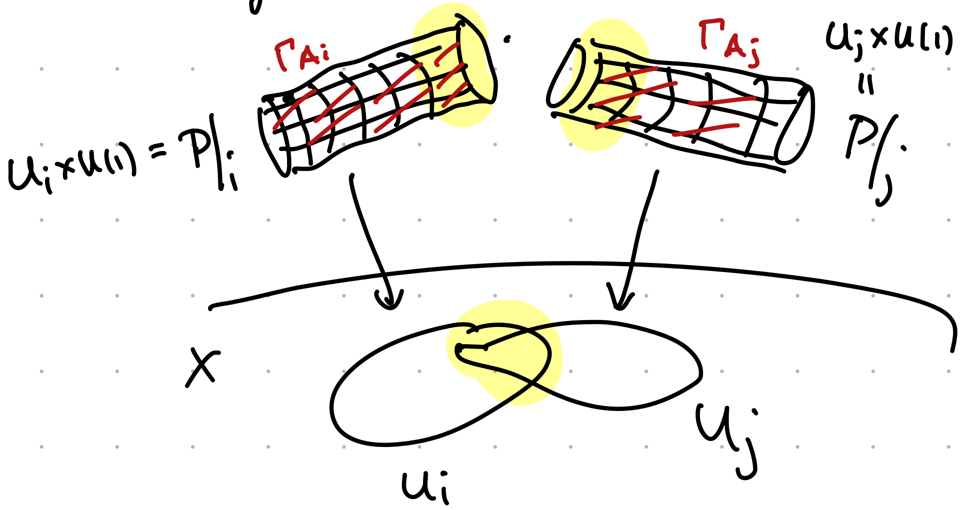
$TP|_{U_i} = TX \oplus \mathbb{R}$



locally we choose $A_i \in \Omega^1(U_i, \mathbb{R})$

in order for Γ_{A_i} and Γ_{A_j} to coincide

on $P\big|_{U_i \cap U_j}$



gluing $\underset{U_i \times U(1)}{\overset{\parallel}{P|_{U_i}}} \xrightarrow{\varphi_{ij}} \underset{U_j \times U(1)}{\overset{\parallel}{P|_{U_j}}}$

$$\varphi_{ij}(x, e^{i\theta}) = (x, g_{ij}(x) \cdot e^{i\theta})$$

$$d\varphi_{ij} : \begin{array}{ccc} T_{(x,e^{i\theta})} P_i & \longrightarrow & T_{(x, g_{ij} e^{i\theta})} P_j \\ \cup & & \cup \\ \Gamma_{A_i} & \longmapsto & \Gamma_{A_j} \end{array}$$

$d\varphi_{ij}$ sends Γ_{A_i} to Γ_{A_j} $\Longleftrightarrow$ $i(A_j - A_i) = g_{ij}^{-1} dg_{ij}$

Result:

The selection of 1-forms $\{A_i \in \Omega^1(U_i, \mathbb{R})\}$

s.t. $i(A_j - A_i) = g_{ij}^{-1} dg_{ij}$

defines a connection on P

The curvature of this connection is a measure of failure of Horizontals to be involutive:

$\Omega^2(X,\mathbb{R}) \ni B$

horizontal lift uses $H \xrightarrow[\cong]{\pi_*} TX$

$$B(X,Y) = [X^H, Y^H] - [X,Y]^H$$

Prop: $B = dA_i \quad \left(= dA_j \text{ since } g^{-1}dg \text{ is closed} \right)$

Q.: Under what condition on B is it possible to find

- P
- $\nabla \quad (\{A_i\})$

st. curvature$(\nabla) = B$

A: B must have integral periods, i.e

$[B] \in H^2(X,\mathbb{R})$ must be in the image of
$H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathbb{R})$

in such a case P, ∇ exist.

(Kostant's prequantization theorem)

So modern view of magnetism:

To describe a magnetic system, need

① $\begin{array}{c} P \circlearrowleft U(1) \\ \downarrow \\ X \end{array}$

② Connection ∇ on it.

The corresponding curvature is usual B

To get EM only need to replace X by $X \times (\mathbb{R}, t)$ $\quad dt^2 - g_X =$ Lorentz. metric

$$\begin{array}{c} P \circlearrowleft U(1) \\ \downarrow \\ X \times \mathbb{R} \end{array}, \quad \nabla = \{A_i\}$$

$$F \in \Omega^2(X \times \mathbb{R})$$

$$\boxed{F = B_{ij}\, dx^i \wedge dx^j + E_i\, dx^i \wedge dt}$$

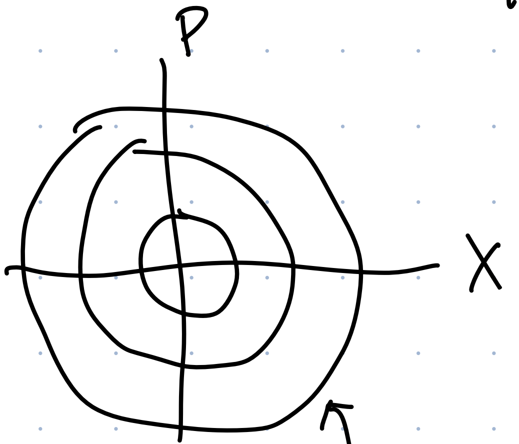
$$\underbrace{TP}_{\text{Lorentzian}} \overset{\nabla}{=} \underbrace{TX \oplus \mathbb{R}\partial_t}_{\substack{\text{Lorentz} \\ dt^2 - dx^2}} \oplus \underbrace{\mathbb{R}\partial_\theta}_{-d\theta^2}$$

Geodesic flow $\Longrightarrow$ Lorentz eqn of motion for charged particle (relativistic)

Idea (Futile)

$$\text{Classical Ham. system} \xrightarrow[\text{Quantiz.}]{\text{Geometric}} \text{Quantum system}$$

Bohr model quantized oscillator



$E_n = n \cdot h$

integer multiples of a basic unit.

Classical system involves symplectic form ω on phase space $\longrightarrow$ Pre-Quantiz.

ω = curv. of Principal $U(1)$ bdle

$\longrightarrow$ Quantum system.

Bohr-Sommerfeld quantiz. rules

$$\text{Principal } U(1)\text{-bundles}/\text{iso} \ni P$$

$$\begin{tikzcd}
\text{Principal } U(1)\text{-bundles}/\text{iso} \arrow[d, leftrightarrow] & \ni & P \arrow[d, mapsto] \\
H^2(X, \mathbb{Z}) & \ni & c_1(P)
\end{tikzcd}$$

complex mflds $\quad \mathrm{Pic} = \text{Principal } \mathbb{C}^* \text{ bundles}$

$$\begin{tikzcd}
H^1(X, \mathcal{O}^*) \arrow[d] \\
H^2(X, \mathbb{Z}) \arrow[d] \\
H^2(X, \mathbb{C}) \ni \omega
\end{tikzcd}$$