4.1 The exterior derivative

Differential forms are equipped with a natural differential operator, which extends the exterior derivative of functions to all forms: $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$. The exterior derivative is uniquely specified by the following requirements: first, it satisfies $d(df) = 0$ for all functions f. Second, it is a graded derivation of the algebra of exterior differential forms of degree 1, i.e.

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta.$$

This allows us to compute its action on any 1-form $d(\xi_i dx^i) = d\xi_i \wedge dx^i$, and hence, in coordinates, we have

$$d(\rho dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \sum_k \tfrac{\partial \rho}{\partial x^k} dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Extending by linearity, this gives a local definition of d on all forms. Does it actually satisfy the requirements? this is a simple calculation: let $\tau_p = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ and $\tau_q = dx^{j_1} \wedge \cdots \wedge dx^{j_q}$. Then

$$d((f\tau_p) \wedge (g\tau_q)) = d(fg\tau_p \wedge \tau_q) = (gdf + fdg) \wedge \tau_p \wedge \tau_q = d(f\tau_p) \wedge g\tau_q + (-1)^p f\tau_p \wedge d(g\tau_q),$$

as required.

Therefore we have defined d, and since the definition is coordinate-independent, we can be satisfied that d is well-defined.

Definition 29. d is the unique degree $+1$ graded derivation of $\Omega^\bullet(M)$ such that $df(X) = X(f)$ and $d(df) = 0$ for all functions f.

Example 4.1. *Consider $M = \mathbb{R}^3$. For $f \in \Omega^0(M)$, we have*

$$df = \tfrac{\partial f}{\partial x^1} dx^1 + \tfrac{\partial f}{\partial x^2} dx^2 + \tfrac{\partial f}{\partial x^3} dx^3.$$

Similarly, for $A = A_1 dx^1 + A_2 dx^2 + A_3 dx^3$, we have

$$dA = (\tfrac{\partial A_2}{\partial x^1} - \tfrac{\partial A_1}{\partial x^2}) dx^1 \wedge dx^2 + (\tfrac{\partial A_3}{\partial x^1} - \tfrac{\partial A_1}{\partial x^3}) dx^1 \wedge dx^3 + (\tfrac{\partial A_3}{\partial x^2} - \tfrac{\partial A_2}{\partial x^3}) dx^2 \wedge dx^3$$

Finally, for $B = B_{12} dx^1 \wedge dx^2 + B_{13} dx^1 \wedge dx^3 + B_{23} dx^2 \wedge dx^3$, we have

$$dB = (\tfrac{\partial B_{12}}{\partial x^3} - \tfrac{\partial B_{13}}{\partial x^2} + \tfrac{\partial B_{23}}{\partial x^1}) dx^1 \wedge dx^2 \wedge dx^3.$$

Definition 30. The form $\rho \in \Omega^\bullet(M)$ is called *closed* when $d\rho = 0$ and *exact* when $\rho = d\tau$ for some τ.

Example 4.2. *A function $f \in \Omega^0(M)$ is closed if and only if it is constant on each connected component of M: This is because, in coordinates, we have*

$$df = \tfrac{\partial f}{\partial x^1} dx^1 + \cdots + \tfrac{\partial f}{\partial x^n} dx^n,$$

and if this vanishes, then all partial derivatives of f must vanish, and hence f must be constant.

Theorem 4.3. *The exterior derivative of an exact form is zero, i.e. $d \circ d = 0$. Usually written $d^2 = 0$.*

Proof. The graded commutator $[d_1, d_2] = d_1 \circ d_2 - (-1)^{|d_1||d_2|} d_2 \circ d_1$ of derivations of degree $|d_1|, |d_2|$ is always (why?) a derivation of degree $|d_1| + |d_2|$. Hence we see $[d, d] = d \circ d - (-1)^{1 \cdot 1} d \circ d = 2d^2$ is a derivation of degree 2 (and so is d^2). Hence to show it vanishes we must test on functions and exact 1-forms, which locally generate forms since every form is of the form $f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

But $d(df) = 0$ by definition and this certainly implies $d^2(df) = 0$, showing that $d^2 = 0$. □