Having defined the integral, we wish to explain the duality between d and ∂: A $n-1$-form α on a n-manifold may be pulled back to the boundary ∂M and integrated. On the other hand, it can be differentiated and integrated over M. The fact that these are equal is Stokes' theorem, and is a generalization of the fundamental theorem of calculus.

First we must some simple observations concerning the behaviour of forms in a neighbourhood of the boundary.

Recall the operation of contraction with a vector field X, which maps $\rho \in \Omega^k(M)$ to $i_X\rho \in \Omega^{k-1}(M)$, defined by the condition of being a graded derivation $i_X(\alpha\wedge\beta) = i_X\alpha\wedge\beta + (-1)^{|\alpha|}\alpha\wedge i_X\beta$ such that $i_X f = 0$ and $i_X df = X(f)$ for all $f \in C^\infty(M,\mathbb{R})$.

Proposition 4.13. *Let M be a manifold with boundary. If M is orientable, then so is ∂M. Furthermore, an orientation on M induces one on ∂M.*

Proof. Given a locally finite atlas (U_i) of ∂M, in each U_i we can pick a nonvanishing outward-pointing vector field X_i in $\Gamma^\infty(U_i, j^*TM)$, for $j : \partial M \longrightarrow M$ the inclusion. Let (θ_i) be a subordinate partition of unity, and form $X = \sum_i \theta_i X_i$. This is a vector field on ∂M, tangent to M and pointing outward everywhere along the boundary.

Given an orientation $[v]$ of M, we can form $[i_X v]$, which is then an orientation of ∂M. This depends only on $[v]$ and X being a nonvanishing outward vector field. $\square$

We now verify a local computation leading to Stokes' theorem. If

$$\alpha = \sum_i a_i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^m$$

is a degree $m-1$ form with compact support in $U \subset H^m$, and if U does not intersect the boundary ∂H^m, then by the fundamental theorem of calculus,

$$\int_U d\alpha = \sum_i (-1)^{i-1} \int_U \tfrac{\partial a_i}{\partial x^i} dx^1 \cdots dx^m = 0.$$

Now suppose that $V = U \cap \partial H^m \neq \emptyset$. Then

$$\begin{aligned}\int_U d\alpha &= \sum_i (-1)^{i-1} \int_U \tfrac{\partial a_i}{\partial x^i} dx^1 \cdots dx^m \\ &= -(-1)^{m-1} \int_V a_m(x_1, \ldots, x_{m-1}, 0) dx^1 \cdots dx^{m-1} \\ &= \int_V a_m(x_1, \ldots, x_{m-1}, 0) i_{-\frac{\partial}{\partial x^m}} (dx^1 \wedge \cdots dx^m) \\ &= \int_V j^*\alpha,\end{aligned}$$

where the last integral is with respect to the orientation induced by the outward vector field.

Theorem 4.14 (Stokes' theorem)**.** *Let M be an oriented manifold with boundary, and let the boundary be oriented with respect to an outward pointing vector field. Then for $\alpha \in \Omega_c^{m-1}(M)$ and $j : \partial M \longrightarrow M$ the inclusion of the boundary, we have*

$$\int_M d\alpha = \int_{\partial M} j^*\alpha.$$

Proof. For a locally finite atlas (U_i, φ_i), we have

$$\int_M d\alpha = \int_M d(\sum_i \theta_i\alpha) = \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* d(\theta_i\alpha)$$

By the local calculation above, if $\varphi_i(U_i) \cap \partial H^m = \emptyset$, the summand on the right hand side vanishes. On the other hand, if $\varphi_i(U_i) \cap \partial H^m \neq \emptyset$, we obtain (letting $\psi_i = \varphi_i|_{U_i \cap \partial M}$ and $j' : \partial H^m \longrightarrow \mathbb{R}^n$), using the local result,

$$\begin{aligned} \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* d(\theta_i \alpha) &= \int_{\varphi_i(U_i) \cap \partial H^m} j'^* (\varphi_i^{-1})^* (\theta_i \alpha) \\ &= \int_{\varphi_i(U_i) \cap \partial H^m} (\psi_i^{-1})^* (j^* (\theta_i \alpha)). \end{aligned}$$

This then shows that $\int_M d\alpha = \int_{\partial M} j^* \alpha$, as desired. □

Corollary 4.15. *If $\partial M = \emptyset$, then for all $\alpha \in \Omega_c^{n-1}(M)$, we have $\int_M d\alpha = 0$.*

Corollary 4.16. *Let M be orientable and compact, and let $v \in \Omega^n(M)$ be nonvanishing. Then $\int_M v > 0$, when M is oriented by $[v]$. Hence, v cannot be exact, by the previous corollary. This tells us that the class $[v] \in H^n_{dR}(M)$ cannot be zero. In this way, integration of a closed form may often be used to show that it is nontrivial in de Rham cohomology.*

4.3 The Mayer-Vietoris sequence

Decompose a manifold M into a union of open sets $M = U \cup V$. We wish to relate the de Rham cohomology of M to that of U and V separately, and also that of $U \cap V$. These 4 manifolds are related by obvious inclusion maps as follows:

$$U \cup V \longleftarrow U \sqcup V \overset{\partial_U}{\underset{\partial_V}{\leftleftarrows}} U \cap V$$

Applying the functor $\Omega^\bullet$, we obtain morphisms of complexes in the other direction, given by simple restriction (pullback under inclusion):

$$\Omega^\bullet(U \cup V) \longrightarrow \Omega^\bullet(U) \oplus \Omega^\bullet(V) \overset{\partial_V^*}{\underset{\partial_U^*}{\rightrightarrows}} \Omega^\bullet(U \cap V)$$

Now we notice the following: if forms $\omega \in \Omega^\bullet(U)$ and $\tau \in \Omega^\bullet(V)$ come from a single global form on $U \cup V$, then they are killed by $\partial_V^* - \partial_U^*$. Hence we obtain a complex of (morphisms of cochain complexes):

$$0 \longrightarrow \Omega^\bullet(U \cup V) \longrightarrow \Omega^\bullet(U) \oplus \Omega^\bullet(V) \xrightarrow{\partial_V^* - \partial_U^*} \Omega^\bullet(U \cap V) \longrightarrow 0 \tag{28}$$

It is clear that this complex is exact at the first position, since a form must vanish if it vanishes on U and V. Similarly, if forms on U, V agree on $U \cap V$, they must glue to a form on $U \cup V$. Hence the complex is exact at the middle position. We now show that the complex is exact at the last position.

Theorem 4.17. *The above complex (of de Rham complexes) is exact. It may be called a "short exact sequence" of cochain complexes.*

Proof. Let $\alpha \in \Omega^q(U \cap V)$. We wish to write α as a difference $\tau - \omega$ with $\tau \in \Omega^q(U)$ and $\omega \in \Omega^q(V)$. Let (ρ_U, ρ_V) be a partition of unity subordinate to (U, V). Then we have $\alpha = \rho_U \alpha - (-\rho_V \alpha)$ in $U \cap V$. Now observe that $\rho_U \alpha$ may be extended by zero in V (call the result τ), while $-\rho_V \alpha$ may be extended by zero in U (call the result ω). Then we have $\alpha = (\partial_V^* - \partial_U^*)(\tau, \omega)$, as required. □

It is not surprising that given an exact sequence of morphisms of complexes

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$$

, we obtain maps between the cohomology groups of the complexes

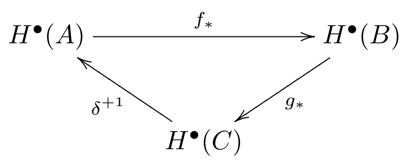
$$H^k(A^\bullet) \xrightarrow{f_*} H^k(B^\bullet) \xrightarrow{g_*} H^k(C^\bullet).$$

And it is not difficult to see that this sequence is exact at the middle term: Let $[\rho] \in H^k(B^\bullet)$, for $\rho \in B^k$ such that $d_B\rho = 0$. Suppose that $g(\rho) = 0$ in C^k, so that there exists $\tau \in A^k$ with $f(\tau) = \rho$. Then since f is a morphism of complexes, it follows that $f(d_A\tau) = d_B f(\tau) = d_B\rho = 0$. Since $f : A^{k+1} \longrightarrow B^{k+1}$ is injective, this implies that $d_A\tau = 0$, so we have $f_*[\tau] = [\rho]$, as required.

The interesting thing is that the maps g_* are not necessarily surjective, nor are f_* necessarily injective. In fact, there is a natural map $\delta : H^k(C^\bullet) \longrightarrow H^{k+1}(A^\bullet)$ (called the connecting homomorphism) which extends the 3-term sequence to a full complex involving all cohomology groups of arbitrary degree:

If $[\alpha] \in H^k(C^\bullet)$, where $d_C\alpha = 0$, then there must exist $\xi \in B^k$ with $g(\xi) = \alpha$, and $g(d_B\xi) = d_C(g(\xi)) = d_C\alpha = 0$, so that there must exist $\beta \in A^{k+1}$ with $f(\beta) = d_B\xi$, and $f(d_A\beta) = d_B(f(\beta) = 0$. Hence this determines a class $[\beta] \in H^{k+1}(A^\bullet)$, and one can check that this does not depend on the choices made. We then define $\delta([\alpha]) = [\beta]$.

Exercise: with this definition of δ, we obtain a "long exact sequence" of vector spaces as follows:



Therefore, from the complex of complexes (28), we immediately obtain a long exact sequence of vector spaces, called the Mayer-Vietoris sequence:

$$\cdots \longrightarrow H^k(U \cup V) \longrightarrow H^k(U) \oplus H^k(V) \longrightarrow H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(U \cup V) \longrightarrow \cdots,$$

where the first map is simply a restriction map, the second map is the difference of the restrictions $\delta_V^* - \delta_U^*$, and the third map is the connecting homomorphism δ, which can be written explicitly as follows:

$$\delta[\alpha] = [\beta], \quad \beta = -d(\rho_V\alpha) = d(\rho_U\alpha).$$

(notice that β has support contained in $U \cap V$.)

4.4 Examples of cohomology computations

Example 4.18 (Circle)**.** *Here we present another computation of $H^\bullet_{dR}(S^1)$, by the Mayer-Vietoris sequence. Express $S_1 = U_0 \cup U_1$ as before, with $U_i \cong \mathbb{R}$, so that $H^0(U_i) = \mathbb{R}$, $H^1_{dR}(U_i) = 0$ by the Poincaré lemma. Since $U_0 \cap U_1 \cong \mathbb{R} \sqcup \mathbb{R}$, we have $H^0(U_0 \cap U_1) = \mathbb{R} \oplus \mathbb{R}$ and $H^1(U_0 \cap U_1) = 0$. Since we know that $H^2_{dR}(S^1) = 0$, the Mayer-Vietoris sequence only has 4 a priori nonzero terms:*

$$0 \longrightarrow H^0(S^1) \longrightarrow \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta_1^* - \delta_0^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta} H^1(S^1) \longrightarrow 0.$$

The middle map takes $(c_1, c_0) \mapsto c_1 - c_0$ and hence has 1-dimensional kernel. Hence $H^0(S^1) = \mathbb{R}$. Furthermore the kernel of δ must only be 1-dimensional, hence $H^1(S^1) = \mathbb{R}$ as well. Exercise: Using a partition of unity, determine an explicit representative for the class in $H^1_{dR}(S^1)$, starting with the function on $U_0 \cap U_1$ which takes values 0,1 on each respective connected component.

Example 4.19 (Spheres)**.** *To determine the cohomology of S^2, decompose into the usual coordinate charts U_0, U_1, so that $U_i \cong \mathbb{R}^2$, while $U_0 \cap U_1 \sim S^1$. The first line of the Mayer-Vietoris sequence is*

$$0 \longrightarrow H^0(S^2) \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R}.$$

The third map is nontrivial, since it is just the subtraction. Hence this first line must be exact, and $H^0(S^2) = \mathbb{R}$ *(not surprising since* S^2 *is connected). The second line then reads (we can start it with zero since the first line was exact)*

$$0 \longrightarrow H^1(S^2) \longrightarrow 0 \longrightarrow H^1(S^1) = \mathbb{R},$$

where the second zero comes from the fact that $H^1(\mathbb{R}^2) = 0$*. This then shows us that* $H^1(S^2) = 0$*. The last term, together with the third line now give*

$$0 \longrightarrow H^1(S^1) = \mathbb{R} \longrightarrow H^2(S^2) \longrightarrow 0,$$

showing that $H^2(S^2) = \mathbb{R}$*.*

Continuing this process, we obtain the de Rham cohomology of all spheres:

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R}, & \text{for } k = 0 \text{ or } n, \\ 0 & \text{otherwise.} \end{cases}$$