In fact, the inverse function theorem leads to a normal form theorem for a more general class of maps:

Theorem 1.19 (Constant rank theorem). *Let V, W be m, n-dimensional vector spaces and $U \subset V$ an open set. If $f : U \longrightarrow W$ is a smooth map such that Df has constant rank k in U, then for each point $p \in U$ there are charts (U, φ) and (V, ψ) containing $p, f(p)$ such that*

$$\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0).$$

Proof. since rk $(f) = k$ at p, there is a $k \times k$ minor of $Df(p)$ with nonzero determinant. Reorder the coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$ so that this minor is top left, and translate coordinates so that $f(0) = 0$. label the coordinates $(x_1, \ldots, x_k, y_1, \ldots y_{m-k})$ on V and $(u_1, \ldots u_k, v_1, \ldots, v_{n-k})$ on W.

Then we may write $f(x, y) = (Q(x, y), R(x, y))$, where Q is the projection to $u = (u_1, \ldots, u_k)$ and R is the projection to v. with $\frac{\partial Q}{\partial x}$ nonsingular. First we wish to put Q into normal form. Consider the map $\phi(x, y) = (Q(x, y), y)$, which has derivative

$$D\phi = \begin{pmatrix} \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \\ 0 & 1 \end{pmatrix}$$

As a result we see $D\phi(0)$ is nonsingular and hence there exists a local inverse $\phi^{-1}(x, y) = (A(x, y), B(x, y))$. Since it's an inverse this means $(x, y) = \phi(\phi^{-1}(x, y)) = (Q(A, B), B)$, which implies that $B(x, y) = y$.

Then $f \circ \phi^{-1} : (x, y) \mapsto (x, \tilde{R} = R(A, y))$, and must still be of rank k. Since its derivative is

$$D(f \circ \phi^{-1}) = \begin{pmatrix} I_{k \times k} & 0 \\ \frac{\partial \tilde{R}}{\partial x} & \frac{\partial \tilde{R}}{\partial y} \end{pmatrix}$$

we conclude that $\frac{\partial \tilde{R}}{\partial y} = 0$, meaning that

$$f \circ \phi^{-1} : (x, y) \mapsto (x, S(x)).$$

We now postcompose by the diffeomorphism $\sigma : (u, v) \mapsto (u, v - s(u))$, to obtain

$$\sigma \circ f \circ \phi^{-1} : (x, y) \mapsto (x, 0),$$

as required. □

As we shall see, these theorems have many uses. One of the most straightforward uses is for defining *submanifolds.*

Definition 8. A *regular submanifold* of dimension k in an n-manifold M is a subspace $S \subset M$ such that $\forall s \in S$, there exists a chart (U, φ) for M, containing s, and with

$$S \cap U = \varphi^{-1}(x_{k+1} = \cdots = x_n = 0).$$

In other words, the inclusion $S \subset M$ is locally isomorphic to the vector space inclusion $\mathbb{R}^k \subset \mathbb{R}^n$.

Of course, the remaining coordinates $\{x_1, \ldots, x_k\}$ define a smooth manifold structure on S itself, justifying the terminology.

Proposition 1.20. *If $f : M \longrightarrow N$ is a smooth map of manifolds, and if $Df(p)$ has constant rank on M, then for any $q \in f(M)$, the inverse image $f^{-1}(q) \subset M$ is a regular submanifold.*

Proof. Let $x \in f^{-1}(q)$. Then there exist charts ψ, φ such that $\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$ and $f^{-1}(q) \cap U = \{x_1 = \cdots = x_k = 0\}$. Hence we obtain that $f^{-1}(q)$ is a codimension k regular submanifold. □

Example 1.21. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be given by $(x_1, \dots, x_n) \mapsto \sum x_i^2$. Then $Df(x) = (2x_1, \dots, 2x_n)$, which has rank 1 at all points in $\mathbb{R}^n \backslash \{0\}$. Hence since $f^{-1}(q)$ contains $\{0\}$ iff $q = 0$, we see that $f^{-1}(q)$ is a regular submanifold for all $q \neq 0$. Exercise: show that this manifold structure is compatible with that obtained in Example 1.9.*

The previous example leads to an observation of the following special case of the previous corollary.

Proposition 1.22. *If $f : M \longrightarrow N$ is a smooth map of manifolds and $Df(p)$ has rank equal to $\dim N$ along $f^{-1}(q)$, then this subset $f^{-1}(q)$ is an embedded submanifold of M.*

Proof. Since the rank is maximal along $f^{-1}(q)$, it must be maximal in an open neighbourhood $U \subset M$ containing $f^{-1}(q)$, and hence $f : U \longrightarrow N$ is of constant rank. □

Definition 9. If $f : M \longrightarrow N$ is a smooth map such that $Df(p)$ is surjective, then p is called a *regular point*. Otherwise p is called a *critical point*. If all points in the level set $f^{-1}(q)$ are regular points, then q is called a *regular value*, otherwise q is called a critical value. In particular, if $f^{-1}(q) = \emptyset$, then q is regular.

It is often useful to highlight two classes of smooth maps; those for which Df is everywhere *injective*, or, on the other hand *surjective*.

Definition 10. A smooth map $f : M \longrightarrow N$ is called a *submersion* when $Df(p)$ is surjective at all points $p \in M$, and is called an *immersion* when $Df(p)$ is injective at all points $p \in M$. If f is an injective immersion which is a homeomorphism onto its image (when the image is equipped with subspace topology), then we call f an *embedding*

Proposition 1.23. *If $f : M \longrightarrow N$ is an embedding, then $f(M)$ is a regular submanifold.*

Proof. Let $f : M \longrightarrow N$ be an embedding. Then for all $m \in M$, we have charts (U, φ), (V, ψ) where $\psi \circ f \circ \varphi^{-1} : (x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$. If $f(U) = f(M) \cap V$, we're done. To make sure that some other piece of M doesn't get sent into the neighbourhood, use the fact that $F(U)$ is open in the subspace topology. This means we can find a smaller open set $V' \subset V$ such that $V' \cap f(M) = f(U)$. Then we can restrict the charts $(V', \psi|_{V'})$, $(U' = f^{-1}(V'), \varphi_{U'})$ so that we see the embedding. □

Having the constant rank theorem in hand, we may also apply it to study manifolds *with boundary*. The following two results illustrate how this may easily be done.

Proposition 1.24. *Let M be a smooth n-manifold and $f : M \longrightarrow \mathbb{R}$ a smooth real-valued function, and let a, b, with $a < b$, be regular values of f. Then $f^{-1}([a, b])$ is a cobordism between the $n-1$-manifolds $f^{-1}(a)$ and $f^{-1}(b)$.*

Proof. The pre-image $f^{-1}((a, b))$ is an open subset of M and hence a submanifold of M. Since p is regular for all $p \in f^{-1}(a)$, we may (by the constant rank theorem) find charts such that f is given near p by the linear map

$$(x_1, \dots, x_m) \mapsto x_m.$$

Possibly replacing x_m by $-x_m$, we therefore obtain a chart near p for $f^{-1}([a, b])$ into H^m, as required. Proceed similarly for $p \in f^{-1}(b)$. □

Example 1.25. *Using $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $(x_1, \dots, x_n) \mapsto \sum x_i^2$, this gives a simple proof for the fact that the closed unit ball $\overline{B(0, 1)} = f^{-1}([-1, 1])$ is a manifold with boundary.*

Example 1.26. *Consider the C^∞ function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by $(x, y, z) \mapsto x^2 + y^2 - z^2$. Both $+1$ and -1 are regular values for this map, with pre-images given by 1- and 2-sheeted hyperboloids, respectively. Hence $f^{-1}([-1, 1])$ is a cobordism between hyperboloids of 1 and 2 sheets. In other words, it defines a cobordism between the disjoint union of two closed disks and the closed cylinder (each of which has boundary $S^1 \sqcup S^1$). Does this cobordism tell us something about the cobordism class of a connected sum?*

Proposition 1.27. *Let $f : M \longrightarrow N$ be a smooth map from a manifold with boundary to the manifold N. Suppose that $q \in N$ is a regular value of f and also of $f|_{\partial M}$. Then the pre-image $f^{-1}(q)$ is a regular submanifold with boundary (i.e. locally modeled on $\mathbb{R}^k \subset \mathbb{R}^n$ or the inclusion $H^k \subset H^n$ given by $(x_1, \ldots x_k) \mapsto (0, \ldots, 0, x_1, \ldots x_k)$.) Furthermore, the boundary of $f^{-1}(q)$ is simply its intersection with ∂M.*

Proof. If $p \in f^{-1}(q)$ is not in ∂M, then as before $f^{-1}(q)$ is a regular submanifold in a neighbourhood of p. Therefore suppose $p \in \partial M \cap f^{-1}(q)$. Pick charts φ, ψ so that $\varphi(p) = 0$ and $\psi(q) = 0$, and $\psi f \varphi^{-1}$ is a map $U \subset H^m \longrightarrow \mathbb{R}^n$. Extend this to a smooth function $\tilde{f}$ defined in an open set $\tilde{U} \subset \mathbb{R}^m$ containing U. Shrinking $\tilde{U}$ if necessary, we may assume $\tilde{f}$ is regular on $\tilde{U}$. Hence $\tilde{f}^{-1}(0)$ is a regular submanifold of $\mathbb{R}^m$ of dimension $m - n$.

Now consider the real-valued function $\pi : \tilde{f}^{-1}(0) \longrightarrow \mathbb{R}$ given by the restriction of $(x_1, \ldots, x_m) \mapsto x_m$. $0 \in \mathbb{R}$ must be a regular value of π, since if not, then the tangent space to $\tilde{f}^{-1}(0)$ at 0 would lie completely in $x_m = 0$, which contradicts the fact that q is a regular point for $f|_{\partial M}$.

Hence, by Proposition 1.24, we have expressed $f^{-1}(q)$, in a neighbourhood of p, as a regular submanifold with boundary given by $\{\varphi^{-1}(x) \ : \ x \in \tilde{f}^{-1}(0) \text{ and } \pi(x) \geq 0\}$, as required. □