The fact that $d^2 = 0$ is dual to the fact that $\partial(\partial M) = \emptyset$ for a manifold with boundary M. We will see later that Stokes' theorem explains this duality. Because of the fact $d^2 = 0$, we have a very special algebraic structure: we have a sequence of vector spaces $\Omega^k(M)$, and maps $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ which are such that any successive composition is zero. This means that the image of d is contained in the kernel of the next d in the sequence. This arrangement of vector spaces and operators is called a *cochain complex* of vector spaces [1]. We often simply refer to this as a "complex" and omit the term "cochain". The reason for the "co" is that the differential increases the degree k, which is opposite to the usual boundary map on manifolds, which decreases k. We will see chain complexes when we study homology.

A complex of vector spaces is usually drawn as a linear sequence of symbols and arrows as follows: if $f : U \longrightarrow V$ is a linear map and $g : V \longrightarrow W$ is a linear map such that $g \circ f = 0$, then we write

$$U \xrightarrow{f} V \xrightarrow{g} W$$

In general, this simply means that $\mathrm{im} f \subset \ker g$, and to measure the difference between them we look at the quotient $\ker g/\mathrm{im} f$, which is called the **cohomology** of the complex at the position V (or homology, if d decreases degree). If we are lucky, and the complex has no cohomology at V, meaning that $\ker g$ is precisely equal to $\mathrm{im} f$, then we say that the complex is **exact** at V. If the complex is exact everywhere, we call it an exact sequence (and it has no cohomology!) In our case, we have a longer cochain complex:

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \longrightarrow 0$$

There is a bit of terminology to learn: we have seen that if $d\rho = 0$ then ρ is called *closed*. But these are also called **cocycles** and denoted $Z^k(M)$. Similarly the exact forms $d\alpha$ are also called **coboundaries**, and are denoted $B^k(M)$. Hence the cohomology groups may be written $H^k_{dR}(M) = Z^k_{dR}(M)/B^k_{dR}(M)$.

Definition 31. The de Rham complex is the complex $(\Omega^\bullet(M), d)$, and its cohomology at $\Omega^k(M)$ is called $H^k_{dR}(M)$, the de Rham cohomology.

Exercise: Check that the graded vector space $H^\bullet_{dR}(M) = \bigoplus_{k\in\mathbb{Z}} H^k(M)$ inherits a product from the wedge product of forms, making it into a $\mathbb{Z}$-graded ring. This is called the de Rham cohomology ring of M, and the product is called the *cup product*.

It is clear from the definition of d that it commutes with pullback via diffeomorphisms, in the sense $f^* \circ d = d \circ f^*$. But this is only a special case of a more fundamental property of d:

Theorem 4.4. *Exterior differentiation commutes with pullback: for $f : M \longrightarrow N$ a smooth map, $f^* \circ d_N = d_M \circ f^*$.*

Proof. We need only check this on functions g and exact 1-forms dg: let X be a vector field on M and $g \in C^\infty(N, \mathbb{R})$.

$$f^*(dg)(X) = dg(f_*X) = \pi_2 g_* f_* X = \pi_2(g \circ f)_* X = d(f^*g)(X),$$

giving $f^*dg = df^*g$, as required. For exact 1-forms we have $f^*d(dg) = 0$ and $d(f^*dg) = d(df^*g) = 0$ by the result for functions. ☐

This theorem may be interpreted as follows: The differential forms give us a $\mathbb{Z}$-graded ring, $\Omega^\bullet(M)$, which is equipped with a differential $d : \Omega^k \longrightarrow \Omega^{k+1}$. This sequence of vector spaces and maps which compose to zero is called a *cochain complex*. Beyond it being a cochain complex, it is equipped with a wedge product.

Cochain complexes $(C^\bullet, d_C)$ may be considered as objects of a new category, whose morphisms consist of a sum of linear maps $\psi_k : C^k \longrightarrow D^k$ commuting with the differentials, i.e. $d_D \circ \psi_k = \psi_{k+1} \circ d_C$. The previous theorem shows that pullback f^* defines a morphism of cochain complexes $\Omega^\bullet(N) \longrightarrow \Omega^\bullet(M)$; indeed it even preserves the wedge product, hence it is a morphism of differential graded algebras.

[1] since this complex appears for $\Omega^\bullet(U)$ for any open set $U \subset M$, this is actually a cochain complex of *sheaves* of vector spaces, but this won't concern us right away.

Corollary 4.5. *We may interpret the previous result as showing that $\Omega^\bullet$ is a functor from manifolds to differential graded algebras (or, if we forget the wedge product, to the category of cochain complexes). As a result, we see that the de Rham cohomology $H^\bullet_{dR}$ may be viewed as a functor, from smooth manifolds to $\mathbb{Z}$-graded commutative rings.*

Example 4.6. *S^1 is connected, and hence $H^0_{dR}(S^1) = \mathbb{R}$. So it remains to compute $H^1_{dR}(S^1)$.*

Let $\frac{\partial}{\partial\theta}$ be the rotational vector field on S^1 of unit Euclidean norm, and let $d\theta$ be its dual 1-form, i.e. $d\theta(\frac{\partial}{\partial\theta}) = 1$. Note that θ is not a well-defined function on S^1, so the notation $d\theta$ may be misleading at first.

Of course, $d(d\theta) = 0$, since $\Omega^2(S^1) = 0$. We might ask, is there a function $f(\theta)$ such that $df = d\theta$? This would mean $\frac{\partial f}{\partial\theta} = 1$, and hence $f = \theta + c_2$. But since f is a function on S^1, we must have $f(\theta + 2\pi) = f(\theta)$, which is a contradiction. Hence $d\theta$ is not exact, and $[d\theta] \neq 0$ in $H^1_{dR}(S^1)$.

Any other 1-form will be closed, and can be represented as $gd\theta$ for $g \in C^\infty(S^1, \mathbb{R})$. Let $\overline{g} = \frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi} g(\theta)d\theta$ be the average value of g, and consider $g_0 = g - \overline{g}$. Then define

$$f(\theta) = \int_{t=0}^{t=\theta} g_0(t)dt.$$

Clearly we have $\frac{\partial f}{\partial\theta} = g_0(\theta)$, and furthermore f is a well-defined function on S^1, since $f(\theta + 2\pi) = f(\theta)$. Hence we have that $g_0 = df$, and hence $g = \overline{g} + df$, showing that $[gd\theta] = \overline{g}[d\theta]$.

Hence $H^1_{dR}(S^1) = \mathbb{R}$, and as a ring, $H^0_{dR} + H^1_{dR}$ is simply $\mathbb{R}[x]/(x^2)$.

Note that technically we have proven that $H^1_{dR}(S^1) \cong \mathbb{R}$, but we will see from the definition of integration later that this isomorphism is canonical.

The de Rham cohomology is an important invariant of a smooth manifold (in fact it doesn't even depend on the smooth structure, only the topological structure). To compute it, there are many tools available. There are three particularly important tools: first, there is Poincaré's lemma, telling us the cohomology of $\mathbb{R}^n$. Second, there is integration, which allows us to prove that certain cohomology classes are non-trivial. Third, there is the Mayer-Vietoris sequence, which allows us to compute the cohomology of a union of open sets, given knowledge about the cohomology of each set in the union.

Lemma 4.7. *Consider the embeddings $J_i : M \longrightarrow M \times [0, 1]$ given by $x \mapsto (x, i)$ for $i = 0, 1$. The induced morphisms of de Rham complexes J_0^* and J_1^* are chain homotopic morphisms, meaning that there is a linear map $K : \Omega^k(M \times [0, 1]) \longrightarrow \Omega^{k-1}(M)$ such that*

$$J_1^* - J_0^* = dK + Kd$$

This shows that on closed forms, J_i^* may differ, but only by an exact form.

Proof. Let t be the coordinate on $[0, 1]$. Define $Kf = 0$ for $f \in \Omega^0(M \times [0, 1])$, and $K\alpha = 0$ if $\alpha = f\rho$ for $\rho \in \Omega^k(M)$. But for $\beta = fdt \wedge \rho$ we define

$$K\beta = (\int_0^1 fdt)\rho.$$

Then we verify that

$$dKf + Kdf = 0 + \int_0^1 \tfrac{\partial f}{\partial t}dt = (J_1^* - J_0^*)f,$$

$$dK\alpha + Kd\alpha = 0 + (\int_0^1 \tfrac{\partial f}{\partial t}dt)\rho = (J_1^* - J_0^*)\alpha,$$

$$dK\beta + Kd\beta = (\int_0^1 d_M fdt) \wedge \rho + (\int_0^1 fdt) \wedge d\rho + K(df \wedge dt \wedge \rho - fdt \wedge d\rho) = 0,$$

which agrees with $(J_1^* - J_0^*)\beta = 0 - 0 = 0$. Note that we have used $K(df \wedge dt \wedge \rho) = K(-dt \wedge d_M f \wedge \rho) = -(\int_0^1 d_M f) \wedge \rho$, and the notation $d_M f$ is a time-dependent 1-form whose value at time t is the exterior derivative on M of the function $f(-, t) \in \Omega^0(M)$. □

The previous theorem can be used in a clever way to prove that homotopic maps $M \longrightarrow N$ induce the same map on cohomology:

Theorem 4.8. *Let $f : M \longrightarrow N$ and $g : M \longrightarrow N$ be smooth maps which are (smoothly) homotopic. Then $f^* = g^*$ as maps $H^\bullet(N) \longrightarrow H^\bullet(M)$.*

Proof. Let $H : M \times [0,1] \longrightarrow N$ be a (smooth) homotopy between f, g, and let J_0, J_1 be the embeddings $M \longrightarrow M \times [0,1]$ from the previous result, so that $H \circ J_0 = f$ and $H \circ J_1 = g$. Recall that $J_1^* - J_0^* = dK + Kd$, so we have

$$g^* - f^* = (J_1^* - J_0^*)H^* = (dK + Kd)H^* = dKH^* + KH^*d$$

This shows that f^*, g^* differ, on closed forms, only by exact terms, and hence are equal on cohomology. □

Corollary 4.9. *If M, N are (smoothly) homotopic, then $H^\bullet_{dR}(M) \cong H^\bullet_{dR}(N)$.*

Proof. M, N are homotopic iff we have maps $f : M \longrightarrow N$, $g : N \longrightarrow M$ with $fg \sim 1$ and $gf \sim 1$. This shows that $f^*g^* = 1$ and $g^*f^* = 1$, hence f^*, g^* are inverses of each other on cohomology, and hence isomorphisms. □

Corollary 4.10 (Poincaré lemma)**.** *Since $\mathbb{R}^n$ is homotopic to the 1-point space ($\mathbb{R}^0$), we have*

$$H^k_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{for } k = 0 \\ 0 & \text{for } k > 0 \end{cases}$$

As a note, we should mention that the homotopy in the previous theorem need not be smooth, since any homotopy may be deformed (using a continuous homotopy) to a smooth homotopy, by smooth approximation. Hence we finally obtain that the de Rham cohomology is a homotopy invariant of smooth manifolds.

4.2 Integration

Since we are accustomed to the idea that a function may be integrated over a subset of $\mathbb{R}^n$, we might think that if we have a function on a manifold, we can compute its local integrals and take a sum. This, however, makes no sense, because the answer will depend on the particular coordinate system you choose in each open set: for example, if $f : U \longrightarrow \mathbb{R}$ is a smooth function on $U \subset \mathbb{R}^n$ and $\varphi : V \longrightarrow U$ is a diffeomorphism onto $V \subset \mathbb{R}^n$, then we have the usual change of variables formula for the (Lebesgue or Riemann) integral:

$$\int_U f dx^1 dx^2 \cdots dx^n = \int_V \varphi^* f \left|\det[\tfrac{\partial \varphi_i}{\partial x^j}]\right| dx^1 \cdots dx^n.$$

The extra factor of the absolute value of the Jacobian determinant shows that the integral of f is coordinate-dependant. For this reason, it makes more sense to view the left hand side not as the integral of f but rather as the integral of $\nu = f dx^1 \wedge \cdots \wedge dx^n$. Then, the right hand side is indeed the integral of $\varphi^*\nu$ (which includes the Jacobian determinant in its expression automatically) , as long as φ^* has positive Jacobian determinant.

Therefore, the integral of a differential n-form will be well-defined on an n-manifold M, as long as we can choose an atlas where the Jacobian determinants of the gluing maps are all positive: This is precisely the choice of an *orientation* on M, as we now show.

Definition 32. A n-manifold M is called *orientable* when $\det T^*M := \wedge^n T^*M$ is isomorphic to the trivial line bundle. An orientation is the choice of an equivalence class of nonvanishing sections v, where $v \sim v'$ iff $fv = v'$ for $f \in C^\infty(M, \mathbb{R})$. M is called *oriented* when an orientation is chosen, and if M is connected and orientable, there are two possible orientations.

$\mathbb{R}^n$ has a natural orientation by $dx^1 \wedge \cdots \wedge dx^n$; if M is orientable, we may choose charts which preserve orientation, as we now show.

Proposition 4.11. *If the n-manifold M is oriented by $[v]$, it is possible to choose an orientation-preserving atlas (U_i, φ_i) in the sense that $\varphi_i^* dx^1 \wedge \cdots \wedge dx^n \sim v$ for all i. In particular, the Jacobian determinants for this atlas are all positive.*

Proof. Choose any atlas (U_i, φ_i). For each i, either $\varphi_i^* dx^1 \wedge \cdots \wedge dx^n \sim v$, and if not, replace φ_i with $q \circ \varphi$, where $q : (x^1, \ldots, x^n) \mapsto (-x^1, \ldots, x^n)$. This completes the proof. □

Now we can define the integral on an oriented n-manifold M, by defining the integral on chart images and asking it to be compatible with these charts:

Theorem 4.12. *Let M be an oriented n-manifold. Then there is a unique linear map $\int_M : \Omega_c^n(M) \longrightarrow \mathbb{R}$ on compactly supported n-forms which has the following property: if h is an orientation-preserving diffeomorphism from $V \subset \mathbb{R}^n$ to $U \subset M$, and if $\alpha \in \Omega_c^n(M)$ has support contained in U, then*

$$\int_M \alpha = \int_V h^* \alpha.$$

Proof. Let $\alpha \in \Omega_c^n(M)$ and choose an orientation-preserving, locally finite atlas (U_i, φ_i) with subordinate partition of unity (θ_i). Then using the required properties (and noting that α is nonzero in only finitely many U_i), we have

$$\int_M \alpha = \sum_i \int_M \theta_i \alpha = \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* \theta_i \alpha.$$

This proves the uniqueness of the integral. To show existence, we must prove that the above expression actually satisfies the defining condition, and hence can be used as an explicit definition of the integral.

Let $h : V \longrightarrow U$ be an orientation-preserving diffeomorphism from $V \subset \mathbb{R}^n$ to $U \subset M$, and suppose α has support in U. Then $\varphi_i \circ h$ are orientation-preserving, and

$$\begin{aligned} \int_M \alpha &= \sum_i \int_{\varphi_i(U_i) \cap \varphi_i(U)} (\varphi_i^{-1})^* \theta_i \alpha \\ &= \sum_i \int_{V \cap h^{-1}(U_i)} (\varphi_i \circ h)^* (\varphi_i^{-1})^* \theta_i \alpha \\ &= \sum_i \int_{V \cap h^{-1}(U_i)} h^*(\theta_i \alpha) \\ &= \int_V h^* \alpha, \end{aligned}$$

as required. □