We now proceed with the first step towards showing that transversality is generic.

Theorem 1.43. *Let $F : X \times S \longrightarrow Y$ and $g : Z \longrightarrow Y$ be smooth maps of manifolds where only X has boundary. Suppose that F and ∂F are transverse to g. Then for almost every $s \in S$, $f_s = F(\cdot, s)$ and ∂f_s are transverse to g.*

Proof. The fiber product $W = (X \times S) \times_Y Z$ is a regular submanifold (with boundary) of $X \times S \times Z$ and projects to S via the usual projection map π. We show that any $s \in S$ which is a regular value for both the projection map $\pi : W \longrightarrow S$ and its boundary map $\partial\pi$ gives rise to a f_s which is transverse to g. Then by Sard's theorem the s which fail to be regular in this way form a set of measure zero.

Suppose that $s \in S$ is a regular value for π. Suppose that $f_s(x) = g(z) = y$ and we now show that f_s is transverse to g there. Since $F(x, s) = g(z)$ and F is transverse to g, we know that

$$\mathrm{Im} DF_{(x,s)} + \mathrm{Im} Dg_z = T_y Y.$$

Therefore, for any $a \in T_y Y$, there exists $b = (w, e) \in T(X \times S)$ with $DF_{(x,s)}b - a$ in the image of Dg_z. But since $D\pi$ is surjective, there exists $(w', e, c') \in T_{(x,y,z)}W$. Hence we observe that

$$(Df_s)(w - w') - a = DF_{(x,s)}[(w, e) - (w', e)] - a = (DF_{(x,s)}b - a) - DF_{(x,s)}(w', e),$$

where both terms on the right hand side lie in $\mathrm{Im} Dg_z$.

Precisely the same argument (with X replaced with ∂X and F replaced with ∂F) shows that if s is regular for $\partial\pi$ then ∂f_s is transverse to g. This gives the result. □

The previous result immediately shows that transversal maps to $\mathbb{R}^n$ are generic, since for any smooth map $f : M \longrightarrow \mathbb{R}^n$ we may produce a family of maps

$$F : M \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

via $F(x, s) = f(x) + s$. This new map F is clearly a submersion and hence is transverse to any smooth map $g : Z \longrightarrow \mathbb{R}^n$. For arbitrary target manifolds, we will imitate this argument, but we will require a (weak) version of Whitney's embedding theorem for manifolds into $\mathbb{R}^n$.

1.10 Partitions of unity and Whitney embedding

In this section we develop the tool of partition of unity, which will allow us to *go from local to global*, i.e. to glue together objects which are defined locally, creating objects with global meaning. As a particular case of this, to define a global map to $\mathbb{R}^N$ which is an embedding, thereby proving Whitney's embedding theorem.

Definition 13. A collection of subsets $\{U_\alpha\}$ of the topological space M is called *locally finite* when each point $x \in M$ has a neighbourhood V intersecting only finitely many of the U_α.

Definition 14. A covering $\{V_\alpha\}$ is a *refinement* of the covering $\{U_\beta\}$ when each V_α is contained in some U_β.

Lemma 1.44. *Any open covering $\{A_\alpha\}$ of a topological manifold has a countable, locally finite refinement $\{(U_i, \varphi_i)\}$ by coordinate charts such that $\varphi_i(U_i) = B(0, 3)$ and $\{V_i = \varphi_i^{-1}(B(0, 1))\}$ is still a covering of M. We will call such a cover a* regular *covering. In particular, any topological manifold is* paracompact *(i.e. every open cover has a locally finite refinement)*

Proof. If M is compact, the proof is easy: choosing coordinates around any point $x \in M$, we can translate and rescale to find a covering of M by a refinement of the type desired, and choose a finite subcover, which is obviously locally finite.

For a general manifold, we note that by second countability of M, there is a countable basis of coordinate neighbourhoods and each of these charts is a countable union of open sets P_i with $\overline{P_i}$ compact. Hence M has a countable basis $\{P_i\}$ such that $\overline{P_i}$ is compact.

Using these, we may define an increasing sequence of compact sets which exhausts M: let $K_1 = \overline{P}_1$, and

$$K_{i+1} = \overline{P_1 \cup \cdots \cup P_r},$$

where $r > 1$ is the first integer with $K_i \subset P_1 \cup \cdots \cup P_r$.

Now note that M is the union of ring-shaped sets $K_i \backslash K_{i-1}^\circ$, each of which is compact. If $p \in A_\alpha$, then $p \in K_{i+2} \backslash K_{i-1}^\circ$ for some i. Now choose a coordinate neighbourhood $(U_{p,\alpha}, \varphi_{p,\alpha})$ with $U_{p,\alpha} \subset K_{i+2} \backslash K_{i-1}^\circ$ and $\varphi_{p,\alpha}(U_{p,\alpha}) = B(0,3)$ and define $V_{p,\alpha} = \varphi^{-1}(B(0,1))$.

Letting p, α vary, these neighbourhoods cover the compact set $K_{i+1} \backslash K_i^\circ$ without leaving the band $K_{i+2} \backslash K_{i-1}^\circ$. Choose a finite subcover $V_{i,k}$ for each i. Then $(U_{i,k}, \varphi_{i,k})$ is the desired locally finite refinement. □

Definition 15. A smooth partition of unity is a collection of smooth non-negative functions $\{f_\alpha : M \longrightarrow \mathbb{R}\}$ such that

i) $\{\text{supp} f_\alpha = \overline{f_\alpha^{-1}(\mathbb{R} \backslash \{0\})}\}$ is locally finite,

ii) $\sum_\alpha f_\alpha(x) = 1 \ \ \forall x \in M$, hence the name.

A partition of unity is *subordinate* to an open cover $\{U_i\}$ when $\forall \alpha$, $\text{supp} f_\alpha \subset U_i$ for some i.

Theorem 1.45. *Given a regular covering $\{(U_i, \varphi_i)\}$ of a manifold, there exists a partition of unity $\{f_i\}$ subordinate to it with $f_i > 0$ on V_i and supp$f_i \subset \varphi_i^{-1}(\overline{B(0,2)})$.*

Proof. A *bump function* is a smooth non-negative real-valued function $\tilde{g}$ on $\mathbb{R}^n$ with $\tilde{g}(x) = 1$ for $||x|| \leq 1$ and $\tilde{g}(x) = 0$ for $||x|| \geq 2$. For instance, take

$$\tilde{g}(x) = \frac{h(2 - ||x||)}{h(2 - ||x||) + h(||x|| + 1)},$$

for $h(t)$ given by $e^{-1/t}$ for $t > 0$ and 0 for $t < 0$.

Having this bump function, we can produce non-negative bump functions on the manifold $g_i = \tilde{g} \circ \varphi_i$ which have support supp$g_i \subset \varphi_i^{-1}(\overline{B(0,2)})$ and take the value $+1$ on $\overline{V_i}$. Finally we define our partition of unity via

$$f_i = \frac{g_i}{\sum_j g_j}, \quad i = 1, 2, \ldots.$$

□

We now investigate the embedding of arbitrary smooth manifolds as regular submanifolds of $\mathbb{R}^k$. We shall first show by a straightforward argument that any smooth manifold may be embedded in some $\mathbb{R}^N$ for some sufficiently large N. We will then explain how to cut down on N and approach the optimal $N = 2 \dim M$ which Whitney showed (we shall reach $2 \dim M + 1$ and possibly at the end of the course, show $N = 2 \dim M$.)

Theorem 1.46 (Compact Whitney embedding in $\mathbb{R}^N$)**.** *Any* compact *manifold may be embedded in $\mathbb{R}^N$ for sufficiently large N.*

Proof. Let $\{(U_i \supset V_i, \varphi_i)\}_{i=1}^k$ be a *finite* regular covering, which exists by compactness. Choose a partition of unity $\{f_1, \ldots, f_k\}$ as in Theorem 1.45 and define the following "zoom-in" maps $M \longrightarrow \mathbb{R}^{\dim M}$:

$$\tilde{\varphi}_i(x) = \begin{cases} f_i(x)\varphi_i(x) & x \in U_i, \\ 0 & x \notin U_i. \end{cases}$$

Then define a map $\Phi : M \longrightarrow \mathbb{R}^{k(\dim M + 1)}$ which zooms simultaneously into all neighbourhoods, with extra information to guarantee injectivity:

$$\Phi(x) = (\tilde{\varphi}_1(x), \ldots, \tilde{\varphi}_k(x), f_1(x), \ldots, f_k(x)).$$

Note that $\Phi(x) = \Phi(x')$ implies that for some i, $f_i(x) = f_i(x') \neq 0$ and hence $x, x' \in U_i$. This then implies that $\varphi_i(x) = \varphi_i(x')$, implying $x = x'$. Hence Φ is injective.

We now check that $D\Phi$ is injective, which will show that it is an injective immersion. At any point x the differential sends $v \in T_xM$ to the following vector in $\mathbb{R}^{\dim M} \times \cdots \times \mathbb{R}^{\dim M} \times \mathbb{R} \times \cdots \times \mathbb{R}$.

$$(Df_1(v)\varphi_1(x) + f_1(x)D\varphi_1(v), \ldots, Df_k(v)\varphi_k(x) + f_k(x)D\varphi_1(v), Df_1(v), \ldots, Df_k(v)$$

But this vector cannot be zero. Hence we see that Φ is an immersion.

But an injective immersion from a compact space must be an embedding: view Φ as a bijection onto its image. We must show that Φ^{-1} is continuous, i.e. that Φ takes closed sets to closed sets. If $K \subset M$ is closed, it is also compact and hence $\Phi(K)$ must be compact, hence closed (since the target is Hausdorff). $\square$

Theorem 1.47 (Compact Whitney embedding in $\mathbb{R}^{2n+1}$)**.** *Any* compact *n-manifold may be embedded in* $\mathbb{R}^{2n+1}$.

Proof. Begin with an embedding $\Phi : M \longrightarrow \mathbb{R}^N$ and assume $N > 2n + 1$. We then show that by projecting onto a hyperplane it is possible to obtain an embedding to $\mathbb{R}^{N-1}$.

A vector $v \in S^{N-1} \subset \mathbb{R}^N$ defines a hyperplane (the orthogonal complement) and let $P_v : \mathbb{R}^N \longrightarrow \mathbb{R}^{N-1}$ be the orthogonal projection to this hyperplane. We show that the set of v for which $\Phi_v = P_v \circ \Phi$ fails to be an embedding is a set of measure zero, hence that it is possible to choose v for which Φ_v is an embedding.

Φ_v fails to be an embedding exactly when Φ_v is not injective or $D\Phi_v$ is not injective at some point. Let us consider the two failures separately:

If v is in the image of the map $\beta_1 : (M \times M)\backslash\Delta_M \longrightarrow S^{N-1}$ given by

$$\beta_1(p_1, p_2) = \frac{\Phi(p_2) - \Phi(p_1)}{||\Phi(p_2) - \Phi(p_1)||},$$

then Φ_v will fail to be injective. Note however that β_1 maps a $2n$-dimensional manifold to a $N-1$-manifold, and if $N > 2n + 1$ then baby Sard's theorem implies the image has measure zero.

The immersion condition is a local one, which we may analyze in a chart (U, φ). Φ_v will fail to be an immersion in U precisely when v coincides with a vector in the normalized image of $D(\Phi \circ \varphi^{-1})$ where

$$\Phi \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N.$$

Hence we have a map (letting $N(w) = ||w||$)

$$\frac{D(\Phi \circ \varphi^{-1})}{N \circ D(\Phi \circ \varphi^{-1})} : U \times S^{n-1} \longrightarrow S^{N-1}.$$

The image has measure zero as long as $2n - 1 < N - 1$, which is certainly true since $2n < N - 1$. Taking union over countably many charts, we see that immersion fails on a set of measure zero in S^{N-1}.

Hence we see that Φ_v fails to be an embedding for a set of $v \in S^{N-1}$ of measure zero. Hence we may reduce N all the way to $N = 2n + 1$. $\square$

Corollary 1.48. *We see from the proof that if we do not require injectivity but only that the manifold be* immersed *in $\mathbb{R}^N$, then we can take $N = 2n$ instead of $2n + 1$.*