Theorem 1.49 (noncompact Whitney embedding in $\mathbb{R}^{2n+1}$)**.** *Any smooth n-manifold may be embedded in $\mathbb{R}^{2n+1}$ (or immersed in $\mathbb{R}^{2n}$).*

Proof. We saw that any manifold may be written as a countable union of increasing compact sets $M = \cup K_i$, and that a regular covering $\{(U_{i,k} \supset V_{i,k}, \varphi_{i,k})\}$ of M can be chosen so that for fixed i, $\{V_{i,k}\}_k$ is a finite cover of $K_{i+1} \backslash K_i^\circ$ and each $U_{i,k}$ is contained in $K_{i+2} \backslash K_{i-1}^\circ$.

This means that we can express M as the union of 3 open sets W_0, W_1, W_2, where

$$W_j = \bigcup_{i \equiv j (\mathrm{mod} 3)} (\cup_k U_{i,k}).$$

Each of the sets $R_i = \cup_k U_{i,k}$ may be injectively immersed in $\mathbb{R}^{2n+1}$ by the argument for compact manifolds, since they have a finite regular cover. Call these injective immersions $\Phi_i : R_i \longrightarrow \mathbb{R}^{2n+1}$. The image $\Phi_i(R_i)$ is bounded since all the charts are, by some radius r_i. The open sets R_i, $i \equiv j(\mathrm{mod}3)$ for fixed j are disjoint, and by translating each Φ_i, $i \equiv j(\mathrm{mod}3)$ by an appropriate constant, we can ensure that their images in $\mathbb{R}^{2n+1}$ are disjoint as well.

Let $\Phi_i' = \Phi_i + (2(r_{i-1} + r_{i-2} + \cdots) + r_i)\overrightarrow{e}_1$. Then $\Psi_j = \cup_{i \equiv j (\mathrm{mod} 3)} \Phi_i' : W_j \longrightarrow \mathbb{R}^{2n+1}$ is an embedding.

Now that we have injective immersions Ψ_0, Ψ_1, Ψ_2 of W_0, W_1, W_2 in $\mathbb{R}^{2n+1}$, we may use the original argument for compact manifolds: Take the partition of unity subordinate to $U_{i,k}$ and resum it, obtaining a 3-element partition of unity $\{f_1, f_2, f_3\}$, with $f_j = \sum_{i \equiv j(\mathrm{mod}3)} \sum_k f_{i,k}$. Then the map

$$\Psi = (f_1 \Psi_1, f_2 \Psi_2, f_3 \Psi_3, f_1, f_2, f_3)$$

is an injective immersion of M into $\mathbb{R}^{6n+3}$. To see that it is in fact an embedding, note that any closed set $C \subset M$ may be written as a union of closed sets $C = C_1 \cup C_2 \cup C_3$, where $C_j = \cup_{i \equiv j(\mathrm{mod}3)} (C \cap K_{i+1} \backslash K_i^\circ)$ is a disjoint union of compact sets. Ψ is injective, hence C_j is mapped to a disjoint union of compact sets, hence a closed set. Then $\Psi(C)$ is a union of 3 closed sets, hence closed, as required.

Using projection to hyperplanes we may again reduce to $\mathbb{R}^{2n+1}$, but if we exclude all hyperplanes perpendicular to $\mathrm{Span}((e_1, 0, 0, 0, 0, 0), (0, e_1, 0, 0, 0, 0), (0, 0, e_1, 0, 0, 0))$, we obtain an injective immersion Ψ' which is *proper*, meaning that inverse images of compact sets are compact. This space of forbidden planes has measure zero as long as $N - 1 > 3$, so that we may reduce to $2n + 1$ for $n > 1$. We leave as an exercise the $n = 1$ case (or see Bredon for a slightly different proof).

The fact that the resulting injective immersion Ψ' is proper implies that it is an embedding, by the closed map lemma, as follows. $\square$

Lemma 1.50 (Closed map lemma for proper maps)**.** *Let $f : X \longrightarrow Y$ be a proper continuous map of topological manifolds. Then f is a closed map.*

Proof. Let $K \subset X$ be closed; we show that $f(K)$ contains all its limit points and hence is closed. Let $y \in Y$ be a limit point for $f(K)$. Choose a precompact neighbourhood U of y, so that y is also a limit point of $f(K) \cap \overline{U}$. Since f is proper, $f^{-1}(\overline{U})$ is compact, and hence $K \cap f^{-1}(\overline{U})$ is compact as well. But then by continuity, $f(K \cap f^{-1}(\overline{U})) = f(K) \cap \overline{U}$ is compact, implying it is closed. Hence $y \in f(K) \cap \overline{U} \subset f(K)$, as required. $\square$

We now use Whitney embedding to extend our understanding of the genericity of transversality. First we need an understanding of the immediate neighbourhood of an embedded submanifold in $\mathbb{R}^N$. For this, we introduce a new manifold associated to an embedded submanifold: its normal bundle (for now we assume the manifold is embedded in $\mathbb{R}^N$).

If $Y \subset \mathbb{R}^N$ is an embedded submanifold, the normal space at $y \in Y$ is defined by $N_y Y = \{v \in \mathbb{R}^N : v \perp T_y Y\}$. The collection of all normal spaces of all points in Y is called the normal bundle:

$$NY = \{(y, v) \in Y \times \mathbb{R}^N \; : \; v \in N_y Y\}.$$

Proposition 1.51. *$NY \subset \mathbb{R}^N \times \mathbb{R}^N$ is an embedded submanifold of dimension N.*

Proof. Given $y \in Y$, choose coordinates $(u^1, \dots u^N)$ in a neighbourhood $U \subset \mathbb{R}^N$ of y so that $Y \cap U = \{u^{n+1} = \cdots = u^N = 0\}$. Define $\Phi : U \times \mathbb{R}^N \longrightarrow \mathbb{R}^{N-n} \times \mathbb{R}^n$ via

$$\Phi(x, v) = (u^{n+1}(x), \dots, u^N(x), \langle v, \tfrac{\partial}{\partial u^1}|_x\rangle, \dots, \langle v, \tfrac{\partial}{\partial u^n}|_x\rangle),$$

so that $\Phi^{-1}(0)$ is precisely $NY \cap (U \times \mathbb{R}^N)$. We then show that 0 is a regular value: observe that, writing v in terms of its components $v^j \frac{\partial}{\partial x^j}$ in the standard basis for $\mathbb{R}^N$,

$$\langle v, \tfrac{\partial}{\partial u^i}|_x\rangle = \langle v^j \tfrac{\partial}{\partial x^j}, \tfrac{\partial x^k}{\partial u^i}(u(x)) \tfrac{\partial}{\partial x^k}|_x\rangle = \sum_{j=1}^N v^j \tfrac{\partial x^j}{\partial u^i}(u(x))$$

Therefore the Jacobian of Φ is the $((N-n)+n) \times (N+N)$ matrix

$$D\Phi(x) = \begin{pmatrix} \frac{\partial u^j}{\partial x^i}(x) & 0 \\ * & \frac{\partial x^j}{\partial u^i}(u(x)) \end{pmatrix}$$

The N rows of this matrix are linearly independent, proving Φ is a submersion. □

The normal bundle NY contains $Y \cong Y \times \{0\}$ as a regular submanifold, and is equipped with a smooth map $\pi : NY \longrightarrow Y$ sending $(y, v) \mapsto y$. The map π is a surjective submersion and is known as the bundle projection. The vector spaces $\pi^{-1}(y)$ for $y \in Y$ are called the fibers of the bundle and we shall see later that NY is an example of a vector bundle.

We may take advantage of the embedding in $\mathbb{R}^N$ to define a smooth map $E : NY \longrightarrow \mathbb{R}^N$ via

$$E(x, v) = x + v.$$

Definition 16. A tubular neighbourhood of the embedded submanifold $Y \subset \mathbb{R}^N$ is a neighbourhood U of Y in $\mathbb{R}^N$ that is the diffeomorphic image under E of an open subset $V \subset NY$ of the form

$$V = \{(y, v) \in NY \ : \ |v| < \delta(y)\},$$

for some positive continuous function $\delta : M \longrightarrow \mathbb{R}$.

If $U \subset \mathbb{R}^N$ is such a tubular neighbourhood of Y, then there does exist a positive continuous function $\epsilon : Y \longrightarrow \mathbb{R}$ such that $U_\epsilon = \{x \in \mathbb{R}^N \ : \ \exists y \in Y \text{ with } |x - y| < \epsilon(y)\}$ is contained in U. This is simply

$$\epsilon(y) = \sup\{r \ : \ B(y, r) \subset U\},$$

which is continuous since $\forall \epsilon > 0, \exists x \in U$ for which $\epsilon(y) \le |x - y| + \epsilon$. For any other $y' \in Y$, this is $\le |y - y'| + |x - y'| + \epsilon$. Since $|x - y'| \le \epsilon(y')$, we have $|\epsilon(y) - \epsilon(y')| \le |y - y'| + \epsilon$.

Theorem 1.52 (Tubular neighbourhood theorem)**.** *Every regular submanifold of $\mathbb{R}^N$ has a tubular neighbourhood.*

Proof. Postpone briefly. □

Corollary 1.53. *Let X be a manifold with boundary and $f : X \longrightarrow Y$ be a smooth map to a manifold Y. Then there is an open ball $S = B(0, 1) \subset \mathbb{R}^N$ and a smooth map $F : X \times S \longrightarrow Y$ such that $F(x, 0) = f(x)$ and for fixed x, the map $f_x : s \mapsto F(x, s)$ is a submersion $S \longrightarrow Y$. In particular, F and ∂F are submersions.*

Proof. Embed Y in $\mathbb{R}^N$, and let $S = B(0, 1) \subset \mathbb{R}^N$. Then use the tubular neighbourhood to define

$$F(y, s) = (\pi \circ E^{-1})(f(y) + \epsilon(y)s),$$

□

The transversality theorem then guarantees that given any smooth $g : Z \longrightarrow Y$, for almost all $s \in S$ the maps $f_s, \partial f_s$ are transverse to g. We improve this slightly to show that f_s may be chosen to be *homotopic* to f.

Corollary 1.54 (Transversality homotopy theorem). *Given any smooth maps $f : X \longrightarrow Y$, $g : Z \longrightarrow Y$, where only X has boundary, there exists a smooth map $f' : X \longrightarrow Y$ homotopic to f with $f', \partial f'$ both transverse to g.*

Proof. Let S, F be as in the previous corollary. Away from a set of measure zero in S, the functions $f_s, \partial f_s$ are transverse to g, by the transversality theorem. But these f_s are all homotopic to f via the homotopy $X \times [0, 1] \longrightarrow Y$ given by

$$(x, t) \mapsto F(x, ts).$$

□

Proof, tubular neighbourhoood theorem. First we show that E is a local diffeomorphism near $y \in Y \subset NY$. if ι is the embedding of Y in $\mathbb{R}^N$, and $\iota' : Y \longrightarrow NY$ is the embedding in the normal bundle, then $E \circ \iota' = \iota$, hence we have $DE \circ D\iota' = D\iota$, showing that the image of $DE(y)$ contains T_yY. Now if ι is the embedding of N_yY in $\mathbb{R}^N$, and $\iota' : N_yY \longrightarrow NY$ is the embedding in the normal bundle, then $E \circ \iota' = \iota$. Hence we see that the image of $DE(y)$ contains N_yY, and hence the image is all of $T_y\mathbb{R}^N$. Hence E is a diffeomorphism on some neighbourhood

$$V_\delta(y) = \{(y', v') \in NY \ : \ |y' - y| < \delta, \ |v'| < \delta\}, \quad \delta > 0.$$

Now for $y \in Y$ let $r(y) = \sup\{\delta \ : \ E|_{V_\delta(y)}$ is a diffeomorphism$\}$ if this is ≤ 1 and let $r(y) = 1$ otherwise. The function $r(y)$ is continuous, since if $|y - y'| < r(y)$, then $V_\delta(y') \subset V_{r(y)}(y)$ for $\delta = r(y) - |y - y'|$. This means that $r(y') \geq \delta$, i.e. $r(y) - r(y') \leq |y - y'|$. Switching y and y', this remains true, hence $|r(y) - r(y')| \leq |y - y'|$, yielding continuity.

Finally, let $V = \{(y, v) \in NY \ : \ |v| < \frac{1}{2}r(y)\}$. We show that E is injective on V. Suppose $(y, v), (y', v') \in V$ are such that $E(y, v) = E(y', v')$, and suppose wlog $r(y') \leq r(y)$. Then since $y + v = y' + v'$, we have

$$|y - y'| = |v - v'| \leq |v| + |v'| \leq \tfrac{1}{2}r(y) + \tfrac{1}{2}r(y') \leq r(y).$$

Hence y, y' are in $V_{r(y)}(y)$, on which E is a diffeomorphism. The required tubular neighbourhood is then $U = E(V)$. □

The last theorem we shall prove concerning transversality is a very useful extension result which is essential for intersection theory:

Theorem 1.55 (Homotopic transverse extension of boundary map). *Let X be a manifold with boundary and $f : X \longrightarrow Y$ a smooth map to a manifold Y. Suppose that ∂f is transverse to the closed map $g : Z \longrightarrow Y$. Then there exists a map $f' : X \longrightarrow Y$, homotopic to f and with $\partial f' = \partial f$, such that f' is transverse to g.*

Proof. First observe that since ∂f is transverse to g on ∂X, f is also transverse to g there, and furthermore since g is closed, f is transverse to g in a neighbourhood U of ∂X. (for example, if $x \in \partial X$ but x not in $f^{-1}(g(Z))$ then since the latter set is closed, we obtain a neighbourhood of x for which f is transverse to g.)

Now choose a smooth function $\gamma : X \longrightarrow [0, 1]$ which is 1 outside U but 0 on a neighbourhood of ∂X. (why does γ exist? exercise.) Then set $\tau = \gamma^2$, so that $d\tau(x) = 0$ wherever $\tau(x) = 0$. Recall the map $F : X \times S \longrightarrow Y$ we used in proving the transversality homotopy theorem 1.54 and modify it via

$$F'(x, s) = F(x, \tau(x)s).$$

Then F' and $\partial F'$ are transverse to g, and we can pick s so that $f' : x \mapsto F'(x, s)$ and $\partial f'$ are transverse to g. Finally, if x is in the neighbourhood of ∂X for which $\tau = 0$, then $f'(x) = F(x, 0) = f(x)$. □

Corollary 1.56. *if $f : X \longrightarrow Y$ and $f' : X \longrightarrow Y$ are homotopic smooth maps of manifolds, each transverse to the closed map $g : Z \longrightarrow Y$, then the fiber products $W = X_f \times_g Z$ and $W' = X_{f'} \times_g Z$ are cobordant.*

Proof. if $F : X \times [0,1] \longrightarrow Y$ is the homotopy between $\{f, f'\}$, then by the previous theorem, we may find a (homotopic) homotopy $F' : X \times [0,1] \longrightarrow Y$ which is transverse to g. Hence the fiber product $U = (X \times [0,1])_{F'} \times_g Z$ is the cobordism with boundary $W \sqcup W'$. $\square$

The previous corollary allows us to make the following definition:

Definition 17. Let $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ be smooth maps with X compact, g closed, and $\dim X + \dim Z = \dim Y$. Then we define the $\pmod 2$ intersection number of f and g to be

$$I_2(f, g) = \sharp(X_{f'} \times_g Z) \pmod 2,$$

where $f' : X \longrightarrow Y$ is any smooth map smoothly homotopic to f but transverse to g, and where we assume the fiber product to consist of a finite number of points (this is always guaranteed, e.g. if g is proper, or if g is a closed embedding).

This allows us to define the notion of intersection $\pmod 2$ of embedded submanifolds: for example,

Example 1.57. *If C_1, C_2 are two distinct great circles on S^2 then they have two transverse intersection points, so $I_2(C_1, C_2) = 0$ in $\mathbb{Z}_2$. Of course we can shrink one of the circles to get a homotopic one which does not intersect the other at all. This corresponds to the standard cobordism from two points to the empty set.*

Example 1.58. *If (e_1, e_2, e_3) is a basis for $\mathbb{R}^3$ we can consider the following two embeddings of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ into $\mathbb{R}P^2$: $\iota_1 : \theta \mapsto \langle \cos(\theta/2)e_1 + \sin(\theta/2)e_2 \rangle$ and $\iota_2 : \theta \mapsto \langle \cos(\theta/2)e_2 + \sin(\theta/2)e_3 \rangle$. These two embedded submanifolds intersect transversally in a single point $\langle e_2 \rangle$, and hence $I_2(\iota_1, \iota_2) = 1$ in $\mathbb{Z}_2$. As a result, there is no way to deform ι_i so that they intersect transversally in zero points.*

Example 1.59. *Given a smooth map $f : X \longrightarrow Y$ for X compact and $\dim Y = 2 \dim X$, we may consider the self-intersection $I_2(f, f)$. In the previous examples we may check $I_2(C_1, C_1) = 0$ and $I_2(\iota_1, \iota_1) = 1$.*