Example 3.4. *From a computational point of view, given an atlas* $(\tilde{U}_i, \varphi_i)$ *for* M*, let* $U_i = \varphi_i(\tilde{U}_i) \subset \mathbb{R}^n$ *and let* $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$*. Then a global vector field* $X \in \Gamma^\infty(M, TM)$ *is specified by a collection of vector-valued functions* $X_i : U_i \longrightarrow \mathbb{R}^n$ *such that* $D\varphi_{ij}(X_i(x)) = X_j(\varphi_{ij}(x))$ *for all* $x \in \varphi_i(\tilde{U}_i \cap \tilde{U}_j)$*.*

For example, if $S^1 = U_0 \sqcup U_1 / \sim$*, with* $U_0 = \mathbb{R}$ *and* $U_1 = \mathbb{R}$*, with* $x \in U_0 \backslash \{0\} \sim y \in U_1 \backslash \{0\}$ *whenever* $y = x^{-1}$*, then* $\varphi_{01} : x \mapsto x^{-1}$ *and* $D\varphi_{01}(x) : v \mapsto -x^{-2}v$*. Then if we define (letting* x *be the standard coordinate along* $\mathbb{R}$*)*

$$X_0 = \frac{\partial}{\partial x}$$
$$X_1 = -y^2 \frac{\partial}{\partial y},$$

we see that this defines a global vector field, which does not vanish in U_0 *but vanishes to order 2 at a single point in* U_1*. Find the local expression in these charts for the rotational vector field on* S^1 *given in polar coordinates by* $\frac{\partial}{\partial \theta}$*.*

3.1 Properties of vector fields

The space $C^\infty(M, \mathbb{R})$ of smooth functions on M is not only a vector space but also a ring, with multiplication $(fg)(p) := f(p)g(p)$. That this defines a smooth function is clear from the fact that it is a composition of the form

$$M \xrightarrow{\Delta} M \times M \xrightarrow{f \times g} \mathbb{R} \times \mathbb{R} \xrightarrow{\times} \mathbb{R}\ .$$

Given a smooth map $\varphi : M \longrightarrow N$ of manifolds, we obtain a natural operation $\varphi^* : C^\infty(N, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$, given by $f \mapsto f \circ \varphi$. This is called the pullback of functions, and defines a homomorphism of rings since $\Delta \circ \varphi = (\varphi \times \varphi) \circ \Delta$.

The association $M \mapsto C^\infty(M, \mathbb{R})$ and $\varphi \mapsto \varphi^*$ is therefore a *contravariant* functor from the category of manifolds to the category of rings, and is the basis for algebraic geometry, the algebraic representation of geometrical objects.

It is easy to see from this that any diffeomorphism $\varphi : M \longrightarrow M$ defines an automorphism φ^* of $C^\infty(M, \mathbb{R})$, but actually all automorphisms are of this form (Exercise!).

The concept of derivation of an algebra A is the infinitesimal version of an automorphism of A. That is, if $\phi_t : A \longrightarrow A$ is a family of automorphisms of A starting at Id, so that $\phi_t(ab) = \phi_t(a)\phi_t(b)$, then the map $a \mapsto \frac{d}{dt}|_{t=0}\phi_t(a)$ is a derivation.

Definition 22. A derivation of the $\mathbb{R}$-algebra A is a $\mathbb{R}$-linear map $D : A \longrightarrow A$ such that $D(ab) = (Da)b + a(Db)$. The space of all derivations is denoted $\mathrm{Der}(A)$.

In the following, we show that derivations of the algebra of functions actually correspond to vector fields.

The vector fields $\Gamma^\infty(M, TM)$ form a vector space over $\mathbb{R}$ of infinite dimension (unless $\dim M = 0$). They also form a module over the ring of smooth functions $C^\infty(M, \mathbb{R})$ via pointwise multiplication: for $f \in C^\infty(M, \mathbb{R})$ and $X \in \Gamma^\infty(M, TM)$, we claim that $fX : x \mapsto f(x)X(x)$ defines a smooth vector field: this is clear from local considerations: if $\{X_i\}$ is a local description of X and $\{f_i\}$ is a local description of f with respect to a cover, then

$$D\varphi_{ij}(f_i(x)X_i(x)) = f_i(x)D\varphi_{ij}X_i(x) = f_j(\varphi_{ij}(x))X_j(\varphi_{ij}(x)).$$

The important property of vector fields which we are interested in is that they act as $\mathbb{R}$-derivations of the algebra of smooth functions. Locally, it is clear that a vector field $X = \sum_i a^i \frac{\partial}{\partial x^i}$ gives a derivation of the algebra of smooth functions, via the formula $X(f) = \sum_i a^i \frac{\partial f}{\partial x^i}$, since

$$X(fg) = \sum_i a^i (\tfrac{\partial f}{\partial x^i} g + f \tfrac{\partial g}{\partial x^i}) = X(f)g + fX(g).$$

We wish to verify that this local action extends to a well-defined global derivation on $C^\infty(M, \mathbb{R})$.

Lemma 3.5. *Let f be a smooth function on $U \subset \mathbb{R}^n$, and $X : U \longrightarrow \mathbb{R}^n$ a vector field. Then $Df : TU \longrightarrow T\mathbb{R} = \mathbb{R} \times \mathbb{R}$ and let $Df_2 : TM \longrightarrow \mathbb{R}$ be the composition of Df with the projection to the fiber $T\mathbb{R} \longrightarrow \mathbb{R}$. Then*

$$X(f) = Df_2(X).$$

Proof. In local coordinates, we have $X(f) = \sum_i a^i \frac{\partial f}{\partial x^i}$ whereas $Df : X(x) \mapsto (f(x), \sum_i \frac{\partial f}{\partial x^i} a^i)$, so that we obtain the result by projection. □

Proposition 3.6. *Local partial differentiation extends to an injective map $\Gamma^\infty(M, TM) \longrightarrow Der(C^\infty(M, \mathbb{R}))$.*

Proof. Globally, we verify that

$$\begin{aligned} X_j(f_j) = X_j(f_i \circ \varphi_{ij}^{-1}) &= ((\varphi_{ij})_* X_i)(f_i \circ \varphi_{ij}^{-1}) && (20)\\ &= D(f_i \circ \varphi_{ij}^{-1})_2((\varphi_{ij})_* X_i) && (21)\\ &= (Df_i)_2(X_i) = X_i(f_i). && (22) \end{aligned}$$

□

In fact, vector fields provide all possible derivations of the algebra $A = C^\infty(M, \mathbb{R})$:

Theorem 3.7. *The map $\Gamma^\infty(M, TM) \longrightarrow Der(C^\infty(M, \mathbb{R}))$ is an isomorphism.*

Proof. First we prove the result for an open set $U \subset \mathbb{R}^n$. Let D be a derivation of $C^\infty(U, \mathbb{R})$ and define the smooth functions $a^i = D(x^i)$. Then we claim $D = \sum_i a^i \frac{\partial}{\partial x^i}$. We prove this by testing against smooth functions. Any smooth function f on $\mathbb{R}^n$ may be written

$$f(x) = f(0) + \sum_i x^i g_i(x),$$

with $g_i(0) = \frac{\partial f}{\partial x^i}(0)$ (simply take $g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(tx)dt$). Translating the origin to $y \in U$, we obtain for any $z \in U$

$$f(z) = f(y) + \sum_i (x^i(z) - x^i(y)) g_i(z), \qquad g_i(y) = \tfrac{\partial f}{\partial x^i}(y).$$

Applying D, we obtain

$$Df(z) = \sum_i (Dx^i) g_i(z) - \sum_i (x^i(z) - x^i(y)) Dg_i(z).$$

Letting z approach y, we obtain

$$Df(y) = \sum_i a^i \tfrac{\partial f}{\partial x^i}(y) = X(f)(y),$$

as required.

To prove the global result, let $(V_i \subset U_i, \varphi_i)$ be a regular covering and θ_i the associated partition of unity. Then for each i, $\theta_i D : f \mapsto \theta_i D(f)$ is also a derivation of $C^\infty(M, \mathbb{R})$. This derivation defines a unique derivation D_i of $C^\infty(U_i, \mathbb{R})$ such that $D_i(f|_{U_i}) = (\theta_i Df)|_{U_i}$, since for any point $p \in U_i$, a given function $g \in C^\infty(U_i, \mathbb{R})$ may be replaced with a function $\tilde{g} \in C^\infty(M, \mathbb{R})$ which agrees with g on a small neighbourhood of p, and we define $(D_i g)(p) = \theta_i(p) D\tilde{g}(p)$. This definition is independent of $\tilde{g}$, since if $h_1 = h_2$ on an open set, $Dh_1 = Dh_2$ on that open set (let $\psi = 1$ in a neighbourhood of p and vanish outside U_i; then $h_1 - h_2 = (h_1 - h_2)(1 - \psi)$ and applying D we obtain zero).

The derivation D_i is then represented by a vector field X_i, which must vanish outside the support of θ_i. Hence it may be extended by zero to a global vector field which we also call X_i. Finally we observe that for $X = \sum_i X_i$, we have

$$X(f) = \sum_i X_i(f) = \sum_i D_i(f) = D(f),$$

as required. □

Since vector fields are derivations, we have a natural source of examples, coming from infinitesimal automorphisms of M:

Example 3.8. *Let φ_t : be a smooth family of diffeomorphisms of M with $\varphi_0 = \mathrm{Id}$. That is, let $\varphi : (-\epsilon, \epsilon) \times M \longrightarrow M$ be a smooth map and $\varphi_t : M \longrightarrow M$ a diffeomorphism for each t. Then $X(f)(p) = \frac{d}{dt}|_{t=0}(\varphi_t^* f)(p)$ defines a smooth vector field. A better way of seeing that it is smooth is to rewrite it as follows: Let $\frac{\partial}{\partial t}$ be the coordinate vector field on $(-\epsilon, \epsilon)$ and observe $X(f)(p) = \frac{\partial}{\partial t}(\varphi^* f)(0, p)$.*

In many cases, a smooth vector field may be expressed as above, i.e. as an infinitesimal automorphism of M, but this is not always the case. In general, it gives rise to a "local 1-parameter group of diffeomorphisms", as follows:

Definition 23. A local 1-parameter group of diffeomorphisms is an open set $U \subset \mathbb{R} \times M$ containing $\{0\} \times M$ and a smooth map

$$\begin{aligned} \Phi : & U \longrightarrow M \\ & (t, x) \mapsto \varphi_t(x) \end{aligned}$$

such that $\mathbb{R} \times \{x\} \cap U$ is connected, $\varphi_0(x) = x$ for all x and if $(t, x), (t + t', x), (t', \varphi_t(x))$ are all in U then $\varphi_{t'}(\varphi_t(x)) = \varphi_{t+t'}(x)$.

Then the local existence and uniqueness of solutions to systems of ODE implies that every smooth vector field $X \in \Gamma^\infty(M, TM)$ gives rise to a local 1-parameter group of diffeomorphisms (U, Φ) such that the curve $\gamma_x : t \mapsto \varphi_t(x)$ is such that $(\gamma_x)_*(\frac{d}{dt}) = X(\gamma_x(t))$ (this means that γ_x is an integral curve or "trajectory" of the "dynamical system" defined by X). Furthermore, if (U', Φ') are another such data, then $\Phi = \Phi'$ on $U \cap U'$.

Definition 24. A vector field $X \in \Gamma^\infty(M, TM)$ is called *complete* when its local 1-parameter group of diffeomorphisms has $U = \mathbb{R} \times M$.

Theorem 3.9. *If M is compact, then every smooth vector field is complete.*

Example 3.10. *The vector field $X = x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$ is not complete. For initial condition x_0, have integral curve $\gamma(t) = x_0(1 - tx_0)^{-1}$, which gives $\Phi(t, x_0) = x_0(1 - tx_0)^{-1}$, which is well-defined on $\{1 - tx > 0\}$.*

3.2 Vector bundles

Definition 25. A smooth real vector bundle of rank k over the base manifold M is a manifold E (called the total space), together with a smooth surjection $\pi : E \longrightarrow M$ (called the bundle projection), such that

- $\forall p \in M$, $\pi^{-1}(p) = E_p$ has the structure of k-dimensional vector space,
- Each $p \in M$ has a neighbourhood U and a diffeomorphism $\Phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ (called a local trivialization of E over U) such that $\pi_1(\Phi(\pi^{-1}(x))) = x$, where $\pi_1 : U \times \mathbb{R}^k \longrightarrow U$ is the first projection, and also that $\Phi : \pi^{-1}(x) \longrightarrow \{x\} \times \mathbb{R}^k$ is a linear map, for all $x \in M$.

Given two local trivializations $\Phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^k$ and $\Phi_j : \pi^{-1}(U_j) \longrightarrow U_j \times \mathbb{R}^k$, we obtain a smooth gluing map $\Phi_j \circ \Phi_i^{-1} : U_{ij} \times \mathbb{R}^k \longrightarrow U_{ij} \times \mathbb{R}^k$, where $U_{ij} = U_i \cap U_j$. This map preserves images to M, and hence it sends (x, v) to $(x, g_{ji}(v))$, where g_{ji} is an invertible $k \times k$ matrix smoothly depending on x. That is, the gluing map is uniquely specified by a smooth map

$$g_{ji} : U_{ij} \longrightarrow GL(k, \mathbb{R}).$$

These are called transition functions of the bundle, and since they come from $\Phi_j \circ \Phi_i^{-1}$, they clearly satisfy $g_{ij} = g_{ji}^{-1}$ as well as the "cocycle condition"

$$g_{ij} g_{jk} g_{ki} = \mathrm{Id}|_{U_i \cap U_j \cap U_k}.$$