4.2 Vector fields vs. derivations

The space $C^\infty(M, \mathbb{R})$ of smooth functions on M is not only a vector space but also a ring, with multiplication $(fg)(p) := f(p)g(p)$. Given a smooth map $\varphi : M \longrightarrow N$ of manifolds, we obtain a natural operation $\varphi^* : C^\infty(N, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$, given by $f \mapsto f \circ \varphi$. This is called the pullback of functions, and defines a homomorphism of rings.

The association $M \mapsto C^\infty(M, \mathbb{R})$ and $\varphi \mapsto \varphi^*$ is therefore a *contravariant* functor from the category of manifolds to the category of rings, and is the basis for algebraic geometry, the algebraic representation of geometrical objects.

It is easy to see from this that any diffeomorphism $\varphi : M \longrightarrow M$ defines an automorphism φ^* of $C^\infty(M, \mathbb{R})$, but actually all automorphisms are of this form (Exercise!).

The concept of derivation of an algebra A is the infinitesimal version of an automorphism of A. That is, if $\phi_t : A \longrightarrow A$ is a family of automorphisms of A starting at Id, so that $\phi_t(ab) = \phi_t(a)\phi_t(b)$, then the map $a \mapsto \frac{d}{dt}|_{t=0}\phi_t(a)$ is a derivation.

Definition 4.12. A derivation of the $\mathbb{R}$-algebra A is a $\mathbb{R}$-linear map $D : A \longrightarrow A$ such that $D(ab) = (Da)b + a(Db)$. The space of all derivations is denoted $\mathrm{Der}(A)$.

If automorphisms of $C^\infty(M, \mathbb{R})$ correspond to diffeomorphisms, then it is natural to ask what derivations correspond to. We now show that they correspond to vector fields.

The vector fields $\mathfrak{X}(M)$ form a vector space over $\mathbb{R}$ of infinite dimension (unless M is a finite set). They also form a module over the ring of smooth functions $C^\infty(M, \mathbb{R})$ via pointwise multiplication: for $f \in C^\infty(M, \mathbb{R})$ and $X \in \mathfrak{X}(M)$, $fX : x \mapsto f(x)X(x)$ is a smooth vector field.

The important property of vector fields which we are interested in is that they act as derivations of the algebra of smooth functions. Locally, it is clear that a vector field $X = \sum_i a^i \frac{\partial}{\partial x^i}$ gives a derivation of the algebra of smooth functions, via the formula $X(f) = \sum_i a^i \frac{\partial f}{\partial x^i}$, since

$$X(fg) = \sum_i a^i \left(\tfrac{\partial f}{\partial x^i} g + f \tfrac{\partial g}{\partial x^i}\right) = X(f)g + fX(g).$$

We wish to verify that this local action extends to a well-defined global derivation on $C^\infty(M, \mathbb{R})$.

Definition 4.13. The differential of a function $f \in C^\infty(M, \mathbb{R})$ is the function on TM given by composing $Tf : TM \to T\mathbb{R}$ with the second projection $p_2 : T\mathbb{R} = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

$$df = p_2 \circ Tf \tag{78}$$

To remove any confusion, df evaluates at the point $(x, v) \in TM$ to give the derivative of f at x in the direction v:

$$df(x, v) = Df|_x(v).$$

Definition 4.14. Let X be a vector field. Then we define

$$X(f) = df \circ X.$$

This is called the directional (or Lie) derivative of f along X.

In coordinates, if $X = \sum a_i \partial/\partial x_i$, then $X(f) = \sum a_i \partial f/\partial x_i$, coinciding with the usual directional derivative mentioned above. This shows that $f \mapsto X(f)$ has the derivation property (since it satisfies it locally), but we can alternatively see that it is a derivation by using the property

$$d(fg) = fdg + gdf$$

of the differential of a product (here fdg is really $(\pi^* f)dg$).

Theorem 4.15. *The map $X \mapsto (f \mapsto X(f))$ is an isomorphism*

$$\mathfrak{X}(M) \to Der(C^\infty(M, \mathbb{R})).$$

Proof. First we prove the result for an open set $U \subset \mathbb{R}^n$. Let D be a derivation of $C^\infty(U, \mathbb{R})$ and define the smooth functions $a^i = D(x^i)$. Then we claim $D = \sum_i a^i \frac{\partial}{\partial x^i}$. We prove this by testing against smooth functions. Any smooth function f on $\mathbb{R}^n$ may be written

$$f(x) = f(0) + \sum_i x^i g_i(x),$$

with $g_i(0) = \frac{\partial f}{\partial x^i}(0)$ (simply take $g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(tx)dt$). Translating the origin to $y \in U$, we obtain for any $z \in U$

$$f(z) = f(y) + \sum_i (x^i(z) - x^i(y))g_i(z), \qquad g_i(y) = \tfrac{\partial f}{\partial x^i}(y).$$

Applying D, we obtain

$$Df(z) = \sum_i (Dx^i)g_i(z) - \sum_i (x^i(z) - x^i(y))Dg_i(z).$$

Letting z approach y, we obtain

$$Df(y) = \sum_i a^i \tfrac{\partial f}{\partial x^i}(y) = X(f)(y),$$

as required.

To prove the global result, let $(V_i \subset U_i, \varphi_i)$ be a regular covering and θ_i an associated partition of unity. Then for each i, $\theta_i D : f \mapsto \theta_i D(f)$ is also a derivation of $C^\infty(M, \mathbb{R})$. This derivation defines a unique derivation D_i of $C^\infty(U_i, \mathbb{R})$ such that $D_i(f|_{U_i}) = (\theta_i Df)|_{U_i}$, since for any point $p \in U_i$, a given function $g \in C^\infty(U_i, \mathbb{R})$ may be replaced with a function $\tilde{g} \in C^\infty(M, \mathbb{R})$ which agrees with g on a small neighbourhood of p, and we define $(D_i g)(p) = \theta_i(p) D\tilde{g}(p)$. This definition is independent of $\tilde{g}$, since if $h_1 = h_2$ on an open set W, $Dh_1 = Dh_2$ on that open set (let $\psi = 1$ in a neighbourhood of p and vanish outside W; then $h_1 - h_2 = (h_1 - h_2)(1 - \psi)$ and applying D we obtain zero in W).

The derivation D_i is then represented by a vector field X_i, which must vanish outside the support of θ_i. Hence it may be extended by zero to a global vector field which we also call X_i. Finally we observe that for $X = \sum_i X_i$, we have

$$X(f) = \sum_i X_i(f) = \sum_i D_i(f) = D(f),$$

as required. $\square$

5 Vector bundles

Definition 5.1. A smooth real vector bundle of rank k over the base manifold M is a manifold E (called the total space), together with a smooth surjection $\pi : E \longrightarrow M$ (called the bundle projection), such that

- $\forall p \in M$, $\pi^{-1}(p) = E_p$ has the structure of k-dimensional vector space,
- Each $p \in M$ has a neighbourhood U and a diffeomorphism $\Phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ (called a local trivialization of E over U) such that $\pi_1(\Phi(\pi^{-1}(x))) = x$, where $\pi_1 : U \times \mathbb{R}^k \longrightarrow U$ is the first projection, and also that $\Phi : \pi^{-1}(x) \longrightarrow \{x\} \times \mathbb{R}^k$ is a linear map, for all $x \in M$.

Given two local trivializations $\Phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^k$ and $\Phi_j : \pi^{-1}(U_j) \longrightarrow U_j \times \mathbb{R}^k$, we obtain a smooth gluing map $\Phi_j \circ \Phi_i^{-1} : U_{ij} \times \mathbb{R}^k \longrightarrow U_{ij} \times \mathbb{R}^k$, where $U_{ij} = U_i \cap U_j$. This map preserves images to M, and hence it sends (x, v) to $(x, g_{ji}(v))$, where g_{ji} is an invertible $k \times k$ matrix smoothly depending on x. That is, the gluing map is uniquely specified by a smooth map

$$g_{ji} : U_{ij} \longrightarrow GL(k, \mathbb{R}).$$

These are called transition functions of the bundle, and since they come from $\Phi_j \circ \Phi_i^{-1}$, they clearly satisfy $g_{ij} = g_{ji}^{-1}$ as well as the "cocycle condition"

$$g_{ij} g_{jk} g_{ki} = \mathrm{Id}|_{U_i \cap U_j \cap U_k}.$$

Example 5.2. To build a vector bundle, choose an open cover $\{U_i\}$ and form the pieces $\{U_i \times \mathbb{R}^k\}$ Then glue these together on double overlaps $\{U_{ij}\}$ via functions $g_{ij} : U_{ij} \longrightarrow GL(k, \mathbb{R})$. As long as g_{ij} satisfy $g_{ij} = g_{ji}^{-1}$ as well as the cocycle condition, the resulting space has a vector bundle structure.

Example 5.3. Let $S^2 = U_0 \sqcup U_1$ for $U_i = \mathbb{R}^2$, as before. Then on $U_{01} = \mathbb{R}^2 \backslash \{0\} = \mathbb{C} \backslash \{0\}$, define

$$g_{01}(z) = [z^k], \quad k \in \mathbb{Z}.$$

In real coordinates $z = re^{i\theta}$, $g_{01}(r, \theta) = r^k \begin{pmatrix} cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \end{pmatrix}$. This defines a vector bundle $E_k \longrightarrow S^2$ of rank 2 for each $k \in \mathbb{Z}$ (or a complex vector bundle of rank 1, since $g_{01} : U_{01} \longrightarrow GL(1, \mathbb{C})$). Actually, since the map g_{01} is actually holomorphic as a function of z, we have defined *holomorphic* vector bundles on $\mathbb{C}P^1$.

Example 5.4 (The tangent bundle)**.** The tangent bundle TM is indeed a vector bundle, of rank $\dim M$. For any chart (U, φ) of M, there is an associated local trivialization $(\pi^{-1}(U), \Phi)$ of TM, and the transition function $g_{ji} : U_{ij} \longrightarrow GL(n, \mathbb{R})$ between two trivializations obtained from $(U_i, \varphi_i), (U_j, \varphi_j)$ is simply the Jacobian matrix

$$g_{ji} : p \mapsto D(\varphi_j \circ \varphi_i^{-1})(p).$$

Just as for the tangent bundle, we can define the analog of a vector-valued function, where the function has values in a vector bundle:

Definition 5.5. A smooth section of the vector bundle $E \xrightarrow{\pi} M$ is a smooth map $s : M \longrightarrow E$ such that $\pi \circ s = \mathrm{Id}_M$. The set of all smooth sections, denoted $\Gamma^\infty(M, E)$, is an infinite-dimensional real vector space, and is also a module over the ring $C^\infty(M, \mathbb{R})$.

Having introduced vector bundles, we must define the notion of *morphism* between vector bundles, so as to form a category.

Definition 5.6. A smooth bundle map between the bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ is a pair (f, F) of smooth maps $f : M \longrightarrow M'$ and $F : E \longrightarrow E'$ such that $\pi' \circ F = f \circ \pi$ and such that $F : E_p \longrightarrow E'_{f(p)}$ is a linear map for all p.

Example 5.7. I claim that the bundles $E_k \xrightarrow{\pi} S^2$ are all non-isomorphic, except that E_k is isomorphic to E_{-k} over the antipodal map $S^2 \longrightarrow S^2$.

Example 5.8. Suppose $f : M \longrightarrow N$ is a smooth map. Then $f_* : TM \longrightarrow TN$ is a bundle map covering f, i.e. (f_*, f) defines a bundle map.

Example 5.9 (Pullback bundle)**.** if $f : M \longrightarrow N$ is a smooth map and $E \xrightarrow{\pi} N$ is a vector bundle over N, then we may form the fiber product $M_f \times_\pi E$, which then is a bundle over M with local trivializations $(f^{-1}(U_i), f^* g_{ij})$, where (U_i, g_{ij}) is the local transition data for E over N. This bundle is called the pullback bundle and is denoted by f^*E. The natural projection to E defines a vector bundle map back to E:

$$\begin{tikzcd} f^*E \arrow[r, "p_2"] \arrow[d, "p_1"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

There is also a natural pullback map on sections: given a section $s \in \Gamma^\infty(N, E)$, the composition $s \circ f$ gives a map $M \longrightarrow E$. This then determines a smooth map $f^*s : M \longrightarrow f^*E$ by the universal property of the fiber product. We therefore obtain a pullback map

$$f^* : \Gamma^\infty(N, E) \longrightarrow \Gamma^\infty(M, f^*E).$$

Example 5.10. If $f : M \longrightarrow N$ is an embedding, then so is the bundle map $f_* : TM \longrightarrow TN$. By the universal property of the fiber product we obtain a bundle map, also denoted f_*, from TM to f^*TN. This is a vector bundle inclusion and $f^*TN/f_*TM = NM$ is a vector bundle over M called the normal bundle of M. *Note: we haven't covered subbundles and quotient bundles in detail. I'll leave this as an exercise.*

5.1 Associated bundles

We now describe a functorial construction of vector bundles, using functors from vector spaces. Consider the category $\mathbf{Vect}_\mathbb{R}$ of finite-dimensional real vector spaces and linear maps. We will describe several functors from $\mathbf{Vect}_\mathbb{R}$ to itself.

Example 5.11. If $V \in \mathbf{Vect}_{\mathbb{R}}$, then $V^* \in \mathbf{Vect}_{\mathbb{R}}$, and if $f : V \longrightarrow W$ then $f^* : W^* \longrightarrow V^*$. Since the composition of duals is the dual of the composition, duality defines a contravariant functor $* : \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$.

Example 5.12. If $V, W \in \mathbf{Vect}_{\mathbb{R}}$, then $V \oplus W \in \mathbf{Vect}_{\mathbb{R}}$, and this defines a covariant functor $\mathbf{Vect}_{\mathbb{R}} \times \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$.

Example 5.13. If $V, W \in \mathbf{Vect}_{\mathbb{R}}$, then $V \otimes W \in \mathbf{Vect}_{\mathbb{R}}$ and this again defines a covariant functor $\mathbf{Vect}_{\mathbb{R}} \times \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$.

Example 5.14. If $V \in \mathbf{Vect}_{\mathbb{R}}$, then

$$\otimes^\bullet V = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \cdots \oplus (\otimes^k V) \oplus \cdots$$

is an infinite-dimensional vector space, with a product $a \otimes b$. Quotienting by the double-sided ideal $I = \langle v \otimes v \ : \ v \in V \rangle$, we obtain the exterior algebra

$$\wedge^\bullet V = \mathbb{R} \oplus V \oplus \wedge^2 V \oplus \cdots \oplus \wedge^n V,$$

with $n = \dim V$. The product is customarily denoted $(a, b) \mapsto a \wedge b$. The direct sum decompositions above, where $\wedge^k V$ or $\otimes^k V$ is labeled by the integer k, are called $\mathbb{Z}$-gradings, and since the product takes $\wedge^k \times \wedge^l \longrightarrow \wedge^{k+l}$, these algebras are called Z-graded algebras.

If $(v_1, \dots v_n)$ is a basis for V, then $v_{i_1} \wedge \cdots \wedge v_{i_k}$ for $i_1 < \cdots < i_k$ form a basis for $\wedge^k V$. This space then has dimension $\binom{n}{k}$, hence the algebra $\wedge^\bullet V$ has dimension 2^n.

Note in particular that $\wedge^n V$ has dimension 1, is also called the determinant line $\det V$, and a choice of nonzero element in $\det V$ is called an "orientation" on the vector space V.

Recall that if $f : V \longrightarrow W$ is a linear map, then $\wedge^k f : \wedge^k V \longrightarrow \wedge^k W$ is defined on monomials via

$$\wedge^k f(a_1 \wedge \cdots \wedge a_k) = f(a_1) \wedge \cdots \wedge f(a_k).$$

In particular, if $A : V \longrightarrow V$ is a linear map, then for $n = \dim V$, the top exterior power $\wedge^n A : \wedge^n V \longrightarrow \wedge^n V$ is a linear map of a 1-dimensional space onto itself, and is hence given by a number, called $\det A$, the determinant of A.

We may now apply any of these functors to vector *bundles*. The main observation is that if F is a vector space functor as above, we may apply it to any vector bundle $E \xrightarrow{\pi} M$ to obtain a new vector bundle

$$F(E) = \sqcup_{p \in M} F(E_p).$$

If (U_i) is an atlas for M and E has local trivializations $(U_i \times \mathbb{R}^k)$, glued together via $g_{ji} : U_{ij} \longrightarrow GL(k, \mathbb{R})$, then $F(E)$ may be given the local trivialization $(U_i \times F(\mathbb{R}^k))$, glued together via $F(g_{ji})$. This new vector bundle $F(E)$ is called the "associated" vector bundle to E, given by the functor F.

Example 5.15. If $E \longrightarrow M$ is a vector bundle, then $E^* \longrightarrow M$ is the dual vector bundle. If E, F are vector bundles then $E \oplus F$ is called the direct or "Whitney" sum, and has rank rk $E +$ rk F. $E \otimes F$ is the tensor product bundle, which has rank rk $E \cdot$ rk F.

Example 5.16. If $E \longrightarrow M$ is a vector bundle of rank n, then $\otimes^k E$ and $\wedge^k E$ are its tensor power bundles, of rank n^k and $\binom{n}{k}$, respectively. The top exterior power $\wedge^n E$ has rank 1, and is hence a line bundle. If this line bundle is trivial (i.e. isomorphic to $M \times \mathbb{R}$) then E is said to be an orientable bundle.

Example 5.17. Starting with the tangent bundle $TM \longrightarrow M$, we may form the cotangent bundle T^*M, the bundle of tensors of type (r, s), $\otimes^r TM \otimes \otimes^s T^*M$.

We may also form the bundle of multivectors $\wedge^k TM$, which has sections $\Gamma^\infty(M, \wedge^k TM)$ called multivector fields.

Finally, we may form the bundle of k-forms, $\wedge^k T^*M$, whose sections $\Gamma^\infty(M, \wedge^k T^*M) = \Omega^k(M)$ are called differential k-forms, and will occupy us for some time.

We have now produced several vector bundles by applying functors to the tangent bundle. We are familiar with vector fields, which are sections of TM, and we know that a vector field is written locally in coordinates $(x^1, \ldots, x^n)$ as

$$X = \sum_i a^i \tfrac{\partial}{\partial x^i},$$

with coefficients a^i smooth functions.

There is an easy way to produce examples of 1-forms in $\Omega^1(M)$, using smooth functions f. We note that the action $X \mapsto X(f)$ defines a dual vector at each point of M, since $(X(f))_p$ depends only on the vector X_p and not the behaviour of X away from p. Recall that $X(f) = Df_2(X)$.

Definition 5.18. The exterior derivative of a function f, denoted df, is the section of T^*M given by the fiber projection Df_2.

Since $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$, we see that $(dx^1, \ldots, dx^n)$ is the dual basis to $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$. Therefore, a section of T^*M has local expression

$$\xi = \sum_i \xi_i dx^i,$$

for ξ_i smooth functions, given by $\xi_i = \xi(\frac{\partial}{\partial x^i})$. In particular, the exterior derivative of a function df can be written

$$df = \sum_i \tfrac{\partial f}{\partial x^i} dx^i.$$

A section of the tensor bundle $\otimes^r TM \otimes \otimes^s T^*M$ can be written as

$$\Theta = \sum_{\substack{i_1, \cdots, i_r \\ j_1, \cdots, j_s}} a^{i_1 \cdots i_r}_{j_1 \cdots j_s} \tfrac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \tfrac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s},$$

where $a^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ are n^{r+s} smooth functions.

A general differential form $\rho \in \Omega^k(M)$ can be written

$$\rho = \sum_{i_1 < \cdots < i_k} \rho_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

6 Differential forms

There are several properties of differential forms which make them indispensible: first, the k-forms are intended to give a notion of k-dimensional volume (this is why they are multilinear and skew-symmetric, like the determinant) and in a way compatible with the boundary map (this leads to the exterior derivative, which we define below). Second, they behave well functorially, as we see now.

Given a smooth map $f : M \longrightarrow N$, we obtain bundle maps $f_* : TM \longrightarrow TN$ and hence $f^* := \wedge^k (f_*)^* : \wedge^k T^*N \longrightarrow \wedge^k T^*M$. Hence we have the diagram

$$\begin{tikzcd} \wedge^k T^*M \arrow[d, "\pi_M"'] & \wedge^k T^*N \arrow[l, "f^*"'] \arrow[d, "\pi_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

The interesting thing is that if $\rho \in \Omega^k(N)$ is a differential form on N, then it is a section of π_N. Composing with f, f^*, we obtain a section $f^*\rho := f^* \circ \rho \circ f$ of π_M. Hence we obtain a natural map

$$\Omega^k(N) \xrightarrow{f^*} \Omega^k(M).$$

Such a natural map does not exist (in either direction) for multivector fields, for instance.

Suppose that $\rho \in \Omega^k(N)$ is given in a coordinate chart by

$$\rho = \sum \rho_{i_1 \cdots i_k} dy^{i_1} \wedge \cdots \wedge dy^{i_k}.$$

Now choose a coordinate chart for M with coordinates $x^1, \ldots x^m$. What is the local expression for $f^*\rho$? We need only compute $f^* dy_i$. We use a notation where f^k denotes the k^{th} component of f in the coordinates $(y^1, \ldots y^n)$, i.e. $f^k = y^k \circ f$.

$$f^* dy_i(\tfrac{\partial}{\partial x^j}) = dy_i(f_* \tfrac{\partial}{\partial x^j}) \tag{79}$$
$$= dy_i(\sum_k \tfrac{\partial f^k}{\partial x^j} \tfrac{\partial}{\partial y_k}) \tag{80}$$
$$= \tfrac{\partial f^i}{\partial x^j}. \tag{81}$$

Hence we conclude that

$$f^* dy_i = \sum_j \tfrac{\partial f^i}{\partial x^j} dx^j.$$

Finally we compute

$$f^*\rho = \sum_{i_1 < \cdots < i_k} f^* \rho_{i_1 \cdots i_k} f^*(dy^{i_1}) \wedge \cdots \wedge f^*(dy^{i_k}) \tag{82}$$
$$= \sum_{i_1 < \cdots < i_k} (\rho_{i_1 \cdots i_k} \circ f) \sum_{j_1} \cdots \sum_{j_k} \tfrac{\partial f^{i_1}}{\partial x^{j_1}} \cdots \tfrac{\partial f^{i_k}}{\partial x^{j_k}} dx^{j_1} \wedge \cdots dx^{j_k}. \tag{83}$$

6.1 The exterior derivative

Differential forms are equipped with a natural differential operator, which extends the exterior derivative of functions to all forms: $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$. The exterior derivative is uniquely specified by the following requirements: first, it satisfies $d(df) = 0$ for all functions f. Second, it is a graded derivation of the algebra of exterior differential forms of degree 1, i.e.

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta.$$

This allows us to compute its action on any 1-form $d(\xi_i dx^i) = d\xi_i \wedge dx^i$, and hence, in coordinates, we have

$$d(\rho dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \sum_k \tfrac{\partial \rho}{\partial x^k} dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Extending by linearity, this gives a local definition of d on all forms. Does it actually satisfy the requirements? this is a simple calculation: let $\tau_p = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ and $\tau_q = dx^{j_1} \wedge \cdots \wedge dx^{j_q}$. Then

$$d((f\tau_p)\wedge(g\tau_q)) = d(fg\tau_p\wedge\tau_q) = (gdf+fdg)\wedge\tau_p\wedge\tau_q = d(f\tau_p)\wedge g\tau_q+(-1)^p f\tau_p\wedge d(g\tau_q),$$

as required.

Therefore we have defined d, and since the definition is coordinate-independent, we can be satisfied that d is well-defined.

Definition 6.1. d is the unique degree $+1$ graded derivation of $\Omega^\bullet(M)$ such that $df(X) = X(f)$ and $d(df) = 0$ for all functions f.

Example 6.2. Consider $M = \mathbb{R}^3$. For $f \in \Omega^0(M)$, we have

$$df = \tfrac{\partial f}{\partial x^1} dx^1 + \tfrac{\partial f}{\partial x^2} dx^2 + \tfrac{\partial f}{\partial x^3} dx^3.$$

Similarly, for $A = A_1 dx^1 + A_2 dx^2 + A_3 dx^3$, we have

$$dA = (\tfrac{\partial A_2}{\partial x^1} - \tfrac{\partial A_1}{\partial x^2}) dx^1 \wedge dx^2 + (\tfrac{\partial A_3}{\partial x^1} - \tfrac{\partial A_1}{\partial x^3}) dx^1 \wedge dx^3 + (\tfrac{\partial A_3}{\partial x^2} - \tfrac{\partial A_2}{\partial x^3}) dx^2 \wedge dx^3$$

Finally, for $B = B_{12} dx^1 \wedge dx^2 + B_{13} dx^1 \wedge dx^3 + B_{23} dx^2 \wedge dx^3$, we have

$$dB = (\tfrac{\partial B_{12}}{\partial x^3} - \tfrac{\partial B_{13}}{\partial x^2} + \tfrac{\partial B_{23}}{\partial x^1}) dx^1 \wedge dx^2 \wedge dx^3.$$

Definition 6.3. The form $\rho \in \Omega^\bullet(M)$ is called *closed* when $d\rho = 0$ and *exact* when $\rho = d\tau$ for some τ.

Example 6.4. A function $f \in \Omega^0(M)$ is closed if and only if it is constant on each connected component of M: This is because, in coordinates, we have

$$df = \tfrac{\partial f}{\partial x^1} dx^1 + \cdots + \tfrac{\partial f}{\partial x^n} dx^n,$$

and if this vanishes, then all partial derivatives of f must vanish, and hence f must be constant.

Theorem 6.5. *The exterior derivative of an exact form is zero, i.e.* $d \circ d = 0$*. Usually written* $d^2 = 0$*.*

Proof. The graded commutator $[d_1, d_2] = d_1 \circ d_2 - (-1)^{|d_1||d_2|} d_2 \circ d_1$ of derivations of degree $|d_1|, |d_2|$ is always (why?) a derivation of degree $|d_1| + |d_2|$. Hence we see $[d, d] = d \circ d - (-1)^{1 \cdot 1} d \circ d = 2d^2$ is a derivation of degree 2 (and so is d^2). Hence to show it vanishes we must test on functions and exact 1-forms, which locally generate forms since every form is of the form $f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

But $d(df) = 0$ by definition and this certainly implies $d^2(df) = 0$, showing that $d^2 = 0$. $\square$