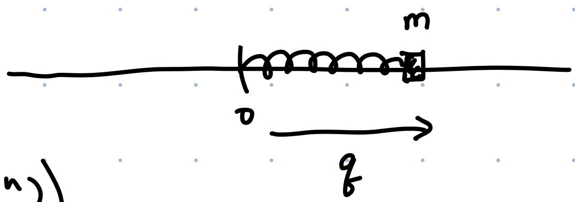
1. Hamiltonian Mechanics

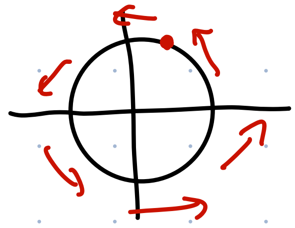
Begin with a smooth manifold X "the configuration space"

eg. $X = (\mathbb{R}, q)$



$X = (\mathbb{R}^n, (q^1, \dots, q^n))$

$X = S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$



q coordinate with $q \sim q + 2\pi$

"$q = \theta$"

$X = S^n \subset \mathbb{R}^{n+1}$ $\quad S^n = \{ (q^0, \dots, q^n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} (q^i)^2 = 1 \}$

Purpose of Ham. mechanics: Predict future/past based upon present.
future not determined simply by "position" i.e. configuration, rather it is determined uniquely by the $\boxed{\text{STATE}}$.

A STATE in Hamiltonian mechanics consists of $\begin{cases} q & \text{position} \\ p & \text{momentum} \end{cases}$

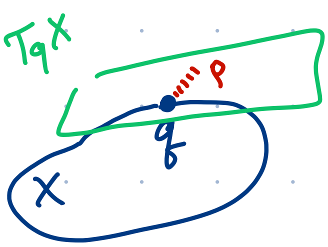
$(q, p) \in T^*X \leftarrow$ set of all (q, p)

State space or Phase space

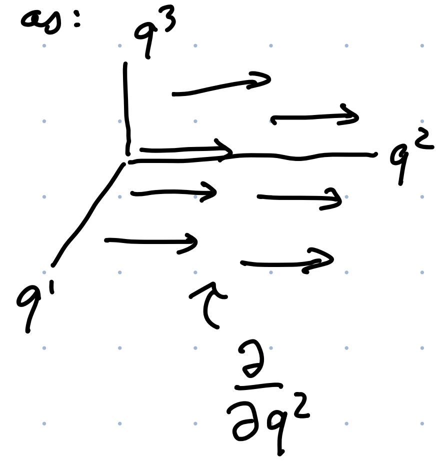
$q \in X$

$p \in T_q^* X$

$\boxed{\text{$p(v)$ is the momentum in direction $v \in T_q X$}}$



A tangent vector at $q \in X$ can be written as:



$$v = v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} + \cdots + v^n \frac{\partial}{\partial q^n}$$

components of v in basis $\left(\frac{\partial}{\partial q^i}\right)_{i=1}^n$

$(v^i) \in \mathbb{R}^n$

A cotangent vector at $q \in X$ . . . . .

$$p = p_1\, dq^1 + p_2\, dq^2 + \cdots + p_n\, dq^n$$

$\mathbb{R}^n \ni (p_i)_{i=1}^n$ components of p in dual basis $(dq^i)_{i=1}^n$

i.e. by choosing coords $(q^1, \dots, q^n)$, automatically obtain coords

$$(q^1, \dots, q^n, p_1, \dots, p_n) \text{ for } T^*X$$

these coordinates specify the point

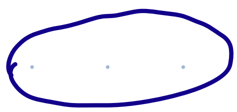
$(q, p) \in T^*X$

$2n$-dimensional smooth manifold. "Cotangent Bundle"

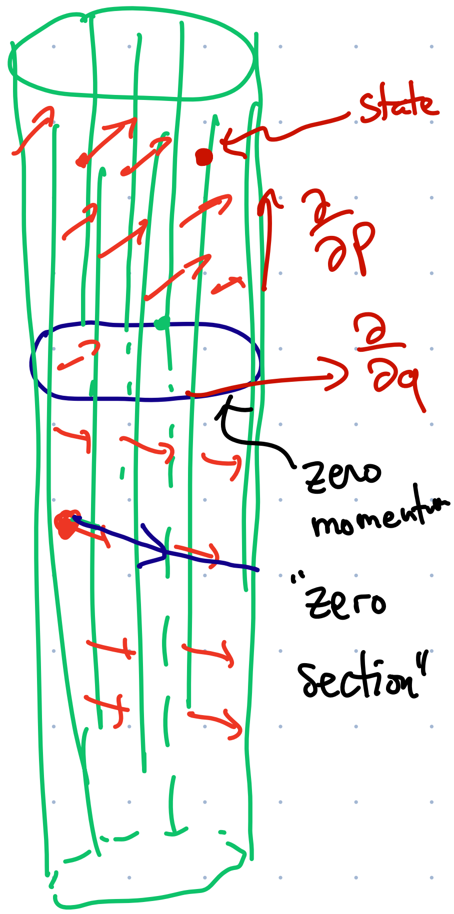
the $(p_1, \dots, p_n)$ are called "the canonically conjugate variables of $(q^1, \dots, q^n)$"

Main Idea: the evolution in time of the state $(q,p) \in T^*X$ is the flow of a vector field on T^*X

$X = S^1$



T^*S^1
$\parallel$
$S^1 \times \mathbb{R}$
cylinder.



$$\begin{array}{cc} T^*X & (q,p) \\ \downarrow \pi & \downarrow \pi \\ X & q \end{array}$$

projection map

The "equation of motion" is the equation for a flow-line of vector field, this uniquely determines dynamics

<u>More precisely:</u> Phase space $M = T^*X$ has a

(natural geometric structure $\omega \in \Omega^2(M)$

called a symplectic form, which determines a Poisson

bracket $\{,\}: C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$)

this provides a mechanism

$$C^\infty(M) \longrightarrow \mathfrak{X}(M)$$

functions on phase space $\qquad\qquad$ Vector fields on Phase space.

$\Downarrow$

$$f \longmapsto X_f$$

vector field on Phase space.

This is used to provide the vector field

determining time evolution.

$$H \longmapsto X_H$$

time evolution

"Hamiltonian" function

= "Energy"

Summary: The mathematical model underlying Hamiltonian mechanics is

$$(X, H)$$

X: smooth mfld

$H \in C^\infty(T^*X)$

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note on dual spaces

$V \in \mathbb{R}\text{-Vect}$

V^* dual of V $\qquad V^* = \mathrm{Hom}(V, \mathbb{R})$ linear maps

(V, V^* not canonically isomorphic)

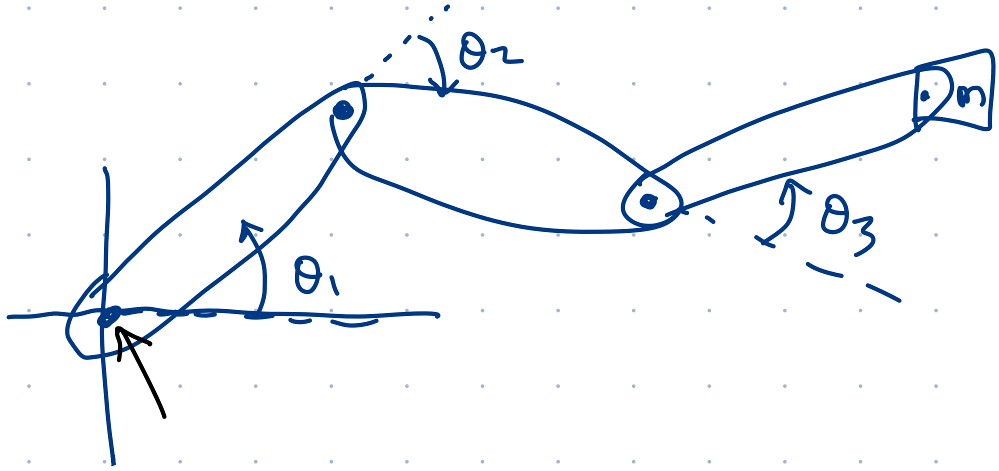
If V is equipped w/ nondeg. inner product

$$\langle , \rangle : V \times V \to \mathbb{R}$$

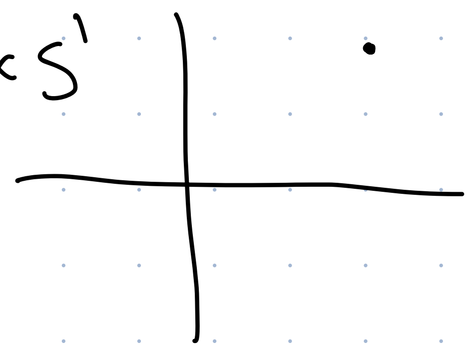
then

$$\begin{aligned} V &\xrightarrow{\cong} V^* \\ u &\longmapsto \langle u, - \rangle \end{aligned}$$

is an isomorphism



$\mathbb{R}^2 \ni (q^1, q^2)$

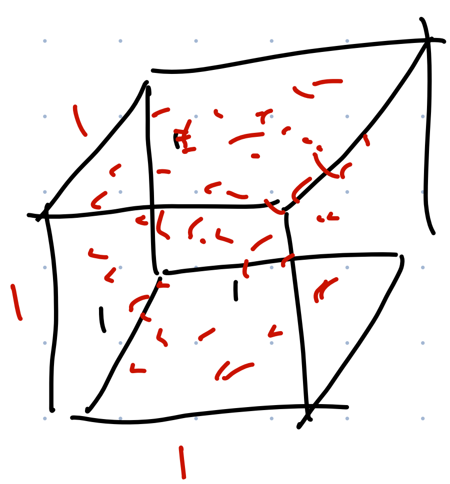


<u>configurations</u>: $X = S^1 \times S^1 \times S^1$

$$q = (q^1, q^2, q^3)$$
$$= (\theta^1, \theta^2, \theta^3)$$

If initial hinge free to move on table.

$$X = \mathbb{R}^2 \times (S^1)^3$$



N particles in a box

$$X = \left([0,1] \times [0,1] \times [0,1]\right)^N$$

dim $= 3N$

$N \sim 10^{23}$