This is the first assignment for the topology half of the course. It will focus on issues related to the fundamental group(oid). Do not use more advanced theorems such as the Van Kampen theorem to compute any fundamental groups in this assignment.

Exercise 1. Hatcher Section 1.1, exercises 8, 13, 16, 17,18.

Exercise 2. Let $p_0, p_1 \in S^2$ be distinct points. Show that any two paths $\gamma_0, \gamma_1 : [0,1] \longrightarrow S^2 \ : \ \gamma_i(i) = p_i, \ i = 0,1$ must be homotopic.

Use this to describe the fundamental groupoid ΠS^2 completely.

Also use this to compute $\pi_1(\mathbb{R}P^2)$. Compute $\pi_1(\mathbb{R}P^n)$.

Exercise 3. In $\mathbb{R}^3$, let C_1 be the z-axis, and let C_2 be the circle $\{x^2 + y^2 = 1 \text{ and } z = 0\}$. Compute $\pi_1(\mathbb{R}^3 \backslash \{C_1 \cup C_2\})$ and express it in terms of generators and relations. Draw a picture of the generators, and draw a picture of the relations.

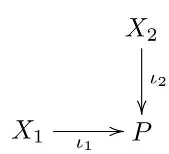
Use this to compute the fundamental group of $\mathbb{R}^3 \backslash \{\text{Hopf link}\}$.

Exercise 4. Find a pointed topological space (X, x_0) such that $\pi_1(X, x_0)$ is not discrete (as a topological group). In your example, is it countable?

Exercise 5. Let $\vec{\mathbf{I}}$ be the category with two objects $\{0,1\}$ and only one non-identity arrow $\iota : 0 \longrightarrow 1$. If $\mathcal{C}, \mathcal{D}$ are categories, then a functor $F : \mathcal{C} \times \vec{\mathbf{I}} \longrightarrow \mathcal{D}$ is called a *natural transformation* from the functor $f_0 : \mathcal{C} \longrightarrow \mathcal{D}$ to the functor $f_1 : \mathcal{C} \longrightarrow \mathcal{D}$, where $f_i(X) = F(X, i), \ i = 0,1$. Prove that there is a category $\text{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$, and whose morphisms are natural transformations. What are the invertible morphisms (isomorphisms) in this category?

Two categories $\mathcal{C}, \mathcal{D}$ are *equivalent* when there are functors $f : \mathcal{C} \longrightarrow \mathcal{D}$ and $g : \mathcal{D} \longrightarrow \mathcal{C}$ such that $f \circ g$ is isomorphic to $\text{Id}_{\mathcal{D}}$ and $g \circ f$ is isomorphic to $\text{Id}_{\mathcal{C}}$, where isomorphism is in the sense above. Give an example of two categories which are equivalent but which have non-bijective objects (consider only small categories, i.e. categories whose objects and morphisms are each a set).

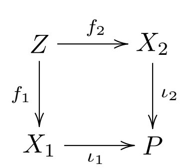
Exercise 6. A *coproduct* or *sum* of two objects X_1, X_2 in a category $\mathcal{C}$ is an object P, equipped with arrows $\iota_i : X_i \longrightarrow P, \ i = 1,2$ such that for any other object Q equipped with arrows $\nu_i : X_i \longrightarrow Q$, there exists a unique arrow $\nu : P \longrightarrow Q$ with $\nu_i = \nu \circ \iota_i$, for $i = 1,2$. We draw the coproduct like this:



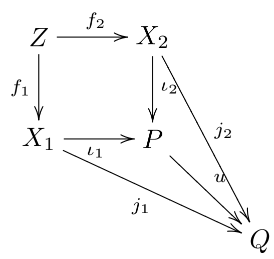
Show that if there are two coproducts of the pair X_1, X_2, then the two coproducts are canonically isomorphic.

Show that the category of sets, topological spaces, pointed topological spaces, groups, (and bonus: groupoids), always have coproducts, i.e. for any pair of objects X_1, X_2, there exists an object which is a coproduct of X_1, X_2.

Exercise 7. Given two objects X_1, X_2 in a category $\mathcal{C}$, and an object Z mapping to both X_i by $f_i : Z \longrightarrow X_i$, the *fibered coproduct* (also called *fibered sum* or *pushout*) of X_1, X_2 over Z is an object P and two morphisms $\iota_i : X_i \longrightarrow P$ such that the diagram commutes:



and such that (P, ι_1, ι_2) is universal for this diagram in the sense that for any other set (Q, j_1, j_2) fitting in the diagram, there must exist $u : P \longrightarrow Q$ making the following diagram commute:



Show that the categories from the previous exercise always have fibered sums.