Under assumptions on X (connected, local simple-connected, and semi-locally simply connected, in order to define the topology of $\tilde{X}$) we constructed a universal covering $(\tilde{X}, p)$, by setting

$$\tilde{X} = \{[\gamma] \ : \ \gamma \text{ is a path in } X \text{ starting at } x_0\}.$$

We also saw that this space has trivial fundamental group, as follows: Any path γ in X may be lifted to $\tilde{X}$ by defining $\tilde{\gamma}(t)$ to be the path γ up to time t (and constant afterwards). If $[\gamma]$ is in the image of p_*, this means that there is a loop in this class, say γ, which lifts to a loop $\tilde{\gamma}$ in $\tilde{X}$. But this means that γ up to time 1 is equal in $\tilde{X}$ (i.e. homotopic to) to γ up to time 0, i.e. $[\gamma] = 0$ in $\pi_1(X)$. Since p_* is injective, it must be that $\pi_1(\tilde{X}) = 0$.

Having the universal cover, we can produce all other coverings via quotients of it, as follows:

surjectivity of functor. Suppose now that (X, x_0) has a (path-connected) universal covering space $(\tilde{X}, \tilde{x}_0)$, and suppose a subgroup $H \subset \pi_1(X, x_0)$ is specified. Then we define an equivalence relation on $\tilde{X}$ as follows: given points $[\gamma], [\gamma'] \in \tilde{X}$, we define $[\gamma] \sim [\gamma']$ iff $\gamma(1) = \gamma'(1)$ and $[\gamma'\gamma^{-1}] \in H$. Because H is a subgroup, this is an equivalence relation. Now set $X_H = \tilde{X}/\sim$. Note that this equivalence relation holds for nearby paths in the sense $[\gamma] \sim [\gamma']$ iff $[\gamma\eta] \sim [\gamma'\eta]$. Therefore, if any two points in $U_{[\gamma]}, U_{[\gamma']}$ are equivalent, then so is every other point in the neighbourhood. This shows that the projection $p : X_H \longrightarrow X$ via $[\gamma] \mapsto \gamma(1)$ is a covering map.

As a basepoint in X_H, pick $[x_0]$, the constant path at x_0. Then the image of p_* is$H$, since the lift of the loop γ is a path beginning at $[x_0]$ and ending at $[\gamma]$, and this is a loop exactly when $[\gamma] \sim [x_0]$, i.e. $[\gamma] \in H$. □

Example 1.37 (Diagram: page 58)**.** *Consider the wedge $S^1 \vee S^1$. Recall that $\pi_1(S^1 \vee S^1) = F_2 =< a, b >$. View it as a graph with one vertex and two edges, labeled by a, b with their appropriate orientations. We can then take any other graph $\tilde{X}$, labeled in this way, and such that each vertex is locally isomorphic to the given vertex, and define a covering map to $S^1 \vee S^1$. The resulting graph $\tilde{X}$ is itself a wedge of k circles, with fundamental group F_k. Hence we obtain a map $F_k \longrightarrow F_2$ which is an injection. Examples (1), (2)*

In fact, every 4-valent graph can be labeled in the way required above: if the graph is finite, take an Eulerian circuit and label the edges $a, b, a, b \ldots$. Then the a edges are a collection of disjoint circles: orient them and do the same for the b edges.

An infinite 4-valent graph may be constructed which is a simply-connected covering space for $S^1 \vee S^1$: it is a fractal 4-branched tree (drawing).

Not only can we have a free group on any number of generators as a subgroup of F_2, but also we can have infinitely many generators (drawing of (10), (11))

Note that changing the basepoint vertex of a covering simply conjugates $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ inside $\pi_1(X, x_0)$. (draw (3), (4)). Isomorphism of coverings (without fixing basepoints) is just a graph isomorphism preserving labeling and orientation.

Note also that characteristic subgroups may be isomorphic *without being* conjugate*. (draw (5),(6)), these are homeomorphic graphs, but not isomorphic as covering spaces.*

Example 1.38. *If X is a path-connected space with fundamental group $\pi_1(X, x_0)$, then by attaching 2-cells e^2_α via maps $\varphi_\alpha : S^1 \longrightarrow X$, then the resulting space Y will have fundamental group which is a quotient of $\pi_1(X, x_0)$ by the normal subgroup N generated by loops of the form $\gamma_\alpha\varphi_\alpha\gamma_\alpha^{-1}$, for any γ_α chosen to join x_0 to $\varphi_\alpha(1)$. This is seen by Van Kampen's theorem applied to a thickened version Z of Y where the paths γ_α are thickened to intervals attached to the discs e_α.*

We can use this construction to obtain any group as a fundamental group. Choose a presentation

$$G = \langle g_\alpha \mid r_\beta \rangle.$$

This is possible since any group is a quotient of a free group. Then we construct X_G from $\vee_\alpha S^1_\alpha$ by attaching 2-cells e^2_β by loops specified by the words r_β. (for example, to obtain $\mathbb{Z}_n = \langle a \mid a^n = 1 \rangle$, attach a single 2-cell to S^1 via the map $z \mapsto z^n$. For $n = 2$ we obtain $\mathbb{R}P^2$.

The Cayley complex is one way of describing the universal cover of X_G. It is a cell complex $\tilde{X}_G$ constructed as follows: The vertices are the elements of G itself. Then at each vertex $g \in G$, attach an edge joining g to gg_α for each generator g_α. The resulting graph is the Cayley graph of G with respect to the generators g_α. Then, each relation r_β determines a loop starting at any $g \in G$, and we attach a 2-cell to all these loops. There is an obvious map to X_G given by quotienting by the action of G on the left, which sends all points to the basepoint, each edge $g \longrightarrow gg_\alpha$ to the edge S^1_α, and each 2-cell associated to r_β to that attached in the construction of X_G.

For example, consider $G = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = 1\rangle$. then the Cayley graph has vertices $\{\ldots, bab, ba, b, e, a, ab, aba, \ldots\}$, and two generators so there will be four edges coming in/out of each vertex g: two outward edges corresponding to right multiplication by a, b to ga, gb, and two inward coming from ga^{-1}, gb^{-1}. We therefore obtain an infinite sequence of tangent circles. We produce the Cayley complex by attaching a 2-cell corresponding to a^2 to the loop produced at each vertex g by following the loop $g \longrightarrow ga \longrightarrow ga^2$. This attaches two 2-cells to each circle, yielding a sequence of tangent 2-spheres, clearly a simply-connected space. The action of G corresponds to an action by even translations (ab) and the antipodal maps, giving the quotient space $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

1.7 Group actions and Deck transformations

In many cases we obtain covering spaces $\tilde{X} \longrightarrow X$ from group actions; if a group A acts on $\tilde{X}$, the quotient map $\tilde{X} \longrightarrow \tilde{X}/A$ may, under some assumptions on A and its action, be a covering.

For example, we can define the n-fold covering $S^1 \longrightarrow S^1$ as simply the quotient of S^1 by the action of $\mathbb{Z}_n$ via $x \mapsto x\sigma^n$ for $\sigma = e^{2\pi i/n}$, or even $\mathbb{R} \longrightarrow S^1$ via the quotient by the $\mathbb{Z}$ action $x \mapsto x + n$.

In general, if $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ is a universal cover, then we can obtain X as a quotient of $\tilde{X}$ by the action of the fundamental group $\pi_1(X, x_0)$ as follows:

Given an element $[\gamma] \in \pi_1(X, x_0)$, γ lifts to a path terminating in $\tilde{x}_0'$ over x_0. Now the covering p has a unique lift to $\tilde{X}$, sending $\tilde{x}_0$ to the alternative basepoint $\tilde{x}_0'$. This lift is a homeomorphism $\tilde{X} \longrightarrow \tilde{X}$, and this defines an action of $\pi_1(X, x_0)$ on $\tilde{X}$. We'll be careful in a moment to show the quotient is X.

In general, not all covering maps p will be the quotient by the action of a group: this will only be the case for *normal* covering maps, i.e. those for which $p_*(\pi_1(\tilde{X}))$ is a normal subgroup N; Then $\pi_1(X, x_0)/N$ is a group, and this will act in the same way as above, with quotient X.

Example 1.39 (Coverings of surfaces)**.** *There are many interesting coverings of surfaces, which can be constructed by acting by symmetry groups:*

An example of a covering of a compact surface: take a genus $mn+1$ surface, draw it as a surface with m genus n legs and a hole in the center. There is an obvious $\mathbb{Z}_n$ symmetry by rotating by $2\pi/m$. The quotient map is then a m-fold covering map to a surface of genus $n+1$.

Consider a genus g surface in $\mathbb{R}^3$ with the holes along an axis, and consider the rotation about this axis by π, giving a $\mathbb{Z}_2$ action with $2(g+1)$ fixed points. Remove the fixed points. The punctured surface then is a 2-sheeted cover of S^2 punctured in $2(g+1)$ points. This is the topological description of an equation $y^2 = f(z)$ with f of degree $2g+1$ (this way, $y^2 = f$ has exactly two solutions except at the $2g+1$ zeros of f and the point at infinity where $f = \infty$. The particular case where f has degree 3 defines a genus 1 surface, which is called an elliptic curve once a complex structure is chosen on it.

Example 1.40. *The antipodal map on S^n is an action of $\mathbb{Z}_2$ with no fixed points; the quotient map is a covering of $\mathbb{R}P^n$. This will imply that $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$. In the case $n = 3$, this 2:1 cover is also known as the sequence of groups*

$$0 \longrightarrow \mathbb{Z}_2 = \{\pm 1\} \longrightarrow SU(2) \xrightarrow{\pi} SO(3) \longrightarrow 0$$

Note that $SO(3)$ has several famous finite subgroups: the cyclic groups A_n, the dihedral groups D_n, and the symmetry groups of the tetrahedron, octahedron, and dodecahedron, E_6, E_7, E_8. In this way we can construct other covering spaces, e.g. $S^3 \longrightarrow S^3/\pi^{-1}(E_8)$, the Poincaré dodecahedral space, a homology 3-sphere.

To formalize the observations above, we wish to answer the following questions: Given a connected covering space (without basepoint), what is its group of automorphisms (deck transformations), and when does this group define the covering as a quotient? And, more generally, when is a group action defining a covering map?

Definition 10. A covering map $p : \tilde{X} \longrightarrow X$ is called normal when, for each $x \in X$ and each pair of lifts $\tilde{x}, \tilde{x}'$ of x, there is an automorphism of p taking $\tilde{x}$ to $\tilde{x}'$.

Theorem 1.41. *If $p : \tilde{X} \longrightarrow X$ is a path-connected covering (of X path-connected and locally path-connected), with characteristic subgroup H, then the group of automorphisms of p is $A = N(H)/H$, and the quotient $\tilde{X}/A$ is the covering with characteristic subgroup $N(H)$. Therefore, a covering is normal precisely when H is normal, and in this case the automorphism group is $A = \pi_1(X)/H$ and $\tilde{X}/A = X$.*

Proof. Changing the basepoint from $\tilde{x}_0 \in p^{-1}(x_0)$ to $\tilde{x}_1 \in p^{-1}(x_0)$ corresponds to conjugating H by $[\gamma] \in \pi_1(X, x_0)$ which lifts to a path $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Therefore, $[\gamma] \in N(H)$ iff $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$, which is the case (by the lifting of maps) iff there is a deck transformation taking $\tilde{x}_0$ to $\tilde{x}_1$. Therefore $\tilde{X}$ is normal iff $N(H) = \pi_1(X, x_0)$, i.e. H is already normal in $\pi_1(X, x_0)$.

In general there is a group homomorphism $\varphi : N(H) \longrightarrow A$, sending $[\gamma]$ to the deck transformation mapping $\tilde{x}_0 \mapsto \tilde{x}_1$ as above. It is surjective by the argument above, and its kernel is precisely the classes $[\gamma]$ lifting to loops, i.e. the elements of H itself. □

Theorem 1.42. *Suppose G acts on Y in a properly discontinuous way, i.e. each $y \in Y$ has a neighbourhood U such that gU are disjoint for all $g \in G$. Then the quotient of Y by G is a normal covering map, and if Y is path-connected then G is the automorphism group of the cover.*

Proof. First we remark that deck transformations of a covering space obviously have the properly discontinuous property.

To prove the result, take any open set U as in the definition of proper discontinuity. Then the quotient map identifies the disjoint homeomorphic neighbourhoods $\{g(U) : g \in G\}$ with $p(U) \subset Y/G$. By the definition of the quotient topology, this gives a homeomorphism on each component, and hence we have a covering.

Certainly G is a subgroup of the deck transformations, and the covering space is normal since $g_2 g_1^{-1}$ takes $g_1(U)$ to $g_2(U)$, and if Y is path-connected then G equals the deck transformations, since if a deck transformation f sends y to $f(y)$, we may simply lift the covering to the alternative point $f(y)$ (the lifting criterion is satisfied since the cover is normal) and this deck transformation must coincide with f by uniqueness. □

Remark 1. *Suppose $p : \tilde{X} \longrightarrow X$ is a finite covering. Fixing $x_0 \in X$, we have two natural permutation actions on the finite set $p^{-1}(x_0)$: one is by $\pi_1(X, x_0)$, via lifting of loops, i.e. given $[\gamma] \in \pi_1(X, x_0)$, the permutation $\sigma([\gamma])$ acts on $\tilde{x}_0$ by $\sigma(\tilde{x}_0) = \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the lift of γ starting at $\tilde{x}_0$. The study of this permutation action is an alternative approach to classifying covering spaces, and this is described in Hatcher. It is useful to understand both approaches.*

The second action is by the group of deck transformations $A = N(H)/H$ (for the characteristic subgroup H). These actions commute. Interestingly, when $\tilde{X}$ is the universal cover, A is $\pi_1(X, x_0)$ as well, and so we have the same group acting in two ways– these actions need not coincide.