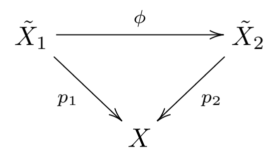
1.6 Covering spaces

Consider the fundamental group $\pi_1(X, x_0)$ of a pointed space. It is natural to expect that the group theory of $\pi_1(X, x_0)$ might be understood geometrically. For example, subgroups may correspond to images of induced maps $\iota_*\pi_1(Y, y_0) \longrightarrow \pi_1(X, x_0)$ from continuous maps of pointed spaces $(Y, y_0) \longrightarrow (X, x_0)$. For this induced map to be an injection we would need to be able to lift homotopies in X to homotopies in Y. Rather than consider a huge category of possible spaces mapping to X, we restrict ourselves to a category of *covering spaces*, and we show that under some mild conditions on X, this category completely encodes the group theory of the fundamental group.

Definition 9. A covering map of topological spaces $p : \tilde{X} \longrightarrow X$ is a continuous map such that there exists an open cover $X = \bigcup_\alpha U_\alpha$ such that $p^{-1}(U_\alpha)$ is a disjoint union of open sets (called *sheets*), each homeomorphic via p with U_α. We then refer to $(\tilde{X}, p)$ (or simply $\tilde{X}$, abusing notation) as a covering space of X.

Let $(\tilde{X}_i, p_i)$, $i = 1, 2$ be covering spaces of X. A morphism of covering spaces is a covering map $\phi : \tilde{X}_1 \longrightarrow \tilde{X}_2$ such that the diagram commutes:

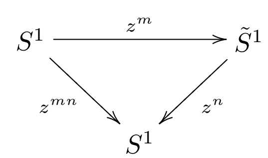


We will be considering covering maps of pointed spaces $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$, and pointed morphisms between them, which are defined in the obvious fashion.

Example 1.30. *The covering space $p : \mathbb{R} \longrightarrow S^1$ has the additional property that $\tilde{X} = \mathbb{R}$ is simply connected. There are other covering spaces $p_n : S^1 \longrightarrow S^1$ given by $z \mapsto z^n$ for $n \in \mathbb{Z}$, and in fact these are the only connected ones up to isomorphism of covering spaces (there are disconnected ones, but they are unions of connected covering spaces).*

Notice that $(p_n)_* : \pi_1(S^1) \longrightarrow \pi_1(S^1)$ maps $[\omega_1] \mapsto [\omega_n] = n[\omega_1]$, hence $(p_n)_*(\pi_1(S^1)) \cong \mathbb{Z}/n\mathbb{Z} \subset \mathbb{Z}$. As a result, we see that there is an isomorphism class of covering space associated to every subgroup of $\mathbb{Z}$: we associate $p : \mathbb{R} \longrightarrow S^1$ to the trivial subgroup.

Note also that we have the commutative diagram



showing that we have a morphism of covering spaces corresponding to the inclusion of groups $mn\mathbb{Z} \subset n\mathbb{Z} \subset \mathbb{Z}$.

There is a natural functor from pointed covering spaces of (X, x_0) to subgroups of $\pi_1(X, x_0)$, as a consequence of the following result:

Lemma 1.31 (Homotopy lifting)**.** *Let $p : \tilde{X} \longrightarrow X$ be a covering and suppose that $\tilde{f}_0 : Y \longrightarrow \tilde{X}$ is a lifting of the map $f_0 : Y \longrightarrow X$. Then any homotopy f_t of f_0 lifts uniquely to a homotopy $\tilde{f}_t$ of $\tilde{f}_0$.*

Proof. The same proof used for the Lemma 1.13 works in this case. □

Corollary 1.32. *The map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ induced by a covering space is injective, and its image $G(p, \tilde{x}_0)$ consists of loops at x_0 whose lifts to $\tilde{X}$ at $\tilde{x}_0$ are loops.*

If we choose a different basepoint $\tilde{x}_0' \in p^{-1}(x_0)$, and if $\tilde{X}$ is path-connected, we see that $G(p, \tilde{x}_0')$ is the conjugate subgroup $\gamma G(p, \tilde{x}_0)\gamma^{-1}$, for $\gamma = p_*[\tilde{\gamma}]$ for $\tilde{\gamma} : \tilde{x}_0 \to \tilde{x}_0'$.

Hence p_* defines a functor as follows:

$$\{\text{ pointed coverings } (\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)\ \} \longrightarrow \{\text{ subgroups } G \subset \pi_1(X, x_0)\}$$

The group $G(p, \tilde{x}_0) = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$ is called the characteristic subgroup of the covering p.

We will prove that under some conditions on X, this is an equivalence:

Theorem 1.33 (injective)**.** *Let X be path-connected and locally path-connected. Then $G(p, \tilde{x}) = G(p', \tilde{x}')$ iff there exists a canonical isomorphism $(p, \tilde{x}) \cong (p', \tilde{x}')$.*

Theorem 1.34 (surjective)**.** *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then for any subgroup $G \subset \pi_1(X, x)$, there exists a covering space $p : (\tilde{X}, \tilde{x}) \longrightarrow (X, x)$ with $G = G(p, \tilde{x})$.*

The first tool is a criterion which decides whether maps to X may be lifted to $\tilde{X}$:

Lemma 1.35 (Lifting criterion)**.** *Let $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ is a covering and let $f : (Y, y_0) \longrightarrow (X, x_0)$ be a a map with Y path-connected and locally path-connected. Then f lifts to $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, x_0))$.*

Proof. It is clear that the group inclusion must hold if f lifts, since $f_* = p_* \tilde{f}_*$. For the converse, we define $\tilde{f}$ as follows: let $y \in Y$ and let $\gamma : y_0 \to y$ be a path. Then take the path $f\gamma$ and lift it at $\tilde{x}_0$, giving $\widetilde{f\gamma}$. Define $\tilde{f}(y) = \widetilde{f\gamma}(1)$.

$\tilde{f}$ is well defined, independent of γ: if we choose $\gamma' : y_0 \to y$, then $(f\gamma')(f\gamma)^{-1}$ is a loop h_0 in the image of f_* and hence is homotopic (via h_t) to a loop h_1 which lifts to a loop $\tilde{h}_1$ at $\tilde{x}_0$. But the homotopy lifts, and hence $\tilde{h}_0$ is a loop as well. By uniqueness of lifted paths, $\tilde{h}_0$ consists of $\widetilde{f\gamma'}$ and $\widetilde{f\gamma}$ (both lifted at $\tilde{x}_0$), traversed as a loop. Since they form a loop, it must be that $\widetilde{f\gamma'}(1) = \widetilde{f\gamma}(1)$.

$\tilde{f}$ is continuous: We show that each $y \in Y$ has a neighbourhood V small enough that $\tilde{f}|_V$ coincides with f. Take a neighbourhood U of $f(y)$ which lifts to $\tilde{f}(y) \in \tilde{U} \subset \tilde{X}$ via $p : \tilde{U} \longrightarrow U$. Then choose a path-connected neighbourhood V of y with $f(V) \subset U$. Fix a path γ from y_0 to y and then for any point $y' \in V$ choose path $\eta : y \to y'$. Then the paths $(f\gamma)(f\eta)$ have lifts $\widetilde{f\gamma}\widetilde{f\eta}$, and $\widetilde{f\eta} = p^{-1} f\eta$. Hence $\tilde{f}(V) \subset \tilde{U}$ and $\tilde{f}|_V = p^{-1} f$, hence continuous. ☐

Lemma 1.36 (uniqueness of lifts)**.** *If $\tilde{f}_1, \tilde{f}_2$ are lifts of a map $f : Y \longrightarrow X$ to a covering $p : \tilde{X} \longrightarrow X$, and if they agree at one point of Y, then $\tilde{f}_1 = \tilde{f}_2$.*

Proof. The set of points in Y where $\tilde{f}_1$ and $\tilde{f}_2$ agree is open and closed: take a neighbourhood U of $f(y)$ such that $p^{-1}(U)$ is a disjoint union of homeomorphic $\tilde{U}_\alpha$, and let $\tilde{U}_1, \tilde{U}_2$ contain $\tilde{f}_1(y), \tilde{f}_2(y)$. Then take $N = \tilde{f}_1^{-1}(\tilde{U}_1) \cap \tilde{f}_2^{-1}(\tilde{U}_2)$. If $\tilde{f}_1, \tilde{f}_2$ agree (disagree) at y, then they must agree (disagree) on all of N. ☐

Proof of injectivity. If there is an isomorphism $f : (\tilde{X}_1, \tilde{x}_1) \longrightarrow (\tilde{X}_2, \tilde{x}_2)$, then taking induced maps, we get $G(p_1, \tilde{x}_1) = G(p_2, \tilde{x}_2)$.

Conversely, suppose $G(p_1, \tilde{x}_1) = G(p_2, \tilde{x}_2)$. By the lifting criterion, we can lift $p_1 : \tilde{X}_1 \longrightarrow X$ to a map $\tilde{p}_1 : (\tilde{X}_1, \tilde{x}_1) \longrightarrow (\tilde{X}_2, \tilde{x}_2)$ with $p_2 \tilde{p}_1 = p_1$. In the other direction we obtain $\tilde{p}_2$ with $p_1 \tilde{p}_2 = p_2$. The composition $\tilde{p}_1 \tilde{p}_2$ is then a lift of p_2 which agrees with the Identity lift at the basepoint, hence it must be the identity. similarly for $\tilde{p}_2 \tilde{p}_1$. ☐

Finally, to show that there is a covering space corresponding to each subgroup $G \subset \pi_1(X, x_0)$, we give a construction. The first step is to construct a simply-connected covering space, corresponding to the trivial subgroup. Note that for such a covering to exist, X must have the property of being semi-locally simply connected, i.e. each point x must have a neighbourhood U such that the inclusion $\iota_* : \pi_1(U, x) \longrightarrow \pi_1(X, x)$ is trivial. In fact this property is equivalent to the requirement that $\pi_1(X, x)$ be *discrete* as a topological group. We prove the existence of a simply-connected covering space when X is path-connected, locally path-connected, and semi-locally simply connected.

Existence of simply-connected covering. Let X be as above, with basepoint x_0. Define

$$\tilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}$$

and let $\tilde{x}_0$ be the trivial path at x_0. Define also the map $p : \tilde{X} \longrightarrow X$ by $p([\gamma]) = \gamma(1)$. p is surjective, since X is path-connected.

We need to define a topology on $\tilde{X}$, show that p is a covering map, and that it is simply-connected.

Topology: Since X is locally path-connected and semilocally simply-connected, it follows that the collection $\mathcal{U}$ of path-connected open sets $U \subset X$ with $\pi_1(U) \longrightarrow \pi_1(X)$ trivial forms a basis for the topology of X. We now lift this collection to a basis for a topology on $\tilde{X}$: Given $U \in \mathcal{U}$ and $[\gamma] \in p^{-1}(U)$, define

$$U_{[\gamma]} = \{[\gamma\eta] \mid \eta \text{ is a path in } U \text{ starting at } \gamma(1)\}$$

Note that $p : U_{[\gamma]} \longrightarrow U$ is surjective since U path-connected and injective since $\pi_1(U) \longrightarrow \pi_1(X)$ trivial. Using the fact that $[\gamma'] \in U_{[\gamma]} \Rightarrow U_{[\gamma]} = U_{[\gamma']}$, we obtain that the sets $U_{[\gamma]}$ form a basis for a topology on $\tilde{X}$. With respect to this topology, $p : U_{[\gamma]} \longrightarrow U$ gives a homeomorphism, since it gives a bijection between subsets $V_{[\gamma']} \subset U_{[\gamma]}$ and the sets $V \in \mathcal{U}$ contained in U ($p(V_{[\gamma']}) = V$ and also $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$ for any $[\gamma'] \in U_{[\gamma]}$ with endpoint in V).

Hence $p : \tilde{X} \longrightarrow X$ is continuous, and it is a covering map, since for fixed $U \in \mathcal{U}$, the sets $\{U_{[\gamma]}\}$ partition $p^{-1}(U)$.

To see that $\tilde{X}$ is simply-connected: Note that for any point $[\gamma] \in \tilde{X}$, we can shrink the path to give a homotopy $t \mapsto [\gamma_t]$ to the constant path $[x_0]$ (this shows $\tilde{X}$ is path-connected). If $[\gamma] \in \pi_1(X, x_0)$ is in the image of p_*, it means that the lift $[\gamma_t]$ is a loop, meaning that $[\gamma_1] = [x_0]$. But $\gamma_1 = \gamma$, this means that $[\gamma] = [x_0]$, hence the image of p_* is trivial. By injectivity of p_*, we get that $\tilde{X}$ is simply-connected. □