Rigid body dynamics:

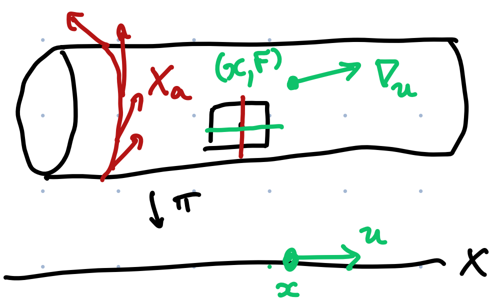
(X, g) Riem n-mfld "space"

$$\begin{array}{c} P \circlearrowleft SO(n) \\ \downarrow \pi \\ X \end{array}$$ Principal frame bundle

infinitesimal rotations

$$0 \to \underbrace{P \times \mathfrak{so}(n)}_{V} \to TP \xrightarrow{\nabla} \pi^* TX \to 0 \qquad \{a \in \mathbb{R}^{n\times n} : a^T = -a\} = \mathfrak{so}(n)$$

V vertical subbundle with basis $\{X_a : a \in \mathfrak{so}(n)\}$



$a \in \mathfrak{so}(n)$

$e^{ta} \in SO(n)$

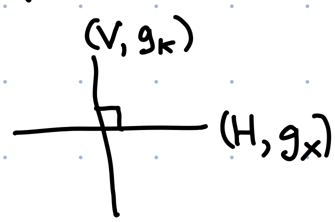
P inherits a <u>connection</u> (Levi-Civita) ∇ which enables the lifting of tgt vectors from X to P.

This means $TP = \overset{\mathfrak{so}(n)}{V} \oplus \overset{\nabla(TX)}{H}$ vertical + horizontal

We have a Riem. metric g_X on TX $\Rightarrow$ on H

We need to $\boxed{\text{choose} \quad K = \langle \cdot, \cdot \rangle}$ a pos.def. inner prod. on $\mathfrak{so}(n)$ ← moment of inertia of the body.

$\Rightarrow$ g_K on V



$\Rightarrow$ $g_P = g_K \oplus g_X$

$\Rightarrow \quad f_{g_p} = \frac{1}{2} g_p^{ij} P_i P_j \quad$ quadratic f^n on T^*P

$\Rightarrow$ Ham. flow generates Rigid body dynamics.

<u>Simplification:</u> Fix $x_0 \in X$ consider $P_{x_0} = \pi^{-1}(x_0)$.

config. space : $P_{x_0} \underset{\uparrow \text{ choose } F_0 \text{ frame}}{\cong} SO(n)$ (all possible frames in $\mathbb{R}^n$)

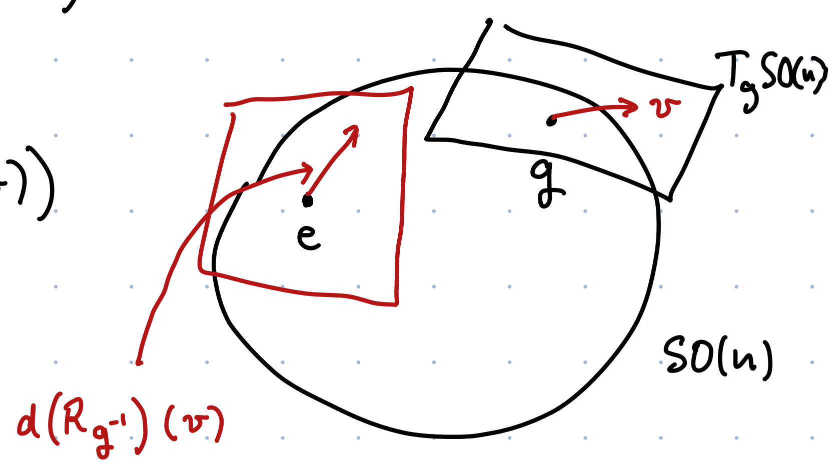
phase space: $M = T^* SO(n)$

$\underset{\underset{SO(n)}{\shortparallel}}{P_{x_0}}$ has free, transitive action by $SO(n)$ $\quad \overset{\overset{SO(n)}{\shortparallel}}{P_{x_0}} \circlearrowleft SO(n)$

$\Rightarrow \quad TP_{x_0} \cong P_{x_0} \times \mathfrak{so}(n)$

$(g, v) \longmapsto (g, dR_{g^{-1}}(v))$

$v \in T_g P_{x_0}$



Once we choose $K = \langle \cdot, \cdot \rangle$ on $\mathfrak{so}(n)$

we obtain a Riem. metric on P_{x_0}

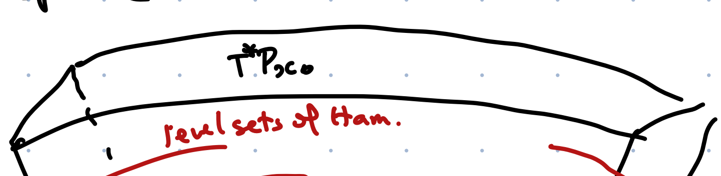
$R_g : SO(n) \xrightarrow{\cong} SO(n)$

$e \longmapsto g.$

This metric is invariant by right action by $SO(n)$.

$dR_g : T_e SO(n) \xrightarrow{\cong} T_g SO(n)$

<u>Consequence:</u> In phase space



$\circlearrowright SO(n)$

using the action

$$T^*SO(n) \circlearrowright SO(n)$$ by symmetries $\begin{cases} \text{of Phase space} \\ \text{of Hamiltonian} \end{cases}$

we have a map

$$J : T^*SO(n) \longrightarrow T_e^*SO(n) = \mathfrak{so}(n)^*$$
$$(g, p) \longmapsto R_g^* p \in \mathfrak{so}(n)^*$$

This map is an example of "momentum map" associated to a group of symmetries $(SO(n) \hookrightarrow T^* SO(n))$.

<u>Def:</u> G Lie group acting on (M, ω) Symplectic by symmetries:

$$\rho : M \times G \longrightarrow M$$
$$: (x, g) \longmapsto \rho_g(x)$$

satisfies $\boxed{\rho_g^* \omega = \omega}$ ☆

This gives an infinitesimal action by

$$\mathfrak{g} = Lie(G) = (T_e G, [\,,\,])$$

$$d\rho : \underset{a}{\overset{\mathfrak{g}}{\cup\!\!\!\!\!\;\; }} \longmapsto X_a \in \mathfrak{X}(M)$$

infinitesimal symmetry gen. by $a \in \mathfrak{g}$

$$\star \Rightarrow L_{X_a}\omega = 0$$
$$\parallel$$
$$d\, i_{X_a}\omega = 0$$

We say the action is <u>Hamiltonian</u> when

$\exists$ a moment map $J : M \longrightarrow \mathfrak{g}^*$

(J is a collection of functions J_a $a \in \mathfrak{g}$)

$$J_a = a \circ J$$

s.t. $X_{J_a} = X_a$

$$\begin{tikzcd} M \arrow[r, "J"] \arrow[dr, "J_a"'] & \mathfrak{g}^* \arrow[d, "a \in \mathfrak{g}"] \\ & \mathbb{R} \end{tikzcd}$$

i.e. $\boxed{i_{X_a}\omega = -dJ_a \quad \forall a}$

(we also require J is G-equivariant

$$G \circlearrowright M \xrightarrow{J} \mathfrak{g}^* \circlearrowleft G$$

Coadjoint action.

i.e. $$\boxed{\{J_a, J_b\} = J_{[a,b]}}$$

Since Riem. metric on P_{x_0} is $SO(n)$-invariant, it follows $\{J_a, f_{g^{-1}}\} = 0 \quad \forall a.$

$\Rightarrow$ if we use $f_{g^{-1}} = H$ as our Hamiltonian, All the J_a are preserved

$$\dot{J}_a = \{H, J_a\} = -\{J_a, H\} = -X_a(H) = 0$$

$\Rightarrow \{J_a : a \in \mathfrak{g}\}$ conserved f^{ns}

$\uparrow$ components of <u>Angular Momentum</u>.

<u>Conservation of angular momentum</u>.

Nöther Theorem

Another example of symmetries $\Longleftrightarrow$ conserved quantities.

<u>Minkowski spacetime:</u> $(\mathbb{R}, t) \times (\mathbb{R}^3, (x,y,z)) = M^4$

$$g = dt^2 - (dx^2 + dy^2 + dz^2)$$

<u>Phase space:</u> $T^*\mathbb{R}^4 \quad (t, E), \ (x, p_x, y, p_y, z, p_z).$

$$\omega = d\tau \wedge dt + dp_x \wedge dx + dp_y \wedge dy + dp_z \wedge dz.$$

quadratic f^n $\quad H = \frac{1}{2}(E^2 - (p_x^2 + p_y^2 + p_z^2))$

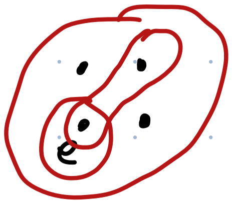
The metric has a large group of symmetries:

- translation $\mathbb{R}^4$ action on $\mathbb{R}^4$
  $(t, \vec{x}) \longmapsto (t + \Delta t, \vec{x} + \Delta\vec{x})$
- Lorentz transformations $O(1,3)$
  $\begin{pmatrix} t \\ \vec{x} \end{pmatrix} \longmapsto L \begin{pmatrix} t \\ \vec{x} \end{pmatrix}$

$$\langle Lx, Ly \rangle_{Mink} = \langle x, y \rangle_{Mink}.$$

In sum, $\quad O(1,3) \ltimes \mathbb{R}^4 = \mathcal{P}$ Poincaré group.

Warning: this is not connected: $O(1,3)$ has 4 connected components



Several subgroups may be used

$$\mathcal{P} \text{ acts on } M^4 \Rightarrow \operatorname{Lie}\mathcal{P} \xrightarrow{\text{infinitesimal action}} \mathfrak{X}(M^4)$$

$\Rightarrow$ $\mathcal{P}$ acts on T^*M^4 in a Hamiltonian fashion!

$$T^*M^4 \xrightarrow{\quad J \quad} \operatorname{Lie}(\mathcal{P})^*$$

$a \in \operatorname{Lie}(\mathcal{P})$

J_a = linear f^n on T^*M^4
corresp. to v.field $X_a \in \mathfrak{X}(M)$

Translations: $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$

Moment map components

$$\underbrace{E\ ,\quad P_x\ ,\quad P_y\ ,\quad P_z}_{\text{functions generating translational symmetries.}}$$

$$\left\{ H = \frac{1}{2}(E^2 - P_x^2 - P_y^2 - P_z^2)\ ,\ E \right\} = 0$$

Free hamiltonian

$$P_x = 0$$
$$P_y = 0$$
$$P_z = 0$$

$\Rightarrow$ $\left.\begin{array}{l} E \\ \vec{p} = (P_x, P_y, P_z) \end{array}\right\}$ conserved in the evol. of system.

E = total or relativistic energy

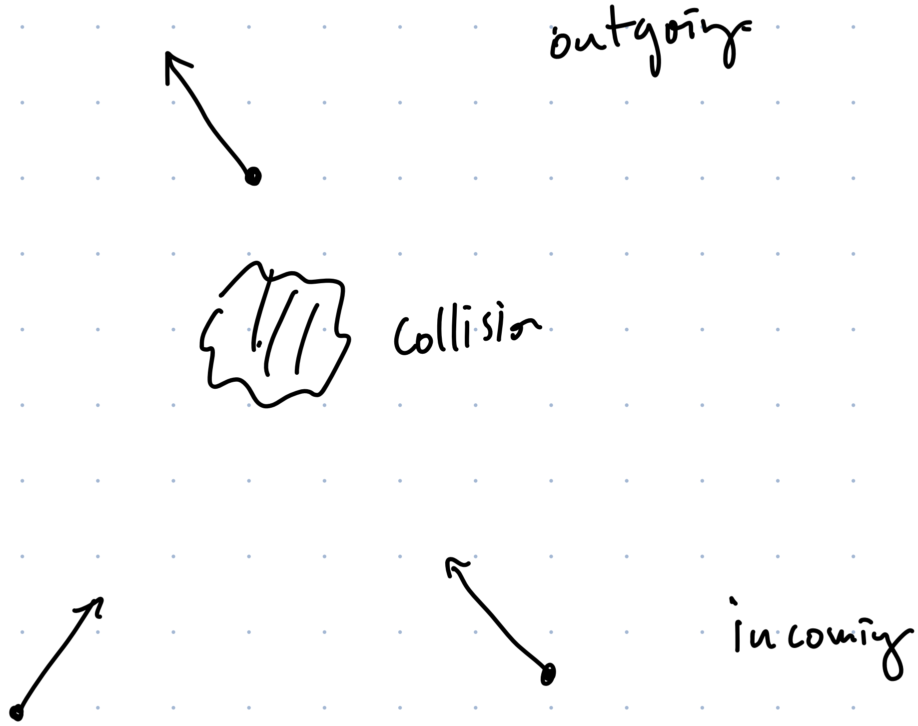
$$m^2 = \|(E, \vec{p})\|^2 = E^2 - |\vec{p}|^2$$ rest mass

coordinate dependent.

$$E = \sqrt{m^2 + |\vec{p}|^2}$$

Rest mass kinetic energy

M^4



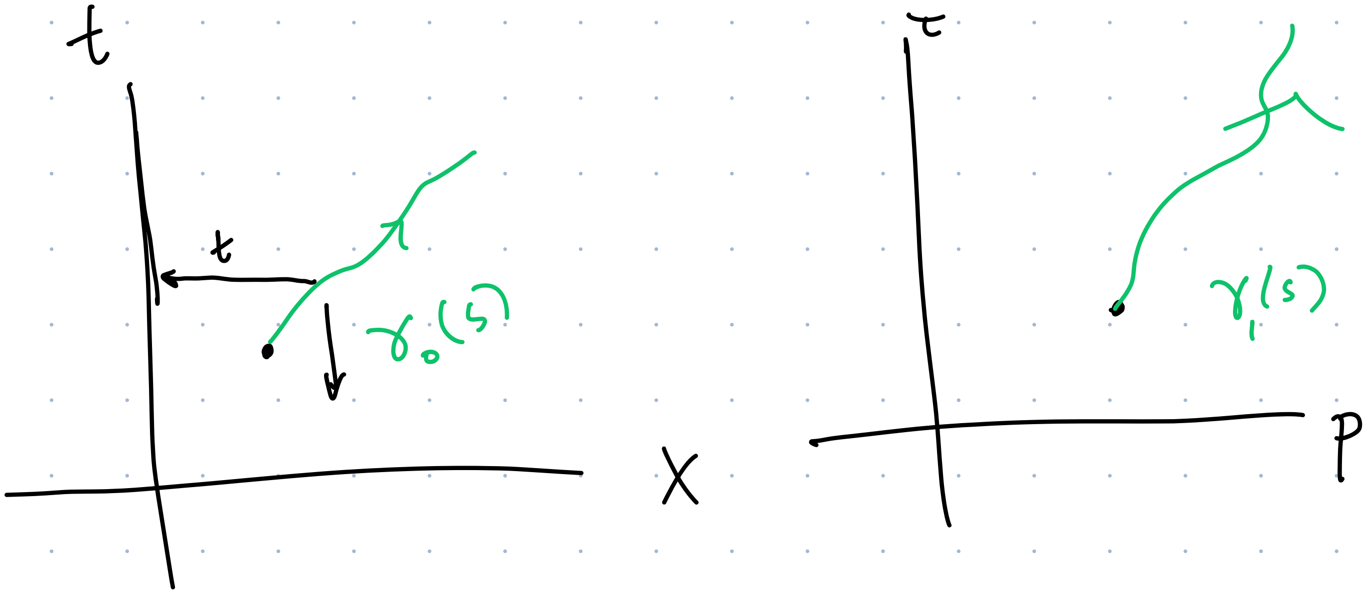
1. rest mass need not be conserved, as Ham. is <u>not</u> free Hamiltonian.

2. If the Hamiltonian is translation-invt,

$$\{H, E\} = \{H, P_i\} = 0$$

$$\Rightarrow (E, \vec{p}) \text{ conserved} \quad \left(\begin{matrix}\text{conserv. of} \\ \text{4-mom.}\end{matrix}\right)$$

Typically mass not conserved, but $\underset{\text{energy}}{E}$ $\underset{\text{momentum.}}{\vec{p}}$



$$\gamma(s) = (\gamma_0(s), \gamma_1(s))$$