Two important pieces of advice: 1. Read the notes!! Make sure you understand everything which is explained in the notes. Please let me know if you find mistakes in them. 2. Make sure you understand how to solve all the homework problems, even if you did not solve them in time for the deadline.

Exercise 1. A partition of unity $\{f_\alpha\}$ is *subordinate* to an open cover $\{U_i\}$ when $\forall\alpha$, $\mathrm{supp} f_\alpha \subset U_i$ for some i.

Show that If M is a manifold and $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ is any open covering, then there exists a partition of unity $\{f_\alpha\}_{\alpha\in A}$ subordinate to $\mathcal{U}$, indexed by the same set.

Finally, show that if $A \subset M$ be any closed subset of a manifold, and if U is any open neighbourhood of A, then there exists a smooth function $f_U : M \longrightarrow \mathbb{R}$ with $f_U \equiv 1$ on A and $\mathrm{supp} f_U \subset U$.

Exercise 2. Let $A \subset M$ be a subset of a smooth manifold. We say $f : A \longrightarrow \mathbb{R}$ is smooth when it has a smooth extension to some neighbourhood of each point $p \in A$.

If $A \subset M$ is closed and $f : A \longrightarrow \mathbb{R}$ is smooth, show that f has a smooth extension to a function $\tilde{f} : M \longrightarrow \mathbb{R}$, and show this extension can be made to vanish outside any open neighbourhood containing A. Is A necessarily closed for these results to hold?

Exercise 3. A *proper map* $f : X \longrightarrow Y$ is a continuous map such that inverse images of compact sets are compact. Show that any manifold M admits a smooth, proper real-valued function f. Can this function always be chosen with only positive values? If so, what constraint would be imposed on the cobordism class of a regular level set of such a function?

Give an explicit example of a function of the above type for $M = TS^2$, and for $M = T(S^1 \times \mathbb{R})$.

Exercise 4. Formulate a tubular neighbourhood theorem for a general regular submanifold M embedded in a manifold N and give a detailed outline of the proof (you may use a Whitney embedding of N to define the normal bundle of M in N).

Exercise 5. Give a detailed outline of a proof of the collar neighbourhood theorem: given a manifold with compact boundary ∂M, this boundary has a "collar" neighbourhood diffeomorphic to $\partial M \times [0, 1)$. You may model the proof on (or use) the steps leading up to and including the tubular neighbourhood theorem.

Exercise 6. Compute the mod 2 self-intersection number of the embedding $S^2 \hookrightarrow TS^2$ given by the zero section. Do the same for $S^1 \hookrightarrow TS^1$.

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Exercise 7. Find an example of a compact manifold X for which the diagonal embedding $\Delta : X \longrightarrow X \times X$ has nonzero $\pmod 2$ self-intersection.

Exercise 8. Let X be compact and $f : X \longrightarrow Y$ smooth with $\dim X = \dim Y$ and Y connected. Choose $y \in Y$ and define $\deg_2(f) = I_2(f, \iota)$, where $\iota : y \mapsto Y$ is the inclusion map.

- Show that $\deg_2(f)$ is independant of the point $y \in Y$.
- Compute $\deg_2(\mathrm{Id})$ for $\mathrm{Id} : X \longrightarrow X$ the identity map.
- Give example of a smooth surjective map $f : S^2 \longrightarrow S^2$ with $\deg_2(f) = 0$.

Exercise 9. Let $X \in C^\infty(M, TM)$ be a vector field, so that it may be viewed as a smooth section $X : M \longrightarrow TM$. The derivative of this map is a map

$$X_* : TM \longrightarrow T(TM).$$

Is this map also a vector field, but on $N = TM$? Prove your result.